27. *On a Characterisation of Order-preserving Mapping-lattice.*

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1. *Introduction.* A mapping f of a lattice L_1 into a lattice L_2 is called order preserving, when for any two elements $a > b$ of L_1, there holds the relation $f(a) > f(b)$ in the order of L_2[1]. If we define $f_1 > f_2$, when for any element a of L_1 $f_1(a) > f_2(a)$ is satisfied, then the set of all order preserving mappings forms a lattice $\{f\}$. The join $f_1 \cup f_2$ and the meet $f_1 \cap f_2$ are respectively defined by the following mappings:

$$(f_1 \cup f_2)(a) = f_1(a) \cup f_2(a),$$
$$(f_1 \cap f_2)(a) = f_1(a) \cap f_2(a).$$

In this paper we are concerned with the problem of a lattice-theoretic characterisation of this order preserving transformation-lattice for the case, when L_2 is the two-element lattice $\{0, 1\}$.

The lattice L^* in the theorem of this paper is isomorphic with the ring of all M-closed subsets of the lattice L of its join-irreducible elements. Evidently we can generalise the theorem for the case, when L is only a partially ordered set in the order of L^*. In this case we can therefore omit the condition (iv) of the theorem[2]. When L_1 is a Boolean algebra, i. e. the lattice of all subsets of a set R, whose order relation is defined by the inclusion relation as usual, then the mapping-lattice is the same as the covering lattice of all subsets of R.

2. *Transformation-lattice.*

Lemma 1. All order preserving mappings $\{f\}$ of a lattice L into the lattice $\{0, 1\}$ form a complete and complete distributive lattice.

Proof. For any subset $\{f_x \mid x \in X\}$ of $\{f\}$ and for any element a of L we have the relations;

$$(\bigvee_{x|X} f_x)(a) = \bigvee_{x|X}\big(f_x(a)\big),$$
$$(\bigwedge_{x|X} f_x)(a) = \bigwedge_{x|X}\big(f_x(a)\big).$$

Furthermore for one element $f_0 \in \{f\}$ we can easily prove

$$\begin{aligned}\big(f_0 \cup (\bigwedge_{x|X} f_x)\big)(a) &= f_0(a) \cup (\bigwedge_{x|X} f_x)(a) \\ &= f_0(a) \cup \big(\bigwedge_{x|X} f_x(a)\big) \\ &= \bigwedge_{x|X}\big((f_0(a) \cup f_x(a)\big) = \bigwedge_{x|X}\big((f_0 \cup f_x)(a)\big),\end{aligned}$$

1) We use the symbol $>$ in the meaning of the usual symbol $\geqq$.
2) See Birkhoff: Lattice Theory, p. 76.

and similarly

$$\big(f_0 \frown (\bigcup_{x|X} f_x)\big)(a) = \big(\bigcup_{x|X} (f_0 \frown f_x)\big)(a)\,.$$

Lemma 2. Every element $f \in \{f\}$ has at least one expression as the join (meet) of some join-irreducible[1] (meet-irreducible) elements.

Proof. The subset $\{a_x \mid X\}$ of L such that $f(a_x)=1$ holds, forms a J-closed subset[2] $f^{-1}(1)$ of L.

Let $\mathfrak{A}_x$ be the dual principal ideal $a_x \smile L$. Now we consider for every x the order preserving transformation f_x such that

$$f_x(a)=1\,, \qquad a \in \mathfrak{A}_x\,,$$
$$f_x(a)=0\,, \qquad a \notin \mathfrak{A}_x\,,$$

then clearly f is the join: $\bigcup_{x|X} f_x$. For $f > f_x$ is clear by the inclusion relation $f^{-1}(1) \supset \mathfrak{A}_x = f_x^{-1}(1)$. And $f < \bigcup_X f_x$ is proved by the fact, that for any element a_x of $f^{-1}(1)$ $f_x(a_x)=1$,

i. e. $$(\bigcup_x f_x)(a_x) = \bigcup_x \big(f_x(a_x)\big) > f_x(a_x) = 1 \qquad \text{holds.}$$

Every element f_x is join-irreducible. For if f_x is a join $\bigcup_{y|Y} g_y$, then $f_x(a_x) = (\bigcup_Y g_y)(a_x) = \bigcup_Y \big(g_y(a_x)\big) = 1$, therefore for some y $g_y(a_x)$ must be 1. Consequently $g_y^{-1}(1) \supset \mathfrak{A}_x$, i. e. $g_y > f_x$. Since $f_x > g_y$, we conclude $f_x = g_y$.

Lemma 3. Every join-irreducible element f of $\{f\}$ has as the set $f^{-1}(1)$ a principal dual ideal $a \smile L$.

Proot. Let $f^{-1}(1) = \{a_x \mid X\}$, and let $\mathfrak{A}_x$ and f_x be defined as in lemma 2. Then

$$f = \bigcup_x f_x\,.$$

Since f is join-irreducible, for some x $f = f_x$. Whence $f^{-1}(1) = f_x^{-1}(1)$ is the principal dual ideal $a_x \smile L$.

3. Having established the above properties of $\{f\}$, we can now consider the inverse problem, that is the characterisation of mapping-lattice $\{f\}$.

