24. Some Metrical Theorems on a Set of Points.

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In this note we will prove some theorems on measurable sets of points.

Theorem I. Let E be a measurable set in an n-dimensional space. We translate E by a vector $\mathfrak{r}$ and $E+\mathfrak{r}$ be the translated set. Then

$$\lim_{|\mathfrak{r}|\to 0} mE(E+\mathfrak{r})=mE. \tag{1}$$

W. H. Young[1] proved the case $n=1$.

Proof. We prove the case $n=2$; the other case can be proved similarly. Let E be a measurable set on the xy-plane and $\varphi(x, y)$ be its characteristic function, then $\varphi(x-h, y-k)$ is the characteristic function of $E+\mathfrak{r}$, where (h, k) are the components of $\mathfrak{r}$, so that $\mathfrak{r}=(h, k)$, $|\mathfrak{r}|=\sqrt{h^2+k^2}$.

(i) First we assume $mE<\infty$. Then

$$mE=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x, y)dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^2(x, y)dxdy,$$

$$mE(E+\mathfrak{r})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x, y)\varphi(x-h, y-k)dxdy,$$

so that

$$|mE(E+\mathfrak{r})-mE|=\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x, y)\big(\varphi(x-h, y-h)-\varphi(x, y)\big)dxdy\right|\leqq$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(x-h, y-k)-\varphi(x, y)|\,dxdy.$$

Since by Lebesgue's theorem[2],

$$\lim_{h^2+k^2\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(x-h, y-k)-\varphi(x, y)|\,dxdy=0,$$

we have $\lim_{|\mathfrak{r}|\to 0} mE(E+\mathfrak{r})=mE$.

(ii) If $mE=\infty$, let E_1 be a bounded sub-set of E, such that $N\leqq mE_1<\infty$. Then by (i), for any $\mathfrak{r}$, such that $|\mathfrak{r}|<\rho$ $mE_1(E_1+\mathfrak{r})\geqq \frac{mE_1}{2}\geqq\frac{N}{2}$, so that $mE(E+\mathfrak{r})\geqq mE_1(E_1+\mathfrak{r})\geqq\frac{N}{2}$. Since N can be taken arbitrarily large, we have $\lim_{|\mathfrak{r}|\to 0} E(E+\mathfrak{r})=\infty$, q. e. d.

Theorem II. Let E_1 and E_2 be measurable sets in an n-dimensional space and one of mE_1, mE_2 be finite. Then

$$\lim_{|\mathfrak{r}|\to 0} mE_1(E_2+\mathfrak{r})=m(E_1\cdot E_2). \tag{2}$$

1) W. H. Young: On a class of parametric integrals and their application in the theory of Fourier series. Proc. Royal Soc. (London) A. **85** (1911).

2) Lebesgue: Lecons sur les séries trigonométriques. p. 15.

Proof. We prove the case $n=2$. Let E_1 and E_2 be measurable sets on the xy-plane and $\varphi_1(x, y)$, $\varphi_2(x, y)$ be the characteristic functions of E_1 and E_2 respectively and $\mathfrak{r}=(h, k)$.

(i) We first assume $mE_2<\infty$. Then

$$m(E_1\cdot E_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_1(x, y)\varphi_2(x, y)dxdy\,,$$

$$mE_1(E_2+\mathfrak{r})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_1(x, y)\varphi_2(x-h, y-k)dxdy\,,$$

so that

$$|mE_1(E_2+\mathfrak{r})-m(E_1\cdot E_2)|=\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_1(x,y)\big(\varphi_2(x-h,y-k)-\varphi_2(x,y)\big)dxdy\right|\leqq$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\,\varphi_2(x-h, y-k)-\varphi_2(x, y)\,|\,dxdy\,.$$

Hence as before we have $\lim_{|\mathfrak{r}|\to 0} mE_1(E_2+\mathfrak{r})=m(E_1\cdot E_2)$.

(ii) If $mE_1<\infty$, then $mE_1(E_2+\mathfrak{r})=m(E_1-\mathfrak{r})E_2$, so that this case reduces to (i), q. e. d.

Hence if we put $\psi(h, k)=mE_1(E_2+\mathfrak{r})$, then $\psi(h, k)$ is a continuous function of (h, k).

Remark. The theorem is not true, if $mE_1=\infty$, $mE_2=\infty$. To see this, we take for E_1 the upper half-plane $y\geqq 0$ and for E_2 the lower half-plane $y\leqq 0$. Then $m(E_1\cdot E_2)=0$. If we trànslate E_2 in the direction of the positive y-axis, and let E_2+y be the translated set. Then $mE_1(E_2+y)=\infty$ for any $y>0$.

Theorem III. Let E_1 and E_2 be measurable sets in an n-dimensional space and $mE_1>0$, $mE_2>0$. Then we can translate E_2 suitably, such that

$$mE_1(E_2+\mathfrak{r}_0)>0\,. \tag{3}$$

Fukamiya[1] proved the case $n=1$.

