PAPERS COMMUNICATED

102. *On the Regular Vector Lattice.*

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Introduction. L. V. Kantrovitch introduced the notion of regularity[1] in vector lattice and applied it to the space of measurable functions. In § 1 of this paper, we prove that the regularity axiom is decomposed into two simple propositions. In the succeeding articles we prove many theorems in Kantrovitch's paper under weaker assumption.

§ 1. Let $\mathfrak{L}$ be a complete vector lattice. Then the regularity axiom due to Kantrovitch reads as follows:

If $E_n \subset \mathfrak{L}$ for $n=1,2,\dots$ and $\sup E_n$ tends to a limit y, then for each n there exists a finite subset E'_n of E_n such that $\lim\limits_{n\to\infty} E'_n = y$.

For regular vector lattice $\mathfrak{L}$, two theorems hold as Kantrovitch shows.

I. If $y_i^{(k)} \to y_i\,(o)$ (as $k\to\infty$) and $y_i \to y\,(o)$ (as $i\to\infty$) in $\mathfrak{L}$, then there exists an increasing sequence of indices $k_1, k_2, \dots$ such that $y_i^{(k_i)} \to y\,(o)$ $(i\to\infty)$[2]

II. For any set $E\subset\mathfrak{L}$, there exists an enumerable subset E' of E such that $\sup E' = \sup E$[3]

Conversely, we can prove the following theorem.

Theorem 1.1. **I** and **II** imply the regularity axiom.

Proof. By **II**, for each E_n there exists an enumerable set $E'_n = \{y_{n,k}\}$ $k=1,2,\dots$, such that $\sup E_n = \sup\,(y_{n,k})$ $k=1,2,\dots$ If we put $y_n^{(k)} = \sup(y_{n,1},\dots,y_{n,k})$, then $y_n^{(k)} \uparrow \sup E_n$ $(n\to\infty)$. Therefore, if $\lim\limits_{n\to\infty}\sup E_n = y_0$, then by **I** we can find an increasing sequence of indices $\{k_n\}$ such that $\lim\limits_{n} y_n^{(k_n)} = y_0$. Hence $\lim\sup\,(y_{n,1},\dots,y_{n,k_n}) = \lim\limits_{n\to\infty}\sup E_n$.

From the proof it is easy to see that in **II** we can replace the condition $y_i^{(k)} \to y_i\,(o)$ (as $k\to\infty$) by $y_n^{(k)} \uparrow y_n\,(o)$ $(k\to\infty)$.

In the space of measurable functions (S), (o)-convergence is equivalent to almost everywhere convergence[4]. Therefore, **I** is nothing but Fréchet's theorem[5].

We can easily verify that the space (S) satisfies **II**. But more generally we can prove

Theorem 1.2. **II** holds in the space of functions with metric function ρ such that 1°. for any $y \geqq 0$, $\rho(y)$ is defined and $\geqq 0$ and $\rho(y)=0$

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1) L. V. Kantrovitch: Lineare halbgeordnete Räume, Recueil Math., **44** (1937), pp. 121–165.
2) loc. cit., Satz 24.
3) loc. cit., Satz 23, a).
4) G. Birkhoff, Lattice theory, Chapter VII.
5) M. Fréchet, Rendiconti di Palermo, **22** (1906), p. 15.

is equivalent to $y=0$, 2°. $y_1 \leqq y_2$ implies $\rho(y_1) < \rho(y_2)$, 3°. $y_n \to y$ (mototonously) implies $\rho(y_n) \to \rho(y)$.

Proof. Let $E \subset \mathfrak{L}$ be an upper bounded set, that is, there exists $y^* \in \mathfrak{L}$ such as $y \leqq y^*$ for any $y \in E$. We can assume that E contains zero-element.

If we put $\bar{y} = \sup(0, y_1, \dots, y_n)$ $(y_i \in E)$, then $\bar{y} \leqq y^*$. Therefore $\rho(\bar{y}) \leqq \rho(y^*)$ by 2°, hence $\{\rho(\bar{y})\}$ is bounded. That is, there exists a number ρ_0 such that $|\rho(\bar{y})| \leqq \rho_0$. If we put $\rho_0 = \text{l. u. b.}\, \rho(\bar{y})$, then there exists $\{\bar{y}_n\}$ such that $\lim_{n\to\infty} \rho(\bar{y}_n) = \rho_0$. We may assume $\bar{y}_1 \leqq \bar{y}_2 \leqq \cdots$. Let $\lim \bar{y}_n = y'$, then $\rho(\bar{y}_n) \to \rho(y')$ by 3°. Thus we have $\rho(y') = \rho_0$.

