26. *On a Characterisation of Join Homomorphic Transformation-lattice*

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1. *Introduction.* A mapping f of a lattice L_1 into a lattice L_2 is called join homomorphic, when for any elements a, b of L_1 there exists the relation

$$f(a \cup b) = f(a) \cup f(b).$$

This mapping is order preserving, for, if $a > b$ in L_1, it follows $f(a) = f(a \cup b) = f(a) \cup f(b)$, i. e. $f(a) > f(b)$ in L_2.

If we define $f_1 > f_2$, when for any element a of L_1 $f_1(a) > f_2(a)$ is satisfied, then the set of all join homomorphic transformations forms a partially ordered set $\{f\}$. If L_2 is complete and completely distributive, then $\{f\}$ is a complete lattice. For there exist the following relations for any element a of L_1

$$(f_1 \cup f_2)(a) = f_1(a) \cup f_2(a),$$

$$\Big(\bigcup_X (f_x \mid X)\Big)(a) = \bigcup_X \big(f_x(a) \mid X\big),$$

$$(f_1 \cap f_2)(a) = \bigcup_X \big(g_x(a) \mid X\big),$$

$$\Big(\bigcap_X (f_x \mid X)\Big)(a) = \bigcup_Y \big(h_y(a) \mid Y\big),$$

where $\{g_x \mid x \in X\}$ is the set of all transformations such that $g_x < f_1, f_2$, and $\{h_y \mid y \in Y\}$ is the set of all transformations such that $h_y < f_x$ for all x of X. This join $f_1 \cup f_2$, meet $f_1 \cap f_2$, complete join $\bigcup_X f_x$ and complete meet $\bigcap_X f_x$ are again clearly join homomorphic transformations.

In this paper we are concerned with the problem of a lattice-theoretic characterisation of this join homomorphic transformation-lattice for the case, when L_2 is the two-element lattice $\{0, 1\}$.

Lemma 1. All ideals in L form a lattice, which is dual isomorphic with the join homomorphic transformation-lattice $\{f\}$ of L into $\{0, 1\}$.

Proof. Let f be a join homomorphic mapping of L into $\{0, 1\}$. Then the set $f^{-1}(0)$ is an ideal in L. For if $a, b \in f^{-1}(0)$, then $f(a \cup b) = f(a) \cup f(b) = 0$; therefore $a \cup b \in f^{-1}(0)$. And if $a \in f^{-1}(0)$, $b < a$, then clearly $f(b) < f(a) = 0$. Hence $f^{-1}(0)$ includes b.

Conversely, let $\mathfrak{A}$ be an ideal in L, then the transformation f such that

$$f(a) = 0, \qquad a \in \mathfrak{A},$$

$$f(a) = 1, \qquad a \notin \mathfrak{A},$$

1) Cf. A. Komatu. On a Characterisation of Order Preserving Transformation-lattice. Proc. **19** (1943), 27.

is clearly join homomorphic. Hence the correspondence between an ideal $\mathfrak{A}$ in L and a join homomorphic transformation of L into $\{0, 1\}$ is one to one.

Furthermore this correspondence is a dual lattice isomorphism. Let f_1, f_2 be any two such transformations, and let $\mathfrak{A}_1, \mathfrak{A}_2$ be respectively the ideals $f_1^{-1}(0)$, $f_2^{-1}(0)$. Now if $(f_1 \cup f_2)(a) = f_1(a) \cup f_2(a) = 0$, then a is included in the ideal $\mathfrak{A}_1 \frown \mathfrak{A}_2$. Conversely, if $a \in \mathfrak{A}_1 \frown \mathfrak{A}_2$, then $f_1(a) = 0$ and $f_2(a) = 0$; therefore