Proof. We prove the case $n=2$. Let E_1 and E_2 be measurable sets on the xy-plane and $\varphi_1(x, y)$, $\varphi_2(x, y)$ be the characteristic functions of E_1 and E_2 respectively and $\mathfrak{r}=(h, k)$.

(i) First we assume $mE_1<\infty$, $mE_2<\infty$. Then by Theorem II,

$$\psi(h, k)=mE_1(E_2+\mathfrak{r})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_1(x, y)\varphi_2(x-h, y-k)dxdy$$

is a continuous function of (h, k), so that by Fubini's theorem,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(h, k)dhdk=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_1(x, y)dxdy\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_2(x-h, y-k)dhdk$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_1(x, y)dxdy\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_2(h, k)dhdk=mE_1\cdot mE_2>0\,.$$

Hence $\psi(h_0, k_0)=mE_1(E_2+\mathfrak{r}_0)>0$ for a suitable $\mathfrak{r}_0=(h_0, k_0)$.

(ii) In the general case, we take bounded sub-sets E_1' of E_1 and E_2' of E_2, such that $0<mE_1'<\infty$, $0<mE_2'<\infty$, then by (i),

3) M. Fukamiya: Sur une propriété des ensembles measurables. Sci. Rep. Tohoku Imp. Univ. **24** (1935).

$mE_1'(E_2'+\mathfrak{r}_0)>0$ for a suitable $\mathfrak{r}_0=(h_0, k_0)$, so that $mE_1(E_2+\mathfrak{r}_0)\geqq mE_1'(E_2'+\mathfrak{r}_0)>0$, q. e. d.

Theorem IV (Steinhaus)[4]. *Let E be a measurable set in an n-dimensional space and $mE>0$. Let $a\in E$, $b\in E$. We translate the vector $\overrightarrow{ab}$, such that its initial point a coincides with the origin of the coordinates and $\mathfrak{r}(a, b)$ be the translated vector. Let E_0 be the set of end points of $\mathfrak{r}(a, b)$. Then E_0 contains a certain n-dimensional sphere about the origin.*

Proof. By Theorem I, for any vector $\mathfrak{r}$, such that $|\mathfrak{r}|<\rho$, $mE(E+\mathfrak{r})>0$, so that $E(E+\mathfrak{r})\neq 0$. Hence there are two points, $a\in E$, $b=a+\mathfrak{r}\in E$, so that E_0 contains a sphere of radius ρ about the origin, q. e. d.

Theorem V (Steinhaus)[5]. *Let E_1 and E_2 be measurable sets in an n-dimensional space and $mE_1>0$, $mE_2>0$ and E_0 be the set of end points of $\mathfrak{r}(a, b)$, where $a\in E_1$, $b\in E_2$. Then E_0 contains a certain n-dimensional sphere.*

Proof. By Theorem III, $mE_1(E_2+\mathfrak{r}_0)>0$ for some $\mathfrak{r}_0=(h_0, k_0)$. Let $E'=E_1(E_2+\mathfrak{r}_0)$ and E_0' be the set of end points of $\mathfrak{r}(a, b)$, where $a\in E'$, $b\in E'$. Then by Theorem IV, E_0' contains a certain n-dimensional sphere K of radius ρ about the origin. Hence for any $\mathfrak{r}$, such that $|\mathfrak{r}|<\rho$, there are two points $a\in E'$, $b=a+\mathfrak{r}\in E'$. Since $a\in E_2+\mathfrak{r}_0$, there exists a point $a_1\in E_2$, such that $a=a_1+\mathfrak{r}_0$, so that $\overrightarrow{a_1b}=\mathfrak{r}+\mathfrak{r}_0$. Hence E_0 contains a sphere $K+\mathfrak{r}_0$, q. e. d.

Theorem VI. Let E_1 and E_2 be measurable sets in an n-dimensional space and $mE_1=mE_2$. Then we can decompose E_1 and E_2, such that

$$E_1=e_1^{(0)}+\sum_{n=1}^{\infty}e_1^{(n)}\,, \qquad E_2=e_2^{(0)}+\sum_{n=1}^{\infty}e_2^{(n)}\,,$$

where $me_1^{(0)}=0$, $me_2^{(0)}=0$ and $e_1^{(n)}$ is congruent with $e_2^{(n)}$ by a translation. Fukamiya[6] proved the case $n=1$.

Proof. We prove the case $n=2$. Let E_1 and E_2 be measurable sets on the xy-plane.

(i) First we assume that E_1 and E_2 are bounded, so that E_1 and E_2 are contained in a square: $|x|<L$, $|y|<L$. Let $\varphi_1(x, y)$, $\varphi_2(x, y)$ be the characteristic functions of E_1 and E_2 respectively and $\mathfrak{r}=(h, k)$ be such a vector, that $|h|\leqq 2L$, $|k|\leqq 2L$. We put

$$\psi(h, k)=mE_1(E_2+\mathfrak{r})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_1(x, y)\varphi_2(x-h, y-k)dxdy\,.$$

Since $\varphi_2(x, y)=0$ for $|x|\geqq L$, $|y|\geqq L$, we have by Fubini's theorem,

$$\int_{-2L}^{2L}\int_{-2L}^{2L}\psi(h, k)dhdk=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_1(x, y)dxdy\int_{-2L}^{2L}\int_{-2L}^{2L}\varphi_2(x-h, y-k)dhdk$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_1(x, y)dxdy\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_2(h, k)dhdk=mE_1mE_2=(mE_1)^2. \quad (4)$$

4), 5) Steinhaus: Sur les distances des points des ensembles de measure positive. Fund. Math. **1** (1920), Rademacher: Über eine Eigenschaft von messbaren Mengen positiven Masses. Jahresbericht d. D. M. V. **30** (1921).

