133. On the Phenomena of Instability in Undamped Quasi-harmonic Vibration. Part I.

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The quasi-harmonic vibration, namely, the vibration of a system with periodically varying elasticity or damping coefficient or inertia mass, is present in such widely different kinds of engineering problems as, for example, a two-pole turbo-generator, a condenser microphone, an electric locomotive of the side-rod type, a two-blade propeller, an internal combustion engine with cranks and pistons, etc. It is possible to show that the equation of undamped quasi-harmonic vibration of any case is generally involved in the expression of the type:

$$\frac{d}{dt}\left\{P(t)\frac{dX}{dt}\right\}+Q(t)X=0\,, \tag{1}$$

where $P(t)$, $Q(t)$ are periodic functions of time. For meeting every practical need, it is advisable to decompose equation (1) to three simple cases, namely

$$P(t)=\text{const.},\qquad Q(t)=Q_0+Q_1\cos 2pt\,, \tag{1 a}$$

$$P(t)=\text{const.},\qquad Q(t)=1/R(t)=1/(R_0+R_1\cos 2pt)\,, \tag{1 b}$$

$$Q(t)=\text{const.},\qquad P(t)=P_0+P_1\cos 2pt\,, \tag{1 c}$$

where $2p$ is the frequency of periodic variation of such coefficient as elasticity or damping or inertia mass.

Case (1 a), having already called attention of many investigators, is well known to be solved with Mathieu's functions, whereas cases (1 b), (1 c) are not treated as simple as in case (1 a). If however R_1, P_1 were small quantities, their solutions would be represented in the forms of expansion in series or in other approximate ones. Since with such restriction as R_1, P_1 being small, the problem is liable to be outside the theoretical interest and also to be remote from practical use, it is now of pressing importance to obtain more satisfactory solutions that should be adapted to any value of R_1 or P_1.

Upon examining the nature of the equations, it has been found that transformation of certain variables aids us to formulate such solutions as will answer, at least, to some cases of ripple in periodically varying coefficient, as a result of which it is possible for the restriction of R_1, P_1 just mentioned to be precluded.

The expression (1 b) can also be written

$$\frac{d^2X}{d\tau^2}+\frac{\omega_{0p}^2}{1-k^2\sin^2\tau}X=0\,, \tag{2}$$

where

$$pt=\tau\,,\quad \frac{1}{P(R_0+R_1)}=\omega_0^2\,,\quad \frac{1}{P(R_0-R_1)}=\omega_1^2\,,\quad \frac{2R_1}{R_0+R_1}=k^2\,,$$

$$\frac{\omega_0}{p}=\omega_{0p}\,,\quad \frac{\omega_1}{p}=\omega_{1p}\,.$$

Writing
$$x=\int_0^\tau \frac{d\tau}{\sqrt{1-k^2\sin\tau}} \tag{3}$$

and transforming τ to x in (2), we get

$$\frac{d^2X}{dx^2}+k^2\frac{\operatorname{sn}x\operatorname{cn}x}{\operatorname{dn}x}\frac{dX}{dx}+\omega_{0p}^2X=0\,. \tag{2.1}$$

If we apply Landen's transformation[1]

$$\operatorname{sn}\left[(1+k')x,\ \frac{1-k'}{1+k'}\right]=\frac{(1+k')\operatorname{sn}(x,k)\operatorname{cn}(x,k)}{\operatorname{dn}(x,k)}\,, \tag{4}$$

in which $k'=\sqrt{1-k^2}$, (2.1) reduces to

$$\frac{d^2X}{dy^2}+k_1\operatorname{sn}(y,k_1)\frac{dX}{dy}+\omega_{0k}^2X=0\,, \tag{2.2}$$

where
$$y=(1+k')x\,,\quad k_1=\frac{1-k'}{1+k'}\,,\quad \omega_{0k}=\frac{\omega_{0p}}{1+k'}\,.$$

Since the coefficient of the second term in equation (2.2) is a periodic function, the solution of that equation should be a periodic function of the second kind (sort) as having the relation

$$X(y+4K_1)=\lambda X(y)\,, \tag{5}$$

where λ represents the coefficient of stability and K_1 is such a quarter period of $\operatorname{sn}(y,k_1)$ as $\operatorname{sn}(K_1,k_1)=1$, that is to say

$$K_1=\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-k_1^2\sin^2\varphi}}$$

It will be seen that equation (2.2) is simplified greatly compared with the original one.