$$(f_1 \cup f_2)(a) = 0 .$$

Hence $$(f_1 \cup f_2)^{-1}(0) = \mathfrak{A}_1 \frown \mathfrak{A}_2 .$$

And if $(f_1 \frown f_2)(a) = 0$, then a is included in all such ideals $\mathfrak{B}_x$ that $\mathfrak{B}_x \supset \mathfrak{A}_1, \mathfrak{A}_2$, i.e. $\mathfrak{B}_x \supset \mathfrak{A}_1 \cup \mathfrak{A}_2$[1]. When we denote by $\mathfrak{A}_1 \uplus \mathfrak{A}_2$ the least ideal $\mathfrak{B}$ such that $\mathfrak{B} \supset \mathfrak{A}_1 \cup \mathfrak{A}_2$, i.e. $\mathfrak{A}_1 \uplus \mathfrak{A}_2 = \bigcap_X \mathfrak{B}_x$, then $a \in \mathfrak{A}_1 \uplus \mathfrak{A}_2$. Conversely if $a \in \mathfrak{A}_1 \uplus \mathfrak{A}_2$, then $a \in \mathfrak{B}_x$ for any ideal $\mathfrak{B}_x$. Hence for any transformation g_x such that $g_x < f_1, f_2$, we have $g_x(a) = 0$,

i.e. $$(f_1 \frown f_2)(a) = \bigcup_X \big(g_x(a)\big) = 0 .$$

Therefore we conclude

$$(f_1 \frown f_2)^{-1}(0) = \mathfrak{A}_1 \uplus \mathfrak{A}_2 .$$

2. *Transformation-lattice.*

Lemma 2. Every element f of $\{f\}$ has at least one expression as the meet of some meet-irreducible[2] elements.

Proof. Let $f^{-1}(0) = \{a_x \mid X\}$, $\mathfrak{A}_x = a_x \frown L$, and let f_x be the join homomorphic transformation such that

$$f_x^{-1}(0) = \mathfrak{A}_x .$$

Then $f = \bigcap_X f_x$. For from $f^{-1}(0) \supset f_x^{-1}(0)$ it follows $f < f_x$, i.e. $f < \bigcap_X f_x$. And if $g < \bigcap_X f_x$, then $g^{-1}(0) \supset f_x^{-1}(0)$, i.e. $g^{-1}(0) \supset \biguplus_X \mathfrak{A}_x = f^{-1}(0)$. Hence $g < f$. Therefore it must be $f = \bigcap_X f_x$.

Every f_x is meet-irreducible or finite-meet-reducible into some meet-irreducible elements[3]. For if

$$f_x = \bigcap_Y \{g_y \mid Y\}\,, \quad g_y^{-1}(0): \text{ principal ideal,}$$

then $f_x < g_y$; hence $f_x^{-1}(0) = \mathfrak{A}_x \supset g_y^{-1}(0)$. If $\mathfrak{A}_x \neq g_y^{-1}(0)$ for all y, then $\mathfrak{A}_x \neq \bigcup_Y \big(g_y^{-1}(0)\big)$. But $\mathfrak{A}_x = (\bigcap_Y g_y)^{-1}(0)$ is the least ideal, which includes all the ideal $g_y^{-1}(0)$. Whence for some finite elements $b_{y_j} \in g_{y_j}^{-1}(0)$

1) $\mathfrak{A}_1 \cup \mathfrak{A}_2$ means the set sum of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

2) a is said meet-irreducible, when, if $a = \bigwedge \{a_x \mid X\}$, then necessarily $a = a_x$ for some x. See. A. Komatu: On a Characterisation of Order Preserving Transformation-lattice. Proc. **19** (1943), 27.

3) a is said finite-meet-reducible or finite-meet-reducible into meet-irreducible elements, when, if $a = \bigwedge \{a_x \mid X\}$ with meet-irreducible elements a_x, then $a = a_{x_1} \frown \cdots \frown a_{x_n}$ for some finite subset $x_1, \ldots, x_n$ of X.

$(j=1, 2, \ldots, n)$ it must be $a_x < b_{y_1} \cup \cdots \cup b_{y_n}$.

