22. A Remark on Ergodic Theorems.

By Shin-ichi IZUMI.
Mathematical Institute, Tohoku Imperial University, Sendai.
(Comm. by M. FUJIWARA, M.I.A., March 12, 1943.)

1. G. D. Birkhoff proved the following theorems.

(*B. 1*) Let T be a measure preserving transformation in $(0,1)$ such that the inverse transformation T^{-1} is also. Then for any $x=x(t)$ in $L=L(0,1)$ the limit

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}x(T^n t) \qquad (1)$$

exists almost everywhere.

(*B, 2*) Let $T^\lambda(-\infty<\lambda<\infty)$ be a set of transformations satisfying above condition such that $T^\lambda(T^\mu t)=T^{\lambda+\mu}t$. If $x(T^\lambda t)$ is measurable in the product space (λ, t) and is integrable in $(0,1)$ with respect to t, then the limit

$$\lim_{N\to\infty}\frac{1}{N}\int_0^N x(T^\lambda t)d\lambda \qquad (2)$$

exists almost everywhere.

These are called individual ergodic theorems. Convergence in (1) and (2) is not dominated by integrable functions in general. But Fukamiya and Wiener proved that

(*FW*) If T $\big($or $T^\lambda(-\infty<\lambda<\infty)\big)$ satisfies above condition and $x(t)\varepsilon L_Z$, that is, $x(t)\log^+|x(t)|$ is integrable, then (1) $\big($or (2)$\big)$ converges dominated by integrable functions almost everywhere.

This is called dominated ergodic theorem. To prove above three theorems Wiener proved the fundamental lemma:

(*W*) Let $x(t)$ be a non-negative integrable function and

$$x^*(t)=\underset{0<N<\infty}{\text{l. u. b.}}\frac{1}{N+1}\sum_{n=0}^{N}x(T^n t) \qquad \Big(\text{or}=\underset{0<N<\infty}{\text{l. u. b.}}\frac{1}{N}\int_0^N x(T^\lambda t)d\lambda\Big)$$

then we have for any $\alpha>0$

$$\big((t;\, x^*(t)>\alpha\big) \leqq \frac{1}{\alpha}\int_0^1 x(t)dt\,.$$

2. In order to prove (B, 1), (B, 2) Wiener proved the mean ergodic theorem in L. But we can prove them directly by using a convergence theorem due to Kantorovitch. Kantorovitch's theorem reads as follows.

(*K*) Let X and Y be regular vector lattices and $\{U_n(x)\}$ be a sequence of (t, t)-continuous operations from X to Y. Then if

1°. for x in a dense set D in X $U_n(x)$ is (o)-$\big($or (t)-$\big)$ convergent, 2°. for each x in X $U_n(x)$ is (o)-(or (t)-) bounded, then $U_n(x)$ is always (o)-(or (t)-) convergent.

In order to prove (B, 1) and (B, 2), we need a lemma:

Lemma. If we put $K(u)=u/(1+\log_2^+ u)^{1+e}$ $(e>0)$, then $x^*\varepsilon L_K$, that is, the integral $\int_0^1 K(x^*(t))dt$, exists, where

$$x^*(t)=\underset{0<N<\infty}{\text{l. u. b.}}\frac{1}{N}\int_0^N x(T^\lambda t)d\lambda\,.$$

Proof. If we put $E_a=(t;\, x^*(t)\geqq a)$, then

$$\int_0^1 K(x^*(t))dt\leqq 1+\sum_{k=2}^{\infty}\int_{E_{2^{k-1}}-E_{2^k}}K(x^*(t))dt$$

$$\leqq 1+\sum_{k=2}^{\infty}\frac{2^k}{(1+\log 2^{k-1})^{1+e}}\,|E_{2^{k-1}}|$$

$$\leqq 1+\sum_{k=2}^{\infty}\frac{2}{k^{1+e}}\int_0^1|x(f)|\,dt$$

by (W). Thus we get the required result.

By (K) and Lemma we can now prove (B, 1) and (B, 2) easily. Let us put $U_n(x)=\frac{1}{N+1}\sum_{m=0}^{N}x(T^m t)$ which transforms L into L_K. L and L_K are regular vector lattices and Lemma gives condition 2° in (K). As D in condition 1° we take the set of bounded measurable functions. For such D we can prove 1° somewhat easily. Thus we get (B, 1). Similarly we can prove (B, 2).

From the proof we know that the convergence in (1) and (2) is dominated by functions in L_K.

3. We will now extend (B, 2) and (B, 2).

Theorem 1. Let T be a linear transformation in L_K (or in L) such that

1°. for any $x\varepsilon L$ (or εL_Z)

$$\limsup_n\left|\frac{1}{n}\sum_{k=1}^{n}T^k x(t)\right|<\infty$$

almost everywhere, and

2°. for any bounded measurable function x there is a constant M such as $|T^n x(t)|\leqq M(n=1, 2, \ldots)$, then for any $x\varepsilon L$ (or εL_Z) the limit

$$\lim_n\frac{1}{n}\sum_{k=1}^{n}T^k x(t)$$

exists almost everywhere.

Proof is done by the method in § 2. This contains (B, 1) and (FW). We can state the theorem containing (B, 2) and the corresponding part of (FW). We are also easy to extend this lattice-theoretically.

Similarly we can prove

Theorem 2. Let T be a linear transformation in L_K (or in L or in L^p $(p>1)$) such as $\|T^n\|$ $(n=1, 2, \dots)$ is bounded. Then for any x in L (or in L_Z or in L^p $(p>1)$) $\frac{1}{n}\sum_{k=1}^{n} T^k x(t)$ converges in L_K-mean (or L-mean or in L^p-mean).