23. *Notes on Differentiation.*

By Shin-ichi IZUMI and Gen-ichirô SUNOUCHI.
Mathematical Institute, Tohoku Imperial University, Sendai.
(Comm. by M. FUJIWARA, M.I.A., March 12, 1943.)

Important theorems concerning differentiation are devided into two classes. The first class consists of theorems of differentiability of indefinite integrals and related theorems. The second is the class of Denjoy's theorem and its analogue. We will give a universal method to prove theorems of the first class, and prove a convergence theorem which contains theorems of the second class. Our method is to use maximal theorem due to Hardy and Littlewood and convergence theorem due to Kantorovitch. This idea is due to K. Yoshida[1] and Kantorovitch[2].

1. *Theorems of Kantorovitch and Hardy-Littlewood.*

Kantorovitch's theorem reads as follows[3].

(*K*) *Let X and Y be regular vector lattices and (U_n) be a sequence of operations from X to Y such that $U_n \in H_t^t$ $(n=1,2,3,\ldots)$ (by the Kantorovitch's notation). If*

1°. *$U_n(x)$ converges in a dense set D in X,*

2°. *for any x in X $\limsup U_n(x)$ and $\liminf U_n(x)$ exists, then $U_n(x)$ (o)-converges for all x in X.*

lim sup and lim inf denote those concerning order topology. If $Y=S$, then the order limit becomes almost everywhere convergence.

On the other hand maximal theorem reads as follows[3].

(*HL*) *We put $y(s)\equiv\sup\left(\frac{1}{|I|}\int_I x(t)dt;\ s\in I\right)$ for integrable function $x(t)$. Then*

1°. *If $x\in L^p$ $(p>1)$, then $y\in L^p$ and $\int_0^1|y(t)|^p dt \leqq A\int_0^1|x(t)|^p dt$.*

2°. *If $x\in L_Z$, then $y\in L$ and $\int_0^1|y(t)|dt\leqq A\int_0^1|x(t)|\log^+|x(t)|dt+B$,*

3°. *If $x\in L$, then $y\in L^\alpha$ $(0<\alpha<1)$ and $\left(\int_0^1|y(t)|^\alpha\right)^{\frac{1}{\alpha}}dt\leqq A\int_0^1|x(t)|dt$,*

where A and B are independent of function $x(t)$, and L_Z denotes the Zygmund class.

The last is due to Privaloff, which is generalized as follows.

3^{∞}. *If $x\in L$, then $y\in L_K$, that is, there exists the integral*

---

1) Yosida's result was not yet published.

2) Kantorovitch, Comptes Rendus Acad. Sci. URSS., **14** (1937), 225 and **14** (1937), 244.

3) Hardy-Littlewood, Acta Math., **54** (1930), 81. See Zygmund, *Trigonometrical Series*, (1935), 150.

$\int_I K(y(t))dt$ *where* $K(u)=u/(1+\log_2^+ u)^{1+e}$ $(e>0)$.

The class L_K was introduced by Kawata (Takahashi)[1].

2. *Functions of a real variable.*

(2.1) *If* $x(t)\in L$, *then the limit*

$$\lim \frac{1}{|I|}\int_I x(t)dt \tag{1}$$

exists and is equal to $x(s)$ *almost everywhere, where the limit is taken such as* $s\in I$ *and* $I\to s$.

This is the fundamental theorem of the Lebesgue integral. If we know the existence of (1), then the remaining is easy. Now existence of (1) follows from (K) and (HL), 3°. For, If we put

$$U_I(x)\equiv U_I(x\,;s)=\frac{1}{|I|}\int_I x(t)dt,$$

where $I=(s-h, s+k)$, h and k being constant, then U_I is (t,t)-continuous operation from L on S. By (HL), 3° sup $U_I(x)<\infty$ almost everywhere. Since class of all continuous functions are dense in S, we get the theorem by (K).

If we use (HL), 3^{∞} instead of (HL), 3°, then we get the theorem due to Kantorovitch.

(2.2) *If* $x(t)\in L$, *then the limit* (1) *exists majorated by function in* L_K.

3. *Functions of many variables.*

(2.1) is not true for functions of many variables in general. But we have

(3.1) *If* $x(s,t)\in L_Z$, *that is* $x(s,t)$ *is measurable and the integral*

$$\int_0^1\int_0^1 |x(s,t)|\log^+|x(s,t)|\,dsdt$$

exists, then the integral $\iint_I x(s,t)dsdt$ *is strongly differentiable.*

This was proved by Jessen, Marcinkiewicz and Zygmund[2]. Their proof is very difficult, but we can give a simple proof by the method of § 2. We will put, as in § 2,

$$U_I(x\,;s,t)\equiv\frac{1}{|I|}\iint_I x(s,t)dsdt.$$

By (L) it is sufficient to prove

$$\limsup U_I(x\,;s,t)<\infty \quad \text{almost everywhere}$$

as $I\to s$.