We will now prove that $y' = \sup E$. $y \in E$ implies $\lim_{n\to\infty} \sup(\bar{y}_n, y) = \sup(y', y)$. Since $\rho\big(\sup(y'_n, y)\big) \leqq \rho_0$, we have $\rho\big(\sup(y', y)\big) \leqq \rho_0$. Obviously, $\sup(y', y) \geqq y'$. Therefore $\rho\big(\sup(y', y)\big) \geqq \rho(y') = \rho_0$. Thus $\sup(y', y) = y'$, namely $y \leqq y'$. Thus we have $y' = \sup E$. (Q. E. D.)

Evidently conditions 1°, 2°, 3° for ρ are satisfied in (S), L^p $(p \geqq 1)$, (s) and l^p $(p \geqq 1)$.

§ **2.** Let $\mathfrak{L}$ be a σ-complete vector lattice for which **I** holds.

Lemma 2.1. σ-complete vector lattice is archimedian, that is, $f > 0$ and $\lambda_n \downarrow 0$ imply $\lambda_n f \downarrow 0$.

For the proof, see Birkhoff, Lattice theory, p. 106, Theorem 7.3.

Lemma 2.2. The sequence $\{f_n\}$ (o)-converges to f if and only if $|f_n - f| \leqq w_n$, for some $w_n \downarrow 0$.

For the proof, see Birkhoff, loc. cit., p. 112, Lemma 2.

Theorem 2.1. If $y_n \to 0$ (o), then there exists a sequence of real numbers $\{\lambda_n\}$ such that $\lambda_n \to \infty$ and $\lambda_n y_n \to 0$ (o).

Proof. If we put $\bar{y}_n = \sup(|y_n|, |y_{n+1}|, \dots)$, then $\bar{y}_n \downarrow 0$. Further put $\dot{y}_n^{(k)} = k\bar{y}_n$ $(k=1, 2, \dots)$, then $\lim \dot{y}_n^{(k)} = 0$ $(k=1, 2, \dots)$. By **I**, there exists an increasing sequence (n_k) of integers such that $\lim_{k\to\infty} \dot{y}_{n_k}^{(k)} = 0$. Therefore $\lim_{k\to\infty} k y_{n_k} = 0$.

Let up put $\lambda_n = k$ if $n_k \leqq n < n_{k+1}$. Evidently $\lambda_n \uparrow +\infty$ and $\lim_{n\to\infty} \lambda_n \bar{y}_n = \lim_{k\to\infty} k\bar{y}_{n_k} = 0$. Hence $\lim \lambda_n y_n = 0$.

Theorem 2.2. In $\mathfrak{L}$ (o)-convergence is equivalent to relative uniform convergence.

Proof. Obviously, relative uniform convergence implies (o)-convergence. Conversely, if $y_n \to y$ (o), then by theorem 2.1 $\lambda_n |y_n - y| \to 0$ for some $\lambda_n \uparrow +\infty$. From Lemma 2.2, there exists $\{w_n\}$ such that $\lambda_n |y_n - y| < w_n$, $(w_n \downarrow 0)$. Putting $1/\lambda_n = \varepsilon_n$, we have $|y_n - y| < \varepsilon_n w_1 (\varepsilon_n \downarrow 0)$. Therefore $\{y_n\}$ converges relative uniformly to y.

Theorem 2.3. If $\lim_{k\to\infty} y_i^{(k)} = y_i$ $(i=1, 2, \dots)$, then for any $\varepsilon > 0$ there exists $y_0 \in \mathfrak{L}$ such that $|y_i^{(k)} - y_i| \leqq \varepsilon y_0$ for $k \geqq K(\varepsilon, 1)$.

Proof. For each i, there exists $y_0^{(i)}$ such that $|y_i^k - y_i| \leqq \varepsilon y_0^{(i)}$ for $k \geqq K(\varepsilon, i)$. By Lemma 2.1 $\lim_{n\to\infty} \frac{1}{n} y_0^{(i)} = 0$ $(i=1, 2, \dots)$ and by **I** $\lim_{i\to\infty} \frac{1}{n_i} y_0^{(i)} = 0$ for some $\{n_i\}$ $(n_1 < n_2 < \cdots)$. Therefore $\left|\frac{1}{n_i} y_0^{(i)}\right| \leqq w_i$ $(w_i \downarrow 0)$. If we put $w_n \leqq w_1 = y_0$, then for each i $\left|\frac{1}{n_i} y_0^{(i)}\right| \leqq y_0$.

§ **3.**

Theorem 3.1. If $\mathfrak{L}$ is a σ-complete vector lattice for which **I** holds, then closure operation defined by (o)-topology satisfies Kuratowski's axiom;

1. if E is one point or vacuous, $\overline{E}=E$,
2. $\overline{E_1 \cup E_2} = \overline{E}_1 \cup \overline{E}_2$,
3. $\overline{\overline{E}} = \overline{E}$.

