31. On the Semi-ordered Ring and its Application to the Spectral Theorem. II.

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In the first place, our note[1] "On the semi-ordered ring and its application to the spectral theorem" contained, in its proof of algebraic part, a falsy argument, which we shall correct here. Namely, its lemma in §1 (p. 557) was incorrect; it ought to have referred only to a normal subgroup generated by positive elements. The following revised proof runs more or less in the same line as Vernikoff-Krein-Tovbin's,[2] but we may put emphasis on that neither associativity (nor commutativity) nor ring property is used; we simply deal with abelian groups with operators. Indeed, as an application of such mode of our approach, we can determine the structure of the additive group of *bounded automorphisms* of a semi-ordered abelian group (satisfying certain conditions); this forming the second purpose of the present supplementary note.

Let G be a semi-ordered abelian group with real multipliers[3], such that[4]

(i) if $x \geqq 0$ and $y \geqq 0$ then $x+y \geqq 0$,

(ii) if $x \geqq 0$ and $-x \geqq 0$ then $x=0$,

(iii) if $x \geqq 0$ and α (real number) $\geqq 0$ then $\alpha x=0$.

Let G possess further an operator domain $\Omega=\{A\}$ which is by itself a semi-ordered abelian group (in the same sense as G) such as

(vii) if $x \geqq 0$ (in G), $A \geqq 0$ (in Ω) then $Ax \geqq 0$ (in G),

(viii) $(A+B)x=Ax+Bx$, $A(x+y)=Ax+Ay$, $A(\alpha x)=\alpha Ax$,

and let moreover

(ix) Ω possess an Archimedean unit I which satisfies $Ix=x$, $x \in G$.

Then we have

Lemma. Every normal subgroup of G generated by a certain system of positive elements is always allowable with respect to Ω.

For, our former proof remains valid certainly in this case.

Suppose now

(iv) G itself possess an Archimedean unit e,

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1) Proc. **18** (1942), 555.

2) Sur les anneaux semi-ordonnés, C. R. URSS, **30** (1941), p. 758.

3) Cf. a remark below

4) The numbers for the conditions are in accordahce with our former note.

and furthermore[5)]

(v) if $-\varepsilon e < x < \varepsilon e$ for every $\varepsilon > 0$ then $x = 0$.

Consider the totality P of elements x in G such that

$$x > -\varepsilon e \quad \text{for every} \quad \varepsilon > 0 .$$

Then we immediately verify the followings: every element $x \geqq 0$ lies in P; if $x, y \in P$ then $x+y \in P$; if both $x, -x \in P$ then $x=0$ (cf. (v)); if $x \in P$ and α (real number) $\geqq 0$ then $\alpha x \in P$; if $x \in P$ and $A (\in \Omega) \geqq 0$ then $Ax \in P$ (cf. (ix)).

But these mean that we may introduce in G a new semi-order, under which P is the totality of positive elements (and 0), every positive element in the original sense is also positive in the new sense, and moreover, (vii) remains true with respect to the new semi-order. Furthermore, from the above construction we readily see that under this new sense of semi-order G satisfies

(vi) if $x < \varepsilon e$ for every $\varepsilon > 0$ then $x \leqq 0$,

or

(vi)′ $\inf_{\varepsilon > 0} \varepsilon e$ exists and $= 0$.

Put now, for an element x in G,

$$\alpha(x) = \inf\ (\alpha ;\ \alpha e > x)^{6)} .$$

Then always $\alpha(x)e - x \geqq 0$ (in the new sense), according to (vi). Hence the normal subgroup generated by an element of the form $\alpha(x)e - x$ is, by the above lemma, allowable (for Ω); we have $\alpha(x)e \equiv x$ modulo this subgroup.

On returning to the original sense of semi-order we deduce from these consideration the following: Let G satisfy (i)-(v), (vii)-(ix). Then, firstly, every element x of G is congruent to the real multiple $\alpha(x)e$ of e modulo a suitable allowable normal subgroup not coinciding with G, whence modulo a certain maximal allowable normal subgroup of G; thus, secondly, the intersection of all the maximal allowable normal subgroups consists of 0 only (cf. (v)); thirdly, the factor group G/M of G modulo a maximal allowable normal subgroup M is isomorphic with the additive group of real numbers. Hence, *G is operator-isomorphic to an additive group of real-valued bounded functions over the space of maximal allowable normal subgroups; order being preserved in the direct sense. If moreover the condition* (vi) *is fulfilled, then this isomorphism is also an order-isomorphism.* For, $x \not\leqq 0$ implies then $\alpha(x) > 0$.

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5) When (v) is not satisfied, the elements x such as $-\varepsilon e < x < \varepsilon e$ for every $\varepsilon > 0$ form a normal subgroup N, the *radical*, which is allowable for Ω, as one sees readily, and the semi-ordered factor group of G modulo N fulfills (v).

6) Either in the new or in the old sense of semi-order; the results are the same.