The expression (1 c) can be written

$$\frac{d}{dt}\left\{(P_0+P_1\cos 2pt)\frac{dX}{dt}\right\}+QX=0\,,$$

that is
$$\frac{d^2X}{dt^2}-\frac{2P_1\sin 2pt}{P_0+P_1\cos 2pt}\frac{dX}{dt}+\frac{Q}{P_0+P_1\cos 2pt}X=0\,.$$

With simplification this reduces to

$$\frac{d^2X}{dt^2}-2k^2\frac{\sin\tau\cos\tau}{1-k^2\sin^2\tau}\frac{dX}{dt}+\frac{\omega_{0p}^2}{1-k^2\sin^2\tau}X=0\,, \tag{6}$$

1) For example, H. Hancock, Theory of Elliptic Functions (1910), 251.

where $pt=\tau$, $\frac{Q}{P_0+P_1}=\omega_0^2$, $\frac{Q}{P_0-P_1}=\omega_1^2$, $\frac{2P_1}{P_0+P_1}=k^2$,

$$\frac{\omega_0}{p}=\omega_{0p}, \quad \frac{\omega_1}{p}=\omega_{1p}.$$

Transforming τ to x by means of (3), we get

$$\frac{d^2X}{dx^2}-k^2\frac{\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x}\frac{dX}{dx}+\omega_{0p}^2X=0. \tag{6.1}$$

Applying again Landen's transformation in (4), we have

$$\frac{d^2X}{dy^2}-k_1\,\mathrm{sn}\,(y, k_1)\frac{dX}{dy}+\omega_{0k}^2X=0, \tag{6.2}$$

where $y=(1+k')x$, $k_1=\frac{1-k'}{1+k'}$, $\omega_{0k}=\frac{\omega_{0p}}{1+k'}$.

It will be seen that equation (6.2) is of the same form as that (2.2) with such distinction as the signs of k_1 in both cases being opposite. If however we write

$$y=y_1+2K_1,$$

equation (6.2) becomes

$$\frac{d^2X}{dy_1^2}+k_1\,\mathrm{sn}\,(y_1, k_1)\frac{dX}{dy_1}+\omega_{0k}^2X=0, \tag{6.3}$$

which is of the same form as (2.2). It follows that for discussing both cases (1 b), (1 c), the equation

$$\frac{d^2X}{dy^2}+k_1\,\mathrm{sn}\,(y, k_1)\frac{dX}{dy}+\omega_{0k}^2X=0 \tag{2.2'}$$

should always be solved. As a matter of fact, with transformation of variable t in our problem the condition of critical stability as resulting from the nature of X is obviously unchanged, from which condition the difficulty in mathematical analysis is fairly diminished. In the present case, particularly, the advantage arising from such type as equation (2.2′) is that the coefficient $\mathrm{sn}\,(y, k_1)$ of the second term can be replaced by a function corresponding to rectangular ripple. $\mathrm{sn}\,(y, k_1)$ is a curve of such character as intermediate between $\mathrm{sn}\,(y, 0)=\sin y$ and $\mathrm{sn}\,(y, 1)=\tanh y$. Since, for example, even in the case of treatment of Mathieu's equation the coefficient $\sin y$ can be replaced by rectangular ripple with a rather satisfactory result to stability condition, it would be possible to conclude straightway that the approximation in the present case is more in line with the answer of the problem than in the case of Mathieu's equation[1].

For the reason above given, we shall now write

$$\mathrm{sn}\,(y, k_1)\rightarrow H(y, k_1), \tag{7}$$

1) Van der Pol, Phil. Mag., [7] **5** (1928), 18; Van den Hartog, Mechanical Vibration (1940), 388.

where

$$\left.\begin{array}{ll} H(y,k_1)=1\,, & [4nK_1<y<(4n+2)K_1] \\ H(y,k_1)=-1\,, & [(4n+2)K_1<y<(4n+4)K_1] \end{array}\right\}\ (n=0,1,2,\ldots)$$

and
$$H(y+4K_1,k_1)=H(y,k_1)\,.$$

It then follows that

$$\frac{d^2X}{dy^2}+k_1H(y,k_1)\frac{dX}{dy}+\omega_{0k}^2X=0\,, \tag{8}$$

the solution of which assumes the forms

$$X=\left\{\begin{array}{ll} X_1=e^{-k_2y}(A\cos\omega y+B\sin\omega y)\,, & [4nK_1<y<(4n+2)K_1] \\ X_2=e^{k_2y}\ (C\cos\omega y+D\sin\omega y)\,, & [(4n+2)K_1<y<(4n+4)K_1] \end{array}\right\} \tag{9}$$

in which $n=0,1,2,\ldots$, $k_2=k_1/2$, $\omega^2=\omega_{0k}^2-k_2^2$, and A,B,C,D are arbitrary constants. The above solutions both in the forms of displacement as well as in those of velocity should satisfy the condition of continuity for $y=2K_1$ and should also be of the type of a periodic function of the second kind as shown in (5). We have then

$$\left.\begin{array}{ll} (X_1)_{y=2K_1}=(X_2)_{y=2K_1}\,, & (X_1')_{y=2K_1}=(X_2')_{y=2K_1}\,, \\ (X_2)_{y=4K_1}=\lambda(X_1)_{y=0}\,, & (X_2')_{y=4K_1}=\lambda(X_1')_{y=0}\,. \end{array}\right\} \tag{10}$$