Proof is evident. Thus we can introduce topological convergence.

Concerning the relation of topological convergence and (*)-convergence, we have

Theorem 3.2. That $\{y_n\}$ is topologically convergent to y $\big(y_n \to y(t)\big)$ is equivalent to that y_n is (*)-convergent to y.

We can prove following series-theorems in our space.

Theorem 3.3. a) In order that $\sum_{i=1}^{\infty} y$ converges, it is necessary and sufficient that $\lim_{m,n\to\infty} (S_m - S_n) = \lim_{m,n\to\infty} \sum_{i=n+1}^{m} y_i = 0$, where $S_n = \sum_{i=1}^{n} y_i$.

b) If $\sum |y_i|$ is convergent, then $\sum y_i$ is also.

c) If $|S_i| \leqq y_0$ and $\lambda_i \downarrow 0$, then $\sum_{i=1}^{\infty} \lambda_i y_i$ is convergent.

d) Whatever be y_i, there exists real numbers $\lambda_i > 0$ such that $\sum_{i=1}^{\infty} \lambda_i |y_i|$ is convergent.

e) If $\sum y_i$ is convergent, then $\sum_{i=1}^{\infty} \lambda_i |y_i|$ is convergent for some real number $\lambda_i \to \infty$.

f) If $y_i \to 0$, then there exists real numbers $\lambda_i > 0$ such that $\sum \lambda_i$ is divergent but $\sum \lambda_i y_i$ is convergent.

§ **4.** We have proved in § 1 that **I** holds in space (S). But more generally we get

Theorem 4.1. **I** holds for the vector lattice with the metric function ρ such that 1°, 3° in Theorem 1.2 and 2'° $y_1 \leqq y_2$ implies $\rho(y_1) \leqq \rho(y_2)$, 4°. $y_n \uparrow +\infty$ not implies $\lim_{n\to\infty} \lim_{p\to\infty} (y_{n+p} - y_n) = 0$.

For the proof we need a lemma.

Lemma 4.1. $y_n \to 0$ is equivalent to $\lim \rho\big(\sup(|y_n|, |y_{n+1}|, \cdots |y_m|)\big) = 0$.

Proof. *Necessity.* $y_n \to 0$ implies $|y_n| \to 0$, therefore $\lim \big(\sup(|y_n|, |y_{n+1}|, \cdots)\big) = 0$. $\sup(|y_n|, |y_{n+1}|, \cdots)$ is monotone decreasing with n, hence by 3° $\lim_{n\to\infty} \rho\big(\sup(|y_n|, |y_{n+1}|, \cdots)\big) = 0$. Hence, for $n \geqq N$ $\rho\big(\sup(|y_N|, |y_{N+1}|, \cdots)\big) < \varepsilon$. Therefore, for $n, m > N$, $\rho\big(\sup(|y_n|, |y_{n+1}|, \cdots, |y_m|)\big) < \varepsilon$.

Sufficiency. For any $\varepsilon \geqq 0$ there is an N such that $n, m \geqq N$

implies $\rho\big(\sup(|y_n|, |y_{n+1}|, \cdots |y_m|)\big) < \varepsilon$. Thus for $n \geqq N$ $\rho\big(\sup(|y_n|, |y_{n+1}|, \ldots)\big) = \lim_{n\to\infty} \rho\big(\sup(|y_n|, |y_{n+1}|, \ldots, |y_m|)\big) < \varepsilon$. Since $\sup(|y_n|, |y_{n+1}|, \ldots)$ is monotone decreasing, there exists a limit. But $\rho\big(\sup(|y_n|, |y_{n+1}|, \ldots)\big) \to 0$ implies $\lim_{n\to\infty} \big(\sup(|y_n|, |y_{n+1}|, \ldots)\big) = 0$. Hence $\lim |y_n| = 0$. Thus $\lim y_n = 0$.

Proof of theorem. We will distinguish four cases.