Therefore $\qquad \mathfrak{A}_x \subset (\bigcap_j g_{y_j})^{-1}(0)$, i. e. $\mathfrak{A}_x = \bigcup_j g_{y_j}^{-1}(0)$.

This shows easily that f_x is finite-meet-reducible into some meet-irreducible elements.

Lemma 3. The subset L' of all meet-irreducible elements and all meet-finite-reducible elements in $\{f\}$ forms a lattice, which is dual isomorphic with L.

Proof. Let f be a meet-irreducible element or a finite-meet-reducible element, i. e. $f \in L'$, and let $f^{-1}(0) = \{a_x \mid X\}$ and $a_x \cap L = \mathfrak{A}_x$. Let f_x be the transformation such that $f_x^{-1}(0) = \mathfrak{A}_x$, then $f = \bigcap_X f_x$ as in lemma 2.

From the finite-meet-reducibility of f we can prove easily

$$f = f_{x_1} \cap \cdots \cap f_{x_n}$$

Whence $f^{-1}(0)$ is the least ideal which includes $f_{x_i}^{-1}(0) = \mathfrak{A}_{x_i}$ $(i = 1, 2, \cdots, n)$. Therefore $f^{-1}(0)$ is the principal ideal

$$(a_{x_1} \cup \cdots \cup a_{x_n}) \cap L.$$

From lemma 1 and 2 we conclude that L is dually lattice isomorphic with L.

Lemma 4. Join in $\{f\}$ is continuous with respect to the generalized (*o*) topology[1] of $\{f\}$. Meet is not necessarily continuous.

Proof. Let a directed set of elements $\{f_x \mid X\}$ converge to f. Then there exist two directed sets of elements $\{\varphi_x \mid X\}$, $\{\psi_x \mid X\}$ such that

$$\left.\begin{array}{l}\varphi_{x_1} < \varphi_{x_2}, \\ \psi_{x_1} > \psi_{x_2},\end{array}\right\} \quad \text{for } x_1 < x_2 \text{ in } X,$$

$$\varphi_x < f_x < \psi_x \quad \text{for any} \quad x \in X,$$

and

$$\bigcup_X \{\varphi_x \mid x \in X\} = \lim f_x = \bigcap_X \{\psi_x \mid x \in X\}.$$

Hence for any element g of $\{f\}$

$$(1) \qquad \left\{\begin{array}{l}\left.\begin{array}{l}\varphi_{x_1} \cup g < \varphi_{x_2} \cup g \\ \psi_{x_1} \cup g > \psi_{x_2} \cup g\end{array}\right\} \quad \text{for any } x_1 < x_2 \text{ in } X, \\ \varphi_x \cup g < f_x \cup g < \psi_x \cup g \quad \text{for any } x \in X, \text{ and}\end{array}\right.$$

$$(2) \qquad \Big(\bigcup_X (\varphi_x \mid X)\Big) \cup g = (\lim f_x) \cup g = \Big(\bigcap_X (\psi_x \mid X)\Big) \cup g.$$

It is clear that $(\bigcup_X \varphi_x) \cup g = \bigcup_X (\varphi_x \cup g)$. Furthermore we can prove easily $(\bigcap_X \psi_x) \cup g = \bigcap_X (\psi_x \cup g)$. For if $a \in \big((\bigcap_X \psi_x) \cup g\big)^{-1}(0)$, then $a \in (\bigcap_X \psi_x)^{-1}(0)$ and $a \in g^{-1}(0)$; by the first relation it follows $a < a_{x_1} \cup a_{x_2} \cup \cdots \cup a_{x_n}$ for some finite $a_{x_i} \in \psi_{x_i}^{-1}(0)$ $(i = 1, \ldots, n)$. Let x be an

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1) Cf. G. Birkhoff: Lattice Theory, p. 32.

element of X such that for every x_i $x>x_i$, then $\psi_x<\psi_{x_i}$, i.e. $\psi_x^{-1}(0)>\psi_{x_i}^{-1}(0)$. Hence every a_{x_i} is included in the ideal $\psi_x^{-1}(0)$ and so is a. Therefore we conclude for this x that $a\in(\psi_x\cup g)^{-1}(0)\subset\bigcup_X\big((\psi_x\cup g)^{-1}(0)\big)$, i.e. $(\bigcap_X \psi_x)\cup g>\bigcap_X(\psi_x\cup g)$.

