106. An Abstract Integral (IX).

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Methods defining integral without use of measure was studied by W. H. Young, P. J. Daniel[1], S. Banach[2], and H. H. Goldsteine[3], S. Izumi[4] extended Banach's method to the case of vector lattice. Many authors defined Lebesgue integral as an extension of Riemann or "abstract" Riemann integral. In this paper, extending Goldsteine's method we give a "Lebesgue integral" as an extension of a certain non-negative functional on a vector lattice.

§ **1.** Let Z be a vector lattice, and X a sublattice of Z which has the following properties: for any $z \in Z$ there exists at least one $x \in X$ such as $|z| \leqq x$. Let $f(x)$ be a linear non-negative functional on the domain X.

If we define the functionals

$$f^0(z) \equiv \underset{z \leqq x \in X}{\text{gr. l. b}}\, f(x), \qquad f_0(z) \equiv \underset{z \geqq x \in X}{\text{l. u. b}}\, f(x)$$

on Z, then we have

(1.1) $f_0(z) \leqq -f^0(-z)$.

(1.2) $f_0(z) \leqq f^0(z)$.

(1.3) $f_0(z_1+z_2) \geqq f_0(z_1)+f_0(z_2), \quad f^0(z_1+z_2) \leqq f^0(z_1)+f^0(z_2)$.

(1.4) $f_0(cz)=cf_0(z)$ and $f^0(cz)=cf^0(z)$, for any real non-negative number c.

(1.5) $f_0(z_1)+f_0(z_2) \leqq f_0(z_1 \wedge z_2)+f_0(z_1 \vee z_2) \leqq f^0(z_1 \wedge z_2) + f^0(z_1 \vee z_2) \leqq f^0(z_1)+f^0(z_2)$.

(1.6) $f^0(|z|)-f_0(|z|) \leqq f^0(z)-f_0(z)$.

We shall prove the last two only. If $x_1 \geqq z_1$, $x_2 \geqq z_2$, $x_1 \in X$, $x_2 \in X$, then we have $x_1 \vee x_2 \geqq z_1 \vee z_2$ and $x_1 \wedge x_2 \geqq z_1 \wedge z_2$, and then

$$f^0(z_1 \wedge z_2)+f^0(z_1 \vee z_2) \leqq f(x_1 \wedge x_2)+f(x_1 \vee x_2)=f(x_1+x_2)=f(x_1)+f(x_2),$$

$$f^0(z_1 \wedge z_2)+f^0(z_1 \vee z_2) \leqq f^0(z_1)+f^0(z_2).$$

Similarly $\quad f_0(z_1 \wedge z_2)+f_0(z_1 \vee z_2) \geqq f_0(z_1)+f_0(z_2)$.

Hence we have the relation (1.5). In the next place by

$$|z|=z \vee (-z), \quad \text{and} \quad -|z|=(-z) \wedge z,$$

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1) Annals of Mathematics, **19** (1918).
2) S. Saks: The theory of integral.
3) Bull. of the Amer. Math. Soc., **47** (1941).
4) S. Izumi: Isōsūgaku 3-2, (1941).

we have

$$f^0(|z|)-f_0(|z|)=f^0(|z|)+f^0(-|z|)=f^0\big(z\vee(-z)\big)$$
$$+f^0\big((-z)\wedge z)\big)\leqq f^0(z)+f^0(-z)=f^0(z)-f_0(z)\,.$$

Thus we get (1.6).

Let us put $Y\equiv\big(y\in Z;\, f^0(y)=f_0(y)\big)$, and we write $F(y)\equiv f^0(y)=f_0(y)$ for $y\in Y$. Then $F(y)$ is evidently a non-negative functional in Y.

Theorem 1. For any real number c and any $y\in Y$, we have $cy\in Y$ and $F(cy)=cF(y)$.

Theorem 2. For any $y_1\in Y$ and $y_2\in Y$, we have $y_1+y_2\in Y$ and $F(y_1+y_2)=F(y_1)+F(y_2)$.

Theorem 3. For any $y\in Y$, we have $|y|\in Y$ and $|F(y)|\leqq F(|y|)$.

Proof. By the use of (1.6) we have immediately $f^0(|y|)=f_0(|y|)$, and so $y\in Y$ implies $|y|\in Y$. Besides we have

$$-F(|y|)=F(-|y|)\leqq F(y)\leqq F(|y|)\,,$$

or

$$|F(y)|\leqq F(|y|)\,.$$

(1.9) If y_1 and $y_2\in Y$ then $y_1\vee y_2$ and $y_1\wedge y_2$ are contained in Y.

(1.10) If $x\in X$ then $F(x)=f(x)$.

From the above discussion we see that the system Y is a vector lattice and $F(y)$ is a linear non-negative functional and Y is an extension of X. If we define $F^0(z)$ and $F_0(z)$ by F and Y, similarly as f_0 and f^0 was defined by f and X, then we have

(1.11) $F^0(z)=f^0(z)$ and $F_0(z)=f_0(z)$.

Proof. Let z, y and x be elements of Z, Y and X, respectively, such as $z\leqq y\leqq x$. Then we have $F^0(z)\leqq F(y)=f^0(y)\leqq f(x)$, and then $F^0(z)\leqq f^0(z)$. On the other hand by the definition, there exists an element y of Y such that $y\geqq z$ and $F^0(z)>F(y)-\varepsilon=f^0(y)-\varepsilon\geqq f^0(z)-\varepsilon$ or $F^0(z)\geqq f^0(z)$. That is $F^0(z)=f^0(z)$. Similarly $F_0(z)=f_0(z)$.

