104. A Note on Infinite Series.

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Introduction. Theorem A. If the infinite series $\sum a_n$ *converges and* $a_n \downarrow 0$ (a_n *tends to zero monotonously), then* $na_n \to 0$.

This is the classical theorem due to Olivier. This is generalized by Cesàro[1], de la Vallée Poussin[2], Rademacher[3], Ostrowski[4], Knopp[5], Izumi[6] and Meyer-König[7]. Ostrowski's theorem reads as follows.

Theorem B. If $a_n \downarrow 0$, *then* $\sum a_n$ *converges when and only when* $s_n - na_n$ converges.

This theorem contains Olivier's theorem. On the other hand Cesàro proved that

Theorem C. Let p_n *and* q_n *be the number of positive and negative terms in*

$$s_n \equiv a_1 + a_2 + \cdots + a_n .$$

If $\sum a_n$ *converges and* $|a_n| \downarrow 0$, *then* $(p_n - q_n)a_n \to 0$.

These theorems suggest us the following theorem.

Theorem D. If $|a_n| \downarrow 0$, $\sum a_n$ *converges when and only when* $s_n - (p_n - q_n)|a_n|$ *converges as* $n \to \infty$.

But we can show that this is not true in general (§ 3). Therefore in order to get the theorem of this type, we need some additional conditions. We give two types of conditions. That is, the one is concerning the magnitude of a_n and the other is concerning the sign of a_n. This is given in Theorem 1 and 2. Conditions in the theorem are the best possible ones in a sense. Incidentally we give a new proof of all above theorems (§ 1). Finally we remark that our problem is transformed into that of function theory.

§ **1.** *Proof of Theorem C.* The identity

$$(\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n)/\lambda_n$$
$$= s_n - \big((\lambda_2 - \lambda_1)s_1 + (\lambda_3 - \lambda_2)s_2 + \cdots + (\lambda_n - \lambda_{n-1})s_{n-1}\big)/\lambda_n \qquad (1)$$

is well known and is easy to verify. Put $a_\nu \equiv |a_\nu| \cdot e_\nu$, $\lambda_\nu = 1/|a_\nu|$, then the left hand side of (1) becomes

$$(e_1 + e_2 + \cdots e_n)|a_n| = (p_n - q_n)|a_n| . \qquad (2)$$

Since $\lambda_n \uparrow \infty$, by the Toeplitz theorem

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1) Cesàro, Rom. Acc. Lincei Rend., **4** (1888), p. 133.
2) de la Vallée-Poussin, Cours d'Analyse Infinitesimale, **1** (1914), p. 408.
3) Rademacher, Math. Zeitschr., **11** (1921), p. 276.
4) Ostrowski, Jahresb. der D.M.V., **34** (1926), p. 161.
5) Knopp, Jahresb. der D.M.V., **37** (1928), p. 325.
6) Izumi, Proc. of the Physico-Math. Soc., **16** (1934), p. 127.
7) Meyer-König, Math. Zeitschr., **45** (1939), p. 751.

$$\big((\lambda_2-\lambda_1)s_1+(\lambda_3-\lambda_2)s_2+\cdots+(\lambda_n-\lambda_{n-1})s_{n-1}\big)/\lambda_n\to s$$

if $s_n\to s$. Thus (1) gives the required result.

Proof of Theorem B. By (1) and (2) (when $a_n>0$)

$$s_n-na_n=\big((\lambda_2-\lambda_1)s_1+(\lambda_3-\lambda_2)s_2+\cdots+(\lambda_n-\lambda_{n-1})s_{n-1}\big)/\lambda_n. \qquad (3)$$

If $a_n>0$, then s_n converges or diverges to infinite. By the regularity of summation-method, we get the theorem.

Similarly we can prove theorems due to de la Vallée Poussin, Rademacher, Knopp and Izumi.

§ **2.** *Theorem 1. If* $|a_n|\downarrow 0$ *and* $1/(|a_n|^{-1}-|a_{n-1}|^{-1})=O(1)$, *then* $s_n-(p_n-q_n)|a_n|\to 0$ *implies* $s_n\to 0$.

Proof. By the Tauberian theorem, if $\lambda_n a_n/(\lambda_n-\lambda_{n-1})=O(1)$ then the right hand side of (3) converges. Now

$$\lambda_n a_n/(\lambda_n-\lambda_{n-1})=e_n/(\lambda_n-\lambda_{n-1})=1/(|a_n|^{-1}-|a_{n-1}|^{-1}).$$

Thus we get the theorem.

Theorem 2. If $\sum(p_n-q_n)^{-2}$ *converges and* $(p_n-q_n)\sum_{m=n}^{\infty}(p_m-q_m)^{-2}$ *tends to zero, then* $s_n-(p_n-q_n)|a_n|\to 0$ *implies* $s_n\to 0$.

Proof. If we put $s_n\equiv(p_n-q_n)t_n$, then

$$\begin{aligned} s_n-(p_n-q_n)|a_n| &= s_n-(p_n-q_n)e_n a_n \\ &= (p_n-q_n)t_n-e_n(p_n-q_n)\big((p_n-q_n)t_n-(p_{n-1}-q_{n-1})t_{n-1}\big) \\ &= e_n(p_n-q_n)(p_{n-1}-q_{n-1})(t_{n-1}-t_n). \end{aligned}$$

Thus we have

$$t_{n-1}-t_n=o\big((p_n-q_n)^{-1}(p_{n-1}-q_{n-1})^{-1}\big)=o\big((p_n-q_n)^{-2}\big).$$

By the convergence of $\sum(p_n-q_n)^{-2}$, t_n tends to a constant A. And then $s_n=(p_n-q_n)A+o(1)$. That is $a_n\to A$, which gives $A=0$. Therefore $s_n\to 0$.