From these follow now immediately the assertions in Theorems 1 and 2 of our former note which we restate here:

Let R be a semi-ordered ring with real multipliers and possessing a ring-unit e which is at the same time an Archimedean unit. *Suppose R satisfy* (v). *Then R is ring-isomorphic to a certain ring $R(\mathfrak{M})$ of real-valued bounded functions over the space $\mathfrak{M}$ of maximal normal ideals: $x \leftrightarrow x(M)$, such that e is represented by* 1: $e(M) \equiv 1$. *In particular, R is both associative and commutative. The order is preserved in the direction $x \rightarrow x(M)$. If R fulfills the stronger condition* (vi), *then R is ring-order-isomorphic with $R(\mathfrak{M})$.*

Remark. The above deduction depends on the existence of real multipliers. But we may replace it, as in our former note[1], by

$$\text{(iii)}' \quad \text{if} \quad nx \geqq 0 \quad \text{for a certain natural number } n, \text{ then} \quad x \geqq 0\,,$$

(and, of course, (v) (or (vi)) by the corresponding condition in our former note). Namely using (iii)′ we extend G so as, firstly, to possess rational multipliers, and then complete it by the norm $\|x\| = \inf(\alpha; -\alpha e < x < \alpha e)$ according to (v). The extension thus obtained has real multipliers (and satisfies the above prescribed conditions), and we may apply our deduction to it.

Now, as an application of our proof to consider groups with operators rather than rings, let us study bounded automorphisms of a semi-ordered group (without operators). Namely, let G be a semi-ordered abelian group (such as (i), (ii), (iii) (or (iii)′)) (without operators), and A be an automorphism of G (simply as a group)[7]. Then, if there exists a natural number n such as

$$nx \geqq x^A \geqq -nx \quad \text{for every } x(\in G) \geqq 0\,,$$

we call A bounded. The totality of the bounded automorphisms of G forms a semi-ordered ring R_G by the usual operations

$$x^{AB} = (x^A)^B\,, \quad x^{A+B} = x^A + x^B$$

and the semi-order:

$$A \geqq 0 \quad \text{meaning:} \quad x^A \geqq 0 \quad \text{for every} \quad x \geqq 0\,.$$

Now we have: *if G satisfies (besides* (i)-(iii)) (iv) *and* (v), *then the ring R_G of bounded automorphisms is, merely as an additive group, isomorphic to a svbgroup of G; order being preserved in the direct sense of correspondence.* To see this, again we first assume the existence of real multipliers. And, we consider R_G as an operator domain Ω for G. Then the above conditions (vii)-(ix) are satisfied. Therefore, the intersection of all the maximal R_G-allowable normal subgroups is 0. Denote their totality by $\mathfrak{M} = \{M\}$. G/M, with a certain $M \in \mathfrak{M}$, is the additive group of real numbers, and every element in G is con-

7) Thus A needs not preserve order relation in G.

gruent modulo M to a real multiple of e. Hence, if an $A \in R_G$ maps e into M, then A maps the whole G into M. Further, if $e^A=0$ then $e^A \in \bigwedge M$, whence $G^A \leqq \bigwedge M=0$, $G^A=0$. Therefore, since $e^{A+B}=e^A+e^B$, $A(\in R_G)$ is determined uniquely by the image e^A of e, and $A \rightarrow e^A$ gives an isomorphism of the additive group R_G with the subgroup $\{e^A\}$ of G. (Indeed, if we put

$$e^A \equiv \omega_A(M)e \quad \text{mod.}\, M,$$

then $\omega_{A+B}(M)=\omega_A(M)+\omega_B(M)$, $\omega_{AB}(M)=\omega_A(M)\omega_B(M)$ and the ring R_G is represented faithfully by the function ring $\{\omega_A(M)\}$ over $\mathfrak{M}$). As to the case without real multipliers, we have only to apply the above remark and to observe that A is a difference of two positive operators, say nI and $nI-A$, which secures (in combination with (v), of course,) the possibility of extending A from G to its complete extension with real multipliers. We see in this way that again A is determined uniquely by $e^A(\in G)$, and therefore, R_G is isomorphic with $\{e^A\}$. Since $A \geq 0$ implies $e^A \geq 0$, the last assertion of our theorem is evident.

Finally, we take this opportunity to correct some misprints in our note[1]: $\underset{M}{\mathfrak{E}}\big(x(M) \geq \lambda\big)$ at the last line of p. 559 reads $\underset{M}{\mathfrak{E}}\big(x(M) \leqq \lambda\big)$; $\sum \lambda_i e'_{\lambda_i}(M)$ and $\sum \lambda_i e_{\lambda_i}(M)$ at the first line of p. 560 read respectively $\sum \lambda_i\big\{e'_{\lambda_i}(M)-e'_{\lambda_{i-1}}(M)\big\}$ and $\sum \lambda_i\big\{e_{\lambda_i}(M)-e_{\lambda_{i-1}}(M)\big\}$.