PAPERS COMMUNICATED

21. *Notes on Banach Space (V): Compactness of Function Spaces.*

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1. We have proved[1] already

Theorem 1. A set $\mathfrak{F}$ in (C) (=family of continuous functions in $(0,1)$) is compact when and only when

1°. $\mathfrak{F}$ is uniformly bounded,

2°. $\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta f(x+t)dt=f(x)$ uniformly for all x in $(0,1)$ and for all f in $\mathfrak{F}$,

Theorem 2. A set $\mathfrak{F}$ in (M) (=family of essentially bounded measurable functions in (0,1)) is compact when and only when 1° and

3°. $\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta f(x+t)dt=f(x)$ uniformly almost everywhere for all x in $(0,1)$ and for all f in F.

On the other hand Phillips[2] proved a compactness theorem in Banach space, whence he derived the Kolmogoroff-Tulajkoff theorem concerning compactness in (L^p) $(p\geqq 1)$. The latter theorem reads as follows

Theorem 3. A set $\mathfrak{F}$ in (L^p) $(p\geqq 1)$ (=family of measurable functions whose p-th power is integrable in $(0,1)$) is compact when and only when

4°. for f in $\mathfrak{F}$ $\int_0^1|f(t)|^p dt$ is uniformly bounded,

5°. $\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta f(x+t)dt=f(x)$ uniformly in the L^p-mean.

Concerning space (S) we proved in § 3

Theorem 4. A set $\mathfrak{F}$ in (S) (=family of measurable functions in $(0,1)$) is compact when and only when

6°. $\operatorname{asy\cdot lim}_{(\delta,N)}\frac{1}{\delta}\int_0^\delta \big(f(x+t)\big)^N dt=f(x)$ uniformly for f in $\mathfrak{F}$, where $\operatorname{asy\cdot lim}_{(\delta,N)}$ is the Moore-Smith limit in measure and

$$\big(f(t)\big)^N=f(t) \quad \text{if} \quad |f(t)|\leqq N \quad \text{and} \quad =0, \quad \text{otherwise.}$$

Summing up above results we get

Theorem 5. A set $\mathfrak{F}$ in E where E is (C), (M), (L^p) $(p\geqq 1)$ or (S), is compact when and only when

7°. $\mathfrak{F}$ is bounded concerning metric in E,

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1) S. Izumi, Proc. **15** (1938).

2) R. Phillips, Trans. Am. Math. Soc., vol. **44** (1940).

8°. $\lim\limits_{(\delta,N)} \frac{1}{\delta}\int_0^\delta \big(f(x+t)\big)^N dt = f(x)$ uniformly in $\mathfrak{F}$, where $\lim\limits_{(\delta,N)}$ is the Moore-Smith limit concerning the metric in E.

In § 2 we prove a key theorem from which all above theorems are derived.

2. Let X be an (F)-space. We suppose that there are sets of operations $U_{\delta,N}(f)=U_\delta(f^N)$ $(\delta>0,\ N>0)$ from X into X such that

a) for fixed δ and N $U_{\delta,N}$ is completely continuous,

b) for a fixed N U_δ is a linear operation and $\| U_\delta \|$ is uniformly bounded, that is there is an M such as $\| U_\delta \| \leqq M$,

c) for each f in X $\lim U_{\delta,N}(f)=f$, limit being taken concerning (F)-metric.

Then we have

Theorem 6. A set S in X is compact when and only when

1′) l. u. b. $(|f|\,;\, f\varepsilon S)<\infty$,

2′) $\lim | U_{\delta,N}(f)-f|=0$ uniformly in S,

where $|\ \ |$ denotes metric in X, such that $|f^N-g^N| \leqq |f-g|$.

Proof. Necessity. Let S be a compact set. Since S is totally bounded, for any $e>0$ there are $f_1, f_2, \dots, f_n$ in S such that for any $f\varepsilon S$ there is a k such that $|f-f_k|<e$.

By c) we can find (δ_e, N_e) such that for all $\delta \leqq \delta_e$ and $N \geqq N_e$

$$| U_{\delta,N}(f_k)-f_k|<e \qquad (k=1, 2, \dots, n).$$

Therefore for any $x\varepsilon S$ and $(\delta, N) \geqq (\delta_e, N_e)$

$$| U_{\delta,N}(f)-f| \leqq | U_\delta(f^N-f_k^N)|+| U_{\delta,N}(f_k)-f_k|+|f_k-f| \leqq e(2+M)$$

by b). Thus we get 2′). 1′) is evident.

Sufficiency. By 2′), for any $e>0$ there exists (δ_e, N_e) such that

$$| U_{\delta_e,N_e}(f)-f|<e \qquad (f\varepsilon S).$$

By 1′) S is bounded, and then by a) $(U_{\delta,N}(f)\,;\, f\varepsilon S)$ is compact. Hence there are $f_1, f_2, \dots, f_n$ in S such that for any f in S there exists k such as

$$U_{\delta_e,N_e}(f)-U_{\delta_e,N_e}(f_k)|<e.$$

Hence

$$|f-f_k| \leqq |f-U_{\delta_e,N_e}(f)|+| U_{\delta_e}(f^{N_e}-f_k^{N_e})|+| U_{\delta_e,N_e}(f_k)-f_k| \leqq 3e.$$

That is, S is totally bounded, and then is compact.

3. We can now prove Theorem 4 by Theorem 6.

In S the metric is defined by

$$|f| \equiv \int_0^1 \frac{|f(t)|}{1+|f(t)|}dt.$$

Convergence by such metric coincides with asymptotic convergence. Condition a) is easily verified in such space. Condition b) is given by

$$\left|\frac{1}{\delta}\int_0^\delta \left(f(x+t)\right)^N dt=\right|=\int_0^1 \frac{\left|\frac{1}{\delta}\int_0^\delta \left(f(x+t)\right)^N dx\right|}{1+\left|\frac{1}{\delta}\int_0^\delta \left(f(x+t)\right)^N dx\right|}dt$$

$$\leqq \int_0^1 \frac{\frac{1}{\delta}\int_0^\delta \left|\left(f(x+t)\right)^N\right| dx}{1+\frac{1}{\delta}\int_0^\delta \left|\left(f(x+t)\right)^N\right| dx}dt \leqq C\int_0^1 \frac{\left|\left(f(t)\right)^N\right|}{1+\left|\left(f(t)\right)^N\right|}dt$$

$$=C|f^N|\leqq C|f|.$$

C being an absolute constant. Finally condition c) is also evident. Thus Theorem 6 gives Theorem 4.