6) M. Fukamiya, l. c. 3).

Let $M=\text{Max.}\,\psi(h,k)$ for $|h|\leqq 2L$, $|k|\leqq 2L$, then $(4L)^2M\geqq(mE_1)^2$, or $M\geqq\dfrac{(mE_1)^2}{(4L)^2}$, so that there exists a vector $\mathfrak{r}_0=(h_0,k_0)$, such that $\psi(h_0,k_0)=mE_1(E_2+\mathfrak{r}_0)\geqq\dfrac{(mE_1)^2}{(4L)^2}$. We put

$$\left.\begin{array}{ll} e_1^{(1)}=E_1(E_2+\mathfrak{r}_0)\,, & e_2^{(1)}=E_1(E_2+\mathfrak{r}_0)-\mathfrak{r}_0\,, \\ E_1^{(1)}=E_1-e_1^{(1)}\,, & E_2^{(1)}=E_2-e_2^{(1)}\,. \end{array}\right\} \tag{5}$$

Then $e_1^{(1)}$ is congruent with $e_2^{(1)}$ by a translation $\mathfrak{r}_0$ and

$$mE_1^{(1)}=mE_2^{(1)}\leqq mE_1-\frac{(mE_1)^2}{(4L)^2}\,. \tag{6}$$

If $mE_1^{(1)}>0$, then we apply the same operation on $E_1^{(1)}$ and $E_2^{(1)}$ and obtain $E_1^{(2)}, E_2^{(2)}, e_1^{(2)}, e_2^{(2)}$, such that $e_1^{(2)}$ is congruent with $e_2^{(2)}$ and

$$mE_1^{(2)}=mE_2^{(2)}\leqq mE_1^{(1)}-\frac{(mE_1^{(1)})^2}{(4L)^2}\,. \tag{7}$$

Repeating the similar operations, after n steps, we obtain $E_1^{(n)}, E_2^{(n)}, e_1^{(n)}, e_2^{(n)}$, where $e_1^{(n)}$ is congruent with $e_2^{(n)}$ by a translation and

$$mE_1^{(n)}=mE_2^{(n)}\leqq mE_1^{(n-1)}-\frac{(mE_1^{(n-1)})^2}{(4L)^2}\,. \tag{8}$$

Since $mE_1^{(n)}$ decreases with n, let $d=\lim\limits_{n\to\infty} mE_1^{(n)}$, then we have from (8), $d\leqq d-\dfrac{d^2}{(4L)^2}$, so that $d=\lim\limits_{n\to\infty} mE_1^{(n)}=0$. Hence if we put $e_1^{(0)}=\lim\limits_{n\to\infty} E_1^{(n)}$, $e_2^{(0)}=\lim\limits_{n\to\infty} E_2^{(n)}$, we have

$$E_1=e_1^{(0)}+\sum_{n=1}^{\infty} e_1^{(n)}\,,\qquad E_2=e_2^{(0)}+\sum_{n=1}^{\infty} e_2^{(n)}\,, \tag{9}$$

where $me_1^{(0)}=0$, $me_2^{(0)}=0$, and $e_1^{(n)}$ is congruent with $e_2^{(n)}$ by a translation.

(ii) In the general case, let $mE_1=mE_2=\sum\limits_{n=1}^{\infty}\eta_n$ $(\eta_n>0)$, where we take $\eta_n=1$, if $mE_1=mE_2=\infty$. Then we can decompose E_1 and E_2 into bounded sub-sets, $E_1^{(n)}$ and $E_2^{(n)}$, such that

$$E_1=\sum_{n=1}^{\infty} E_1^{(n)}\,,\qquad E_2=\sum_{n=1}^{\infty} E_2^{(n)}\,, \tag{10}$$

where $mE_1^{(n)}=mE_2^{(n)}=\eta_n$. To see this, let $Q:|x|\leqq L,\ |y|\leqq L$ be a square and we determine L, so that $mE_1Q=\eta_1+\cdots+\eta_n$ and put $E_1^{(n)}=E_1(Q_n-Q_{n-1})$. Then $mE_1^{(n)}=\eta_n$ and $E_1=\sum\limits_{n=1}^{\infty}E_1^{(n)}$. Similarly we have $E_2=\sum\limits_{n=1}^{\infty}E_2^{(n)}$, $mE_2^{(n)}=\eta_n$. Since by (i), we can decompose $E_1^{(n)}$ and $E_2^{(n)}$ into congruent sub-sets, we can decompose E_1 and E_2 into congruent sub-sets as stated in the Theorem, q. e. d.