The inverse order is obvious from $\psi_x\cup g>(\bigcap_X\psi_x)\cup g$, hence

$$(\bigcap_X \psi_x)\cup g=\bigcap_X(\psi_x\cup g)\,.$$

The formula (2) now takes the form

$$\bigcup_X(\varphi_x\cup g)=(\lim f_x)\cup g=\bigcap_X(\psi_x\cup g)\,. \tag{3}$$

From (1) and (3) we see that $\lim(f_x\cup g)=(\lim f_x)\cup g$, i.e. $\{f_x\cup g\mid X\}$ converges to $f\cup g$.

3. *Characterisation of the transformation-lattice.*

Lemma 5. Let L^* be a lattice with the following properties: i) complete, ii) every element a is a meet of meet-irreducible elements. iii) join is continuous with respect to the generalized (o)-topology of L^*.

Then, if $a=\bigcap_X a_x=\bigcap_Y b_y$ are any two reductions of a into infinite meet-irreducible components, we can select for every y suitably some finite x_i $(i=1,2,\dots,n)$ such that

$$b_y>a_{x_1}\cap\cdots\cap a_{x_x}$$

and for every x some finite y_j $(j=1,2,\dots,m)$ such that

$$a_x>b_{y_1}\cap\cdots\cap b_{y_m}\,.$$

Proof. Let Γ be the set of all finite subsets $\{\alpha\}$ of X, then Γ is a directed set. If $\alpha=\{x_1,x_2,\dots,x_n\}$ and $a_\alpha=a_{x_1}\cap\cdots\cap a_{x_n}$, then for $\alpha<\beta$ in Γ we have $a_\alpha>a_\beta$ in L^*.

Clearly $a<a_\alpha$ for every $\alpha\in\Gamma$, hence

$$a<\bigcap_\Gamma a_\alpha\,. \tag{4}$$

But if we select $\alpha_x\in\Gamma$ suitably for every $x\in X$ such that $x\in\alpha_x$, then $a_x>a_{\alpha_x}$ in L^*; hence

$$a=\bigcap_X a_x>\bigcap_X a_{\alpha_x}>\bigcap_\Gamma a_\alpha\,. \tag{5}$$

From (4) and (5) it follows that the directed set of elements $\{a_\alpha\mid\Gamma\}$ converges to a. From the property iii) of L^*

$$b_y=b_y\cup a=b_y\cup(\bigcap_\Gamma a_\alpha)=\bigcap_\Gamma(a_\alpha\cup b_y)\,.$$

From the property ii)

$$a_\alpha\cup b_y=\bigcap_{Z_\alpha}c_z\,,\qquad c_z:\ \text{meet-irreducible},$$

i.e. $b_y=\bigcap_{\alpha\in\Gamma}(\bigcap_{Z_\alpha}c_z)$. But b_y is meet-irreducible, hence $b_y=c_z>a_\alpha\cup b_y$ for some $z\in Z_\alpha$.

Therefore it must be $b_y=a_\alpha\cup b_y$, i.e.

$$b_y > a_a = a_{x_1} \cap \cdots \cap a_{x_n}.$$

Similarly we can prove for every x with some finite y_j $(j=1,2,\dots,m)$ $a_x > b_{y_1} \cap \cdots \cap b_{y_m}$.

Theorem. Let L^* be a lattice with the following properties: i) complete ii) every element a is a meet of meet-irreducible elements. iii) join is continuous with respect to the generalized (o)-topology of L^*. iv) the set L of all meet-irreducible elements and all finite-meet-reducible elements forms a lattice with the (relative) order of L^*. Then L^* is isomorphic with the join homomorphic transformation-lattice of L' into $\{0, 1\}$, where L' is dual isomorphic to the lattice L.