Lemma 4. Let L^* be a lattice with following properties: i) complete, ii) completely distributive, iii) every element a is a join of join-irreducible elements, iv) the set of all join-irreducible elements forms a lattice L by the order of L^*. Then $a \bigwedge b$ of join-irreducible elements a, b is again join-irreducible.

Proof. Join and meet in L we denote by $\bigvee, \bigwedge$. Clearly $a \bigwedge b < a \frown b$ in the order of L^*. If $a \bigwedge b \neq a \frown b$, then $a \frown b$ must be join-reducible. Therefore

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1) An element f is called join-irreducible, when f is the join of any elements f_x, i. e. $f = \bigcup_{x|X} f_x$, then necessarily for some x $f_x = f$.

2) Cf. Birkhoff: Lattice Theory, p. 14.

$$a \frown b = \bigvee_Z a_z, \qquad a_z: \text{ join-irreducible.}$$

Since $a \wedge b$ is the greatest lower bound of a, b in L,

$$a \wedge b > a_z .$$

And then $a \frown b = \bigvee_Z a_z > a \wedge b > \bigvee_Z a_z = a \frown b$. i.e. $a \frown b$ must be join-irreducible.

Lemma 5. Let $a = \bigvee_X a_x = \bigvee_Y b_y$ be any two experssions by join-irreducible elements in L^*. Then for any a_x we can select suitably some b_y such that $b_y > a_x$ and for any b_y some a_x such that $a_x > b_y$.

Proof. From complete distributivity we can deduce

$$b_y = b_y \frown a = b_y \frown (\bigvee_X a_x) = \bigvee_X (b_y \frown a_x) .$$

Since b_y is join-irreducible, for some x $b_y = b_y \frown a_x$, i.e.

$$a_x > b_y .$$

Theorem. Let L^* be a lattice as in lemma 4. Let L' be the dual ideal of L. Then L^* is lattice isomorph to the order preserving transformation-lattice $\{f\}$ of L' in $\{0, 1\}$.

Proof. i) One to one correspondence. Let $a = \bigvee_X a_x$ be one expression of $a \in L^*$ by join-irreducible elements a_x, and $\mathfrak{A}_x$ the principal ideal $a_x \wedge L$ in L (principal dual ideal in L'). Let f_x be the order preserving transformation of L' in $\{0, 1\}$ such that $f_x^{-1}(1) = \mathfrak{A}_x$, then $f = \bigvee_X f_x$ is clearly an order preserving transformation and $f^{-1}(1) = \sum_X \mathfrak{A}_x$. We consider the correspondence $a \to f$. This correspondence is uniquely determined. For if $a = \bigvee_X a_x = \bigvee_Y b_y$, then from lemma 5 every $\mathfrak{A}_x$ is included by some principal ideal $\mathfrak{B}_y \subset L$, therefore $\sum_X \mathfrak{A}_x \subset \sum_Y \mathfrak{A}_y$, and similary $\sum \mathfrak{B}_y \subset \sum \mathfrak{A}_x$. Hence the inverse set of 1 is equal for any expression of a. This correspondence is one to one, for if $a \neq b$, $a = \bigvee_X a_x$ and $b = \bigvee_Y b_y$, then for at least one element a_x (or b_y) there exists no element b_y (or a_x) such that $b_y > a_x$ (or $a_x > b_y$). Therefore $a_x \notin \sum_Y \mathfrak{B}_y$, this shows that $f_a^{-1}(1) \neq f_b^{-1}(1)$, whence $f_a \neq f_b$.

ii) Let f be any order preserving transformation of L' to $\{0, 1\}$, then the set $f^{-1}(1) = \{a_x \mid X\}$ is a M-closed subset of L. From completeness of L^* the element $a = \bigvee_X a_x$ exists and corresponds to the element f of $\{f\}$.

iii) Lattice-isomorphism. Let $a = \bigvee_X a_x$, $b = \bigvee_Y b_y$, then $a \smile b = (\bigvee_X a_x) \smile (\bigvee_Y b_y)$, $a \frown b = \bigvee_{X,Y} (a_x \frown b_y)$ and let $f_a, f_b, f_{a \smile b}, f_{a \frown b}$ be each the mappings of L' in $\{0, 1\}$.

Now for any element c of L' if $c \in \mathfrak{A}_x$ or $c \in \mathfrak{B}_y$ for some x or y, then $f_{a \smile b}(c) = 1 = f_a(c) \smile f_b(c)$. If $c \notin \mathfrak{A}_x$ and $c \notin \mathfrak{B}_y$ for any x and y, then $f_{a \smile b}(c) = 0 = f_a(c) \smile f_b(c)$. Therefore in all cases

$$f_{a \smile b} = f_a \smile f_b .$$

Similarly $$f_{a\frown b}=f_a \frown f_b\,.$$

From the above proof we can see easily that L^* is isomorphic with the ring of all M-closed subsets of the lattice L of the join-irreducible elements.

Corollary. Let L^* be a lattice as in this theorem. Then every element of L^* is meet of some meet-irreducible elements and the subset of all meet-irreducible elements forms a lattice, which is lattice-isomorphic to the lattice L of all join-irreducible elements.