§ **2.** We will now introduce a new condition (C): Z is a σ-complete vector lattice, and $x_n\in X$ and $x_n\uparrow z$ imply $\lim\limits_{n\to\infty}f(x_n)=f^0(z)$.

(2.1) If $x_n\in X$ and $x_n\uparrow z$, then $z\in Y$ and $\lim\limits_{n\to\infty}f(x_n)=F(z)$.

Proof. $\lim\limits_n f(x_n)=f^0(z)\geqq f_0(z)\geqq\lim\limits_n f(x_n)$, and so $\lim\limits_n f(x_n)=F(z)$.

Theorem 4. If $y_n\in Y$ and $y_n\uparrow y$, then $y\in Y$ and $F(y)=\lim\limits_n F(y_n)$.

Proof. We can choose an element $x_n\in X$, such that for arbitrary ε $x_n\geqq y_n$ and $F(y_n)>f(x_n)-\dfrac{\varepsilon}{2^n}$. Now let a be a fixed element of X such as $a\geqq y$. Then it is evident that such element exists. We can suppose that $X_n\leqq a$, for otherwise it suffices to consider $a\wedge x_n$ instead of x_n.

Putting $x'_n \equiv \bigvee_{n=1}^{n} x_n$ and $\varepsilon_n \equiv \left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)\varepsilon$, we have $F(y_n) > f(x'_n) - \varepsilon_n \geqq f(x'_n) - \varepsilon$. For, this is true for $n=1$. We assure that this is true for $n-1$. By $x'_n \in X$ and $(x'_{n-1} \wedge x_n) \geqq y_{n-1}$ we have

$$f(x'_{n-1} \wedge x_n) \geqq F(y_{n-1}) > f(x'_{n-1}) - \varepsilon_{n-1}.$$

On the other hand

$$x'_{n-1} + x_n = (x'_{n-1} \vee x_n) + (x'_{n-1} \wedge x_n) = x'_n + (x'_{n-1} \wedge x_n),$$

so that

$$f(x'_{n-1}) + f(x_n) = f(x'_n) + f(x'_{n-1} \wedge x_n) > f(x'_n) + f(x'_{n-1}) - \varepsilon_{n-1},$$

or

$$f(x_n) > f(x'_n) - \varepsilon_{n-1}.$$

This is

$$F(y_n) > f(x_n) - \frac{\varepsilon}{2^n} > f(x'_n) - \varepsilon_{n-1} - \frac{\varepsilon}{2^n} = f(x'_n) - \varepsilon_n,$$

and so

$$F(y_n) > f(x'_n) - \varepsilon_n.$$

Since $x'_n \uparrow x'$, $x' \in Y$ by (2.1), and so $x' = \bigvee_{n=1}^{\infty} x'_n \geqq \bigvee_{n=1}^{\infty} y_n = y \geqq y_n$. Consequently

$$F(x') \geqq F^0(y) \geqq F_0(y) \geqq F(y_m) > f(x'_m) - \varepsilon_m > f(x'_m) - \varepsilon,$$

or

$$F(x') = F^0(y) = F_0(y) = \lim_m F(y'_m).$$

Instead of (C), if we suppose that (C'): Z is a σ-complete vector lattice and $0 \leqq x_n \in X$, $X_n \uparrow z$ imply $\lim_n f(x_n) = f^0(z)$, then we have

Theorem 4'. If $0 \leqq y_n \in Y$ and $y_n \uparrow y$, then $y \in Y$ and $F(y) = \lim F(y_n)$.

Theorem 5. If $y_n \in Y$, $\lim_n y_n = y$, $|y_n| \leqq \bar{y}$ $(n=1, 2, \ldots)$, and $\bar{y} \in Y$, then $y \in Y$, and $\lim_n F(y_n) = F(y)$.

Proof. We have

$$\bigvee_{k=n}^{\infty} y_k = \lim_m \bigvee_{k=n}^{m} y_k, \quad \text{and} \quad (\bigvee_n^m y_k) \uparrow (\bigvee_n^{\infty} y_k),$$

consequently $\bigvee_n^{\infty} y_k \in Y$.

Moreover $(\bigvee_n^{\infty} y_k) \downarrow y$ (as $n \to \infty$), so that $-\bigvee_n^{\infty} y_k \uparrow -y$ or $y \in Y$.

$$\lim_n F(-\bigvee_n^{\infty} y_k) = F(-y) \quad \text{or} \quad \lim_n F(\bigvee_n^{\infty} y_k) = F(y).$$

That is, for any $\varepsilon > 0$ there exists $n_1(\varepsilon)$ such that

$$F(y_n) \leqq F(\bigvee_n^{\infty} y_k) < F(y) + \varepsilon \quad \text{for all} \quad n \geqq n_1(\varepsilon),$$

or

$$F(y_n) < F(y) + \varepsilon \quad \text{for all} \quad n \geqq n_1(\varepsilon).$$

Similarly we have

$$F(y_m) > F(y) - \varepsilon \quad \text{for all} \quad m \geqq n_2(\varepsilon).$$

So that $$|F(y_p) - F(y)| \leqq \varepsilon \quad \text{for} \quad p \geqq \max(n_1, n_2).$$

By above theorems 1–5, we see that the functional F has the all properties of the Lebesgue integral[1]. That is, starting from the system X and functional $f(x)$ on X, we can reach the functional $F(y)$ which has the properties of the integral, and the class of integrable functions Y contains X.

1) Fatou's theorem is modified.