We can suppose that $x(t)\geqq 0$ almost everywhere. By the Fubini's theorem there is a set E_1 with measure 1 such that for any fixed t in E_1 $x(s,t)\in L$ concerning s. For $t\in E_1$ we put

1) Takahashi, Sci. Rep. Tohoku Univ., **25** (1936), 56.

2) Jessen, Marcinkiewicz and Zygmund, Fund. Math., **25** (1935), 217. See Saks, *Theory of the Integral*, (1937), 147.

$$y(s, t) \equiv \sup_{h_1, h_2} \frac{1}{h_1+h_2}\int_{s-h_1}^{s+h_2} x(u, t)du .$$

As may easily be seen $y(s, t)$ is measurable. If we apply (HL), 2° to $y(s, t)$ as function of s, then

$$\int_0^1 y(s, t)ds \leqq A \int_0^1 x(s, t) \log^+ x(s, t)ds+B .$$

Integrating by t we get

$$\int_0^1\int_0^1 y(s, t)dsdt \leqq A\int_0^1\int_0^1 x(s, t) \log^+ x(s, t)dsdt+B . \tag{2}$$

If we put $I=(s-h_1,\ s+h_2;\ t-k_1,\ t+k_2)$, then

$$U_I(x\,; s, t)=\frac{1}{k_1+k_2}\int_{-k_1}^{k_2}\left\{\frac{1}{h_1+h_2}\int_{-h_1}^{h_2} x(s+u, t+v)du\right\}dv$$

$$\leqq \frac{1}{k_1+k_2}\int_{-k_1}^{k_2} y(s, t+v)dv \quad \text{almost everywhere.}$$

Thus we have

$$\limsup U_I(x\,; s, t) \leqq y(s, t) \quad \text{almost everywhere.}$$

Since $y(s, t) \in L$ by (2), we get the required result.

We can prove similarly

(3.2) *If $x(s, t)$ is measurable and the integral*

$$\int_0^1\int_0^1 |x(s, t)|\left(\log^+ |x(s, t)|\right)^2 dsdt$$

exists, then $\iint_I x(s, t)dsdt$ is strongly differentiable majorated by integrable function.

(3.3) *If $x(t_1, t_2, \dots, t_m)$ is measurable and the integral*

$$\int_0^1\cdots\int_0^1 |x(t_1, t_2, \dots, t_m)|\left(\log^+ |x(t_1, t_2, \dots, t_m)|\right)^{m-1}dt_1 \dots dt_m$$

exists, then the indefinite integral of $x(t_1, t_2, \dots, t_m)$ is strongly differentiable.

4. *Extension of (HL) for functions of many variables.*

Let $x(s, t)$ be a function in L and put

$$y(s, t) \equiv \sup\left(\frac{1}{|I|}\iint_I x(u, v)dudv\,;\ (s, t) \in I\right)$$

then $y(s, t)$ does not belong to any $L^a(0 < a < 1)$. For, if not so, we can prove by the method in § 2 that the indefinite integral of functions in L is strongly differentiable. But this is not ture in general[1]. Therefore (HL), 3^{∞} does not ture for functions of two variables in general.

If we restrict to regular intervals, then (HL), 3° holds. More generally we can prove

1) Saks, Fund. Math., **25** (1935), 235.

(4.1) *Let* $x(s,t)\in L$ *and put*

$$y(s,t)\equiv \sup \frac{1}{|I|}\iint_I x(u,v)dudv\,,$$

where $(s,t)\in I$ *and* I *varies on regular intervals. Then* $y(s,t)\in L_K$.

For, if we put $E_a\equiv\big((s,t);|y(s,t)|>a\big)$, then by the Vitali's covering theorem

$$\frac{1}{3}|E_a|\leqq\frac{1}{a}\int_0^1\int_0^1|x(s,t)|\,dsdt\,.$$

Now we have

$$\int_0^1\int_0^1 K\big(y(s,t)\big)dsdt\leqq 1+\sum_{k=2}^{\infty}\iint_{E_{2^{k-1}}-E_{2^k}}K\big(y(s,t)\big)dsdt$$

$$\leqq 1+\sum_{k=2}^{\infty}\frac{2^k}{(1+\log 2^{k-1})^{1+\epsilon}}|E_{2^{k-1}}|$$

$$\leqq 1+\Big(\sum_{k=2}^{\infty}{}'\frac{6}{k^{1+\epsilon}}\Big)\int_0^1\int_0^1|x(s,t)|\,dsdt<\infty\,.$$

Two variables analogy of (HL), 2° is not true in general. But we have

(4.2) *If* $x(s,t)\in L_Z$, *then*

$$\int_0^1\int_0^1|y(s,t)|\,dsdt\leqq A\int_0^1\int_0^1|x(s,t)|\log^+|x(s,t)|\,dsdt+B\,.$$