Proof. (1) One to one Correspondence.

Let $a = \bigcap_X a_x$ be an expression of a with meet-irreducible elements $\{a_x \mid X\}$. Let $a'_x \in L'$ be the element which corresponds to $a_x \in L$, and let f_x be the join homomorphic mapping of L' into $\{0, 1\}$ such that

$$f_x^{-1}(0) = a'_x \cap L' = \mathfrak{A}'_x.$$

Let f be the mapping of L' into $\{0, 1\}$ such that

$$f^{-1}(0) = \bigcup_X \mathfrak{A}'_x.$$

Now we consider the correspondence $a \to f$. Clearly $a_x \to f_x$. This correspondence is uniquely determined. For if $a = \bigcap_X a_x = \bigcap_Y b_y$, then from lemma 5 for every y with some $x_i \in X$ $(i=1, 2, \dots, n)$

$$b_y > a_{x_1} \cap \cdots \cap a_{x_n}.$$

Hence b'_y is included in the ideal $\bigcup_i (a'_{x_i} \cap L') = \bigcup_i \mathfrak{A}'_{x_i}$,

i. e. $$\mathfrak{B}'_y = b'_y \cap L' \subset \bigcup_{i=1}^{n} \mathfrak{A}'_{x_i}.$$

Similarly for every x $\mathfrak{A}'_x \subset \bigcup_j \mathfrak{B}'_{y_j}$, whence

$$\bigcup_X \mathfrak{A}'_x = \bigcup_Y \mathfrak{B}'_y.$$

This correspondence is one to one. For if $a = \bigcap_X a_x$, $b = \bigcap_Y b_y$, $a \neq b$, then at least for one a_x (or b_y) there exist no finite subsets $y_1, \dots, y_m$ (or $x_1, \dots, x_n$) such that

$$a_x > b_{y_1} \cap \cdots \cap b_{y_m}.$$

Hence in L' $a'_x \notin \bigcup_Y \mathfrak{B}'_y$, therefore

$$f_a^{-1}(0) \neq f_b^{-1}(0), \quad \text{i. e.} \quad f_a \neq f_b.$$

(2) Let f be a join homomorphic transformation of L' into $\{0, 1\}$, and let $f^{-1}(0) = \mathfrak{A}' = \{a'_x \mid X\}$. Clearly

$$\mathfrak{A}' = \bigcup_X \mathfrak{A}'_x = \bigcup_Y (a'_x \cap L').$$

From completeness of L^* there exists an element a such that

$$a = \bigcap_X a_x .$$

Hence $$a \to f .$$

(3) Meet homomorphism.

Let $a = \bigcap_X a_x$, $b = \bigcap_Y b_y$, then $a \frown b = (\bigcap_X a_x) \frown (\bigcap_Y b_y)$: Let f_a, f_b, and $f_{a \frown b}$ be respectively the following mappings of L' into $\{0, 1\}$ such that

$$f_a^{-1}(0) = \biguplus_X (a'_x \frown L') ,$$

$$f_b^{-1}(0) = \biguplus_Y (b'_y \frown L') ,$$

$$f_{a \frown b}^{-1}(0) = \biguplus_{X, Y} \{(a_x \frown L'), (b'_y \frown L')\} ,$$

then clearly $$f_{a \frown b} = f_a \frown f_b .$$

The last formula follows from the relation

$$\biguplus_{X, Y} \{a_x \frown L'), (b'_y \frown L')\} = \left(\biguplus_X (a'_x \frown L') \right) \uplus \left(\biguplus_Y (b_y \frown L') \right) .$$

We can easily prove from 1)–3) that this correspondence is isomorphic.

Corollary. The lattice L of all join homomorphic transformations of finite lattice L' into $\{0, 1\}$ is dual isomorphic to L'.