1) $y_n^{(k)} \downarrow y_n$ $(k \to \infty)$ and $y_n \downarrow 0$ $(n \to \infty)$ imply that there exists a sequence of elements $\{y_n^{(k_n)}\}$ tending to 0. In fact, $y_n \downarrow 0$ implies $\rho(y_n) \downarrow 0$, hence, we can find real $\varepsilon_n \to 0$ such that $\rho(y_n) < \varepsilon_n$. $\rho(|y_1^{(k)}|) \to \rho(y_1) < \varepsilon_1$ implies $\rho(|y_1^{(k_1)}|) < \varepsilon_1$ for some index k_1. We have $\rho(|y_1^{(k_1)}| \cup |y_2^{(k)}|) \to \rho(|y_1^{(k_1)}| \cup y_2) \leqq \rho(|y_1^{(k_1)}| \cup y_1) = \rho(|y_1^{(k_1)}|) < \varepsilon_1$, and $\rho(|y_2^{(k)}|) \to \rho(y_2) < \varepsilon_2$ $(k \to \infty)$. Hence, there exists k_2 such that $\rho(|y_1^{(k_1)}| \cup |y_2^{(k_2)}|) < \varepsilon_1$, $\rho(|y_2^{(k_2)}|) < \varepsilon_2$. Thus proceeding we can find $\{k_n\}$ such that $\rho\big(\sup(|y_n^{(k_n)}|, \ldots, |y_{n+p}^{(k_{n+p})}|)\big) < \varepsilon_n$ $(n=1, 2, \ldots;\ p=1, 2, \ldots)$. Lemma 4.1 gives $\lim_{n\to\infty} y_n^{(k_n)} = 0$, which is the required.

2) Let us suppose that $y_n^{(k)} \to y_n$ $(k \to \infty)$ and $y_n \downarrow 0$. By Lemma 2.1, there exists $\{w_n^{(k)}\}$ such that $|y_n^{(k)} - y_n| \leqq w_n^{(k)}\big(w_n^{(k)} \downarrow 0\ (k \to \infty)\big)$. We have $|y_n^{(k)}| \leqq y_n + w_n^{(k)}$, and $y_n + w_n^{(k)} \downarrow y_n$ $(k \to \infty)$, $y_n \downarrow 0$. By the case 1), there exists $\{k_n\}$ such as $y_n + w_n^{(k_n)} \to 0$. Thus $|y_n^{(k_n)}| \to 0$, $y_n^{(k_n)} \to 0$.

3) Let $y_n^{(k)} \to y_n$ $(k \to \infty)$ and $y_n \to 0$. If we put $\bar{y} = \sup(y_n, y_{n+1}, \ldots)$, then $\bar{y}_n \downarrow 0$. Putting $\bar{y}_n^{(k)} = \sup(\bar{y}_n \cup |y_n^{(k)}|)$, we have $|\bar{y}_n^{(k)}| \to \bar{y}_n$ $(k \to \infty)$. Therefore, by the case 2) $|\bar{y}_n^{(k_n)}| \to 0$ $(n \to \infty)$ and then $\bar{y}_n^{(k_n)} \to 0$. Thus we have $y_n^{(k_n)} \to 0$.

4) general case is easily reduced to the case 3).

For the concrete case metric function ρ may be taken as follows.

$$\rho(y) = \int_E \frac{|y(t)|}{1+|y(t)|}\, dt \quad \text{if} \quad \mathfrak{L} \equiv (S)$$

$$\rho(y) = \int_E |y(t)|^p\, dt \quad \text{if} \quad \mathfrak{L} \equiv L^p (p \geqq 1),$$

$$\rho(y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\eta^{(i)}|}{1+|\eta^{(i)}|}, \quad \text{where} \quad y = (\eta^{(1)}, \eta^{(2)}, \ldots) \quad \text{if} \quad \mathfrak{L} = (s),$$

$$\rho(y) = \sum_{i=1}^{\infty} |\eta^{(i)}|^p, \quad \text{where} \quad y = (\eta^{(1)}, \eta^{(2)}, \ldots) \quad \text{if} \quad \mathfrak{L} = l^p \quad (p \geqq 1).$$

In the case of (S), we have from theorem 2.2

Theorem 4.3. (Egoroff) In the space (S), if $\phi_n(t) \to \varphi(t)$ almost everywhere, then there exists a function $\phi_0 \in (S)$ such that $\big(\phi_n(t) - \varphi(t)/\varphi_0(t)\big) \to 0$ almost everywhere uniformly.

From Theorem 3.3, e) and f), we have Steinhaus' theorem.

Theorem 4.4. In the space (S), a) if $\sum_{n=1}^{\infty} \phi_n$ is almost everywhere convergent, then there exists a sequence of real numbers $\lambda_n \to \infty$ such that $\sum \lambda_n \varphi_n$ converges almost everywhere.

b) if $\varphi_n \to 0$ a. e., then there exists real numbers $\lambda_n > 0$ such that $\sum \lambda_n$ is divergent but $\sum \lambda_n \varphi_n$ is a. e. convergent.

When I have written up this paper, Nakano's paper appeared in *Shijô-sûgaku Danwakwai*, 241, where he proved that regularity axiom is equivalent to **II** and regular completeness. $\mathfrak{L}$ is called regularly complete when $y_{i, j_i} \to 0\,(o)$ (as $i \to \infty$) implies the existence of y_0 such as $y_0 \geqq y_{i, j_i}$ $(i=1, 2, \ldots)$. This is equivalent to **I.**