This is due to Wiener[1]. If we drop the regularity of I, then

(4.3) *If* $x(s,t)\in L_Z$ *and we put*

$$z(s,t)\equiv\lim_{k_1,k_2\to 0}\frac{1}{k_1+k_2}\int_{t-k_1}^{t+k_2}\Big\{\sup_{h_1,h_2}\frac{1}{h_1+h_2}\int_{s-h_1}^{s+h_2}|x(u,v)|\,du\Big\}dv\,,$$

then we have

$$\int_0^1\int_0^1 z(s,t)dsdt\leqq A\int_0^1\int_0^1|x(s,t)|\log^+|x(s,t)|\,dsdt+B\,.$$

5. *Regular differentiability of indefinite integral.*

By (4.1) we can prove that

(5.1) *Indefinite integral of integrable functions of many variables is regularly differentiable almost everywhere.*

6. *A convergence theorem.*

(6.1) *Let* $U_m(x)$ *be a sequence of linear transformations in* H_t^t *which transforms a regular vector lattice* X *onto another* Y. *If the conditions :*

i) $U_m(x)$ *(o)-converges in a dense set* D *in* X.

2°. *if* $U_m(x)$ *is (o)-bounded, then there are* (x_n) *in* D *and* (λ_k) *such that* $k\geqq\lambda_k\uparrow\infty$, $\sum\lambda_k|x_k-x_{k+1}|$ *(o)-converges and*

---

1) Wiener, Duke Math. Journ., **5** (1939), 1.

$$\left\{U_m\left(\sum_{k=1}^{m-1}\lambda_k(x_{n_k}-x_{n_{k-1}})\right)\right\}$$

is (*o*)*-bounded for any* (n_k)*, are satisfied, then* $\{U_m(x)\}$ *is not* (*o*) *bounded or* (*o*)*-converges.*

Proof. If the theorem is not true, then there is an $\bar{x}$ in X such that $\{U_m(x)\}$ is (*o*)-bounded but does not (*o*)-converges. Then there is a positive element $\bar{y}$ in Y defined by

$$\bar{y}=\lim\sup U_m(\bar{x})-\lim\inf U_m(\bar{x}).$$

On the other hand there is a sequence (x_n) in D satisfying conditions in 2°. If we put

$$s_{n,m}(x)\equiv\sup\left(U_n(x),\cdots,U_m(x)\right)-\inf\left(U_n(x),\cdots,U_m(x)\right),$$

then

$$\lim s_{n,m}(\bar{x})=\bar{y},\quad \lim s_{n,m}(x_p)=0,\quad \lim s_{n,m}(x_p-x_q)=0.$$

Let us take a sequence (ε_k) of positive number such that $\sum_{k=1}^{\infty}k\varepsilon_k=K<\infty$. There is a y such that above limits exist uniformly relative to y. Now there are n_1 and m_1 such that $|s_{n_1,m_1}(\bar{x})-y|<\varepsilon_1 y$, and then there is a p_1 such that

$$|s_{n_1,m_1}(x_p)-s_{n_1,m_1}(x_q)|<\varepsilon_1 y \text{ for } p,q\geqq p_1,$$

$$|s_{n_1,m_1}(x_{p_1})-\bar{y}|<2\varepsilon_1 y.$$

When (n_i, m_i, p_i) $(i=1,2,\dots,k-1)$ are determined, we can find n_k, m_k and p_k such that $p_k>p_{k-1}$,

$$|s_{n_k,m_k}(x_{p_k})-\bar{y}|<2\varepsilon_k y,$$

$$|s_{n_k,m_k}(x_{p_i})|<\varepsilon_i y \quad (i=1,2,\dots,k-1),$$

$$|s_{n_i,m_i}(x_{p_k})-s_{n_i,n_i}(x_{p_{k-1}})|<\varepsilon_k y \quad (i=1,2,\dots,k-1).$$

By the condition 2°, there is an $\bar{\bar{x}}=\sum\lambda_k(x_{n_k}-x_{n_{k-1}})$ such that $\sum\lambda_k|x_{n_k}-x_{n_{k-1}}|$ (*o*)-converges and $U_n(\bar{\bar{x}})$ is (*o*)-bounded. Now

$$|s_{n_k,m_k}(x)-s_{n_k,m_k}(\lambda_k x_{p_k})|$$
$$\leqq\sum_{i=1}^{k-1}(\lambda_i-\lambda_{i-1})|s_{n_k,m_k}(x_{p_i})|+\sum_{i=k+1}^{\infty}\lambda_i|s_{n_k,m_k}(x_{p_i}-x_{p_{i-1}})|$$
$$\leqq\left(\sum_{i=1}^{k-1}i\varepsilon_i+\sum_{i=k+1}^{\infty}i\varepsilon_k\right)y\leqq Ky.$$

Thus we have

$$|s_{n_k,m_k}(x)-\lambda_k\bar{y}|\leqq Ky+2\lambda_k\varepsilon_k y,$$

which implies $s_{n_k,m_k}(x)$ is not (*o*)-bounded. This is a contradiction. Thus we get the theorem.