§ **3.** We will now give an example that $|a_n|\downarrow 0$, $s_n-(p_n-q_n)\cdot|a_n|\to 0$ do not imply $s_n\to 0$. Let $0<\alpha<1/2$, $[n_{p-1}^{\alpha}]\equiv p$. Let $|a_n|\equiv 1/p$ in $n_p^{\alpha}\leqq n\leqq n_p^{\alpha}+2p-1$ and $1/n^{\alpha}$ otherwise. Then $|a_n|\downarrow 0$. Let us define p_n-q_n such that the curve of $p_\nu-q_\nu$ inscrives $(-1)^{p-1}\nu^{\alpha}$ as closely as possible in $n_p^{\alpha}+2p-1<\nu<n_{p+1}^{\alpha}$ and is linear otherwise. More precisely, let $e_1\equiv 1$. Further let $e_2=-1$, $e_3=+1$, $e_4=-1,\dots$ alternatively till $[n^{\alpha}]$ becomes 2. If $[n_1^{\alpha}]=2$, $e_{n_1-1}=+1$ then $e_{n_1}=e_{n_1+1}=e_{n_1+2}=-1$, and otherwise $e_{n_1+1}=e_{n_1+2}=e_{n_1+3}=-1$. For larger n, e_n is defined $+1$ or -1, alternatively till $[n^{\alpha}]$ becomes 3. If $[n_2^{\alpha}]=3$, $e_{n_2-1}=-1$, then $e_{n_2}=e_{n_2+1}+1=\cdots=e_{n_2+4}=+1$, and otherwise $e_{n_2+1}=e_{n_2+2}=\cdots=e_{n_2+5}=+1$. For larger n, e_n is defined $+1$ or -1 alternatively till $[n^{\alpha}]$ becomes 4. Thus proceeding we can define (e_n) such that $p_n-q_n=O(n^{\alpha})$ and p_n-q_n oscilates between n^{α} and $-n^{\alpha}$.

If we put $a_n\equiv e_n|a_n|$, then $s_n=a_1+a_2+\cdots+a_n$ oscilates boundedly. For $[n^{\alpha}]$ terms are of the same sign in the neighbourhood of n such that n^{α} becomes integer and their sign is $(-1)^{[n^{\alpha}]-1}$ and otherwise alternative. Thus $s_n - \sum_{\nu^{1/a} \leq n} (-1)^{\nu-1}(2\nu-1)a_{\nu a}$ is alternative series. Now $(2\nu-1)a_{\nu a} = (2\nu-1)/\nu = 2-1/\nu$. Hence s_n oscillates boundedly.

On the other hand

$$\lambda_n = \lambda_{n-1} = 1/|a_n| - 1/|a_{n-1}| = n^a - (n-1)^a = an^{a-1} + o(1).$$

If we replace s_n by $s'_n \equiv s_n - \sum_{\nu^{1/a} \leq n} (-1)^{\nu-1}(2\nu-1)a_{\nu a}$ in

$$s_n - (p_n - q_n)|a_n| = \Big((\lambda_2 - \lambda_2)s_1 + (\lambda_3 - \lambda_2)s_2 \cdots + (\lambda_n - \lambda_{n-1})s_{n-1}\Big)/\lambda_n,$$

then the resulting sequence converges. If s_n by $s''_n \equiv \sum_{\nu^{1/a} \leq n} (-1)^{\nu-1}(2\nu-1)a_\nu$, then the resulting sequence also converges. Thus $s_n - (p_n - q_n)|a_n|$ converges.

If $1 > a \geqq \frac{1}{2}$ it is enough to take a subsequence of (n_ν) with sufficiently large gap and follow above method concerning such subsequence.

This example shows that Theorem 1 and 2 are best possible in a sense. For, if sign of a_n are given arbitrarily, then the magnitude of a_n can not be of order $\leqq 1/n^a (a < 1)$. Condition of Theorem 1 becomes $|a_n| = 1/n$ if $|a_n| = 1/n^\beta$. On the other hand if the magnitudes of a_n are given arbitrarily then $p_n - q_n$ cannot be of order $\leqq n^a (a < 1)$.

§ **4.** Let $f(z)$ be an analytic function defined in the unite circle, $r_1, r_2, \dots, r_n$ and $\rho_1, \rho_2, \dots, \rho_m$ be the absolute value of zeros and poles of $f(z)$ in $|z| < R (R < 1)$. By the Jensen's formula

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\varphi})| \, d\varphi = \log \left| \frac{\rho_1 \cdot \rho_2 \cdots \rho_m}{r_1 \cdot r_2 \cdots r_n} \right| f(0) R^{n-m}.$$

Without loss of generality we can suppose that $f(0) = 1$. If we arrange r_i and ρ_j in the increasing order of magnitude and put $\log(1/r_i) \equiv a_k$, $\log \rho_j \equiv -a_j$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\varphi})| \, d\varphi = s_{m+n} - (n-m)|a_{n+m}|$$
$$= s_{m+n} - (p_{m+n} - q_{m+n})|a_{m+n}|$$

by the notation used above. Thus the relation of

$$\lim_{n\to\infty} s_n \quad \text{and} \quad \lim_{R\to 1} \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\varphi})| \, d\varphi$$

is given by our theorem.