109. *On the Distributivity of a Lattice of Lattice-congruences.*

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In a previous note[1] one of us studied the structure of the lattice formed of congruences of a finite-dimensional lattice to prove that it is a distributive lattice. In the following we want to show that the congruences of any lattice, not necessarily finite-dimensional, form always a distributive lattice. The proof is quite simple and direct. Namely:

Let L be a lattice and let $\Phi=\{\varphi\}$ be the (complete) lattice of its congruences; we mean by $\varphi_1 \geqq \varphi_2$ that[2] $a\equiv b$ mod. φ_1 implies $a\equiv b$ mod. φ_2. Thus $a\equiv b$ mod. $\varphi_1 \cup \varphi_2$ when and only when a and b are congruent both mod. φ_1 and mod. φ_2, while $a\equiv b$ mod. $\varphi_1 \cap \varphi_2$ is equivalent to that there exists a finite system of elements $c_1, c_2, \dots, c_n$ in L such that

$$(1) \qquad a\equiv c_1(\varphi_1),\ c_1\equiv c_2(\varphi_2),\ c_2\equiv c_3(\varphi_1), \dots,\ c_{n-1}\equiv c_n(\varphi_1),\ c_n\equiv b(\varphi_2).$$

Consider arbitrary three congruences φ_1, φ_2 and φ_3. Obviously $(\varphi_1 \cap \varphi_2) \cup \varphi_3 \leqq (\varphi_1 \cup \varphi_3) \cap (\varphi_2 \cup \varphi_3)$. In order to prove the converse inclusion, assume

$$(2) \qquad a\equiv b \text{ mod. } (\varphi_1 \cap \varphi_2) \cup \varphi_3$$

for a certain pair $a>b$ of elements in L. Then $a\equiv b$ mod. φ_3 and there is a finite set of elements $c_1, c_2, \dots, c_n$ such that (1) holds. Now, the transformation

$$x \rightarrow x' = (x \cap a) \cup b$$

maps L onto the interval $[b, a]$, and it preserves any congruence relation. On applying this tranformation to (1), we see that we may assume without loss of generality that

$$a \geqq c_i \geqq b \qquad (i=1, 2, \dots, n).$$

But then, since $a\equiv b$ mod. φ_3, the elements a, b and c_i are all congruent mod. φ_3. Hence

$$a\equiv c_1(\varphi_1 \cup \varphi_3),\ c_1\equiv c_2(\varphi_2 \cup \varphi_3),\ c_2 \equiv c_4(\varphi_1 \cup \varphi_3), \dots$$
$$\dots,\ c_{n-1}\equiv c_n (\varphi_1 \cup \varphi_3),\ c_n\equiv b(\varphi_2 \cup \varphi_3),$$

which means

$$a\equiv b \text{ mod. } (\varphi_1 \cup \varphi_3) \cap (\varphi_2 \cup \varphi_3).$$

Since this is the case for every pair $a>b$ in L satisfying (2), we have $(\varphi_1 \cap \varphi_2) \cup \varphi_3 \geqq (\varphi_1 \cup \varphi_3) \cap (\varphi_2 \cup \varphi_3)$ as desired. Thus

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1) N. Funayama, On lattice congruence, Proc. **18** (1942).

2) Contrary to the previous note, l. c. 1).

Theorem. The totality of the congruences of any lattice forms a distributive lattice.

Remark 1. By the same argument we find that in the complete lattice Φ of lattice-congruences the infinite distributive law

$$(\bigcap_{\tau} \varphi_\tau) \cup \varphi = \bigcap (\varphi_\tau \cup \varphi)$$

is valid. But the dual infinite distributivity does not hold in general, as the following example shows:

Let L be the interval $[0, 1]$ of real numbers considered as a linearly ordered lattice. Let S be the set of all the elements (namely, numbers) in L whose triadic expansions have 1 as a coefficient at least once. S consists of infinitely many mutually disjoint intervals (closed on the left and open on the right). Then let φ be a congruence of L which is obtained by defining two numbers belonging to one and the same interval in S to be congruent. On the other hand, let T_n be, for each natural number n, the set of numbers a in L such as

$$\frac{3\nu-1}{3^n} - \frac{1}{3^{n+1}} \leqq a \leqq \frac{3\nu+1}{3^n} + \frac{1}{3^{n+1}} \quad (\nu=0, 1, \ldots, 3^{n-1}).$$

Then T_n consists of $3^{n-1}+1$ mutually disjoint intervals, and the corresponding congruence φ_n can be introduced similarly as above. Since the lengths of intervals in T_n tends to 0 (as $n \to \infty$), we have $\bigcup_n \varphi_n = I$; here I means the unit-congruence (=equality). Thus

$$(\bigcup_n \varphi_n) \cap \varphi = \varphi.$$

On the other hand, L is, for each n, covered by S and T_n, and two elements in L are connected by a finite number of intervals in S and T_n. Hence $\varphi_n \cap \varphi$ is the 0-congruence (by which all the elements are congruent). Therefore

$$\bigcup_n (\varphi_n \cap \varphi) = 0.$$

Remark 2. Our theorem gives, as K. Yosida kindly pointed out, also a new proof to the fact that normal subgroups of a lattice-ordered group G form a distributive lattice; by a normal subgroup we mean an invariant subgroup which induces a congruence of G as a lattice-ordered group. For, a normal subgroup H gives certainly a congruence φ_H of G simply as a lattice, and it is easy to see that the join $\varphi_H \cup \varphi_{H'}$ and the meet $\varphi_H \cap \varphi_{H'}$ of the congruences φ_H and $\varphi_{H'}$, G being considered again simply as a lattice, are respectively the congruences induced by the meet and the join of the normal subgroups H, H'.