From (9) and (10), eliminating $A,B,C,D,$ we get

$$\lambda^2+2N\lambda+1=0\,, \tag{11}$$

where
$$N=2(1+\varkappa^2)\sin^2 2\omega K_1-1\,,\qquad \varkappa=k_2/\omega\,,$$

from which the coefficient of stability becomes

$$\lambda=-N\pm\sqrt{N^2-1}\,. \tag{12}$$

From (12) it will be seen that if $N>1$ or $N<-1$, a one value of λ is greater than unity so that the motion is unstable, whereas if $1>N>-1$, both values of λ are complex, with their real parts being less than unity so that the motion is stable. It then follows that $|N|=1$ represents the condition of the critical stability. The expressions showing the criticals are such that

$$\left.\begin{array}{l} \sin^2 2\omega K_1=\dfrac{1}{1+\varkappa^2}\,,\quad (N=1) \\ \sin 2\omega K_1=0\,.\quad \omega\neq 0\ (N=-1) \end{array}\right\} \tag{13}$$

From $N=-1$ we have

$$2\omega K_1=n\pi\,,\qquad (n=1,2,3,\ldots) \tag{14}$$

and from $N=1$ we get

$$k_1K_1=\frac{(2n+1)\pi}{2}\varkappa\pm\varkappa\tan^{-1}\varkappa\,.\qquad (\varkappa=k_1/2\omega) \tag{15}$$

Substituting now (14), (15) in the following expressions[1)]

$$\omega_{0p}=\frac{\omega_0}{p}=\frac{\sqrt{(2\omega)^2+k_1^2}}{1+k_1}, \qquad \omega_{1p}=\frac{\omega_1}{p}=\frac{\sqrt{(2\omega)^2+k_1^2}}{1-k_1}, \tag{16}$$

we get the values of ω_0/p, ω_1/p. Referring to (9), it is possible for ω to be imaginary, in which case we should write

$$\omega=i\omega_i, \qquad (\omega_i=\sqrt{k_2^2-\omega_{0k}^2})$$

from which it follows

$$N=2\left\{\left(\frac{k_1}{2\omega_i}\right)^2-1\right\}\sinh^2 2\omega_i K_1-1\,.$$

The critical condition $|N|=1$ in this case is given by

$$\left\{\left(\frac{k_1}{2\omega_i}\right)^2-1\right\}\sinh^2 2\omega_i K_1=1\,.$$

Substituting the relation between ω_i and k_1 thus found in

$$\omega_{0p}=\frac{\omega_0}{p}=\frac{\sqrt{k_1^2-(2\omega_i)^2}}{1+k_1}, \qquad \omega_{1p}=\frac{\omega_1}{p}=\frac{\sqrt{k_1^2-(2\omega_i)^2}}{1-k_1}, \tag{16'}$$

we obtain ω_0/p and ω_1/p.

Now, since ω_0/p and ω_1/p are functions of either one of k_1 or ω, it is possible to obtain the relation between ω_0/p and ω_1/p at the transition state $\lambda=\pm 1$, namely, the state at which the vibration changes from stable one to unstable, the result of calculation being shown by series of curves in Fig. 1. The vibrational state indicated by shaded areas is stable and that indicated by blank areas unstable. The critical corresponding to $\lambda=-1$, namely $N=1$, is represented by every curve forming the boundary between shaded and blank areas, whereas the critical corresponding to $\lambda=1$, namely $N=-1$, is indicated by every curve quite within any shaded area, it being revealed that there remains no region for the unstable condition corresponding to $N<-1$. Since furthermore the condition between ω_0/p and ω_1/p can be reversed, the arrangement of curves in Fig. 1 is symmetrical with respect to the diagonal line passing through O; such a figure of symmetrical form has been drawn, as a matter of fact, from need in the application of actual problems. It should be borne in mind that the base lines $\omega_1/p=0$ and $\omega_0/p=0$ are asymptotic to the respective both curves nearest to the same base lines and the lines $\omega_1/p=1/2$ and $\omega_0/p=1/2$ asymptotic to all the remaining curves.

The vibrational condition for any ratio of ω_0/ω_1 is indicated by a line passing through the origin. If this line be OA, the conditions corresponding to segments $O\alpha_1$, $\alpha_1\alpha_2$, $\alpha_2\alpha_3$, ... are stable and those corresponding to segments a_1a_1, a_3a_3, ... as well as points a_2, a_4, ... unstable.

1) For obtaining the expressions, the relations $\omega_{0p}=(1+k')\omega_{0k}$, $\omega^2=\omega_{0k}^2-(k_1/2)^2$, $k_1=(1-k')/(1+k')$, and $\omega_0^2/\omega_1^2=1-k^2$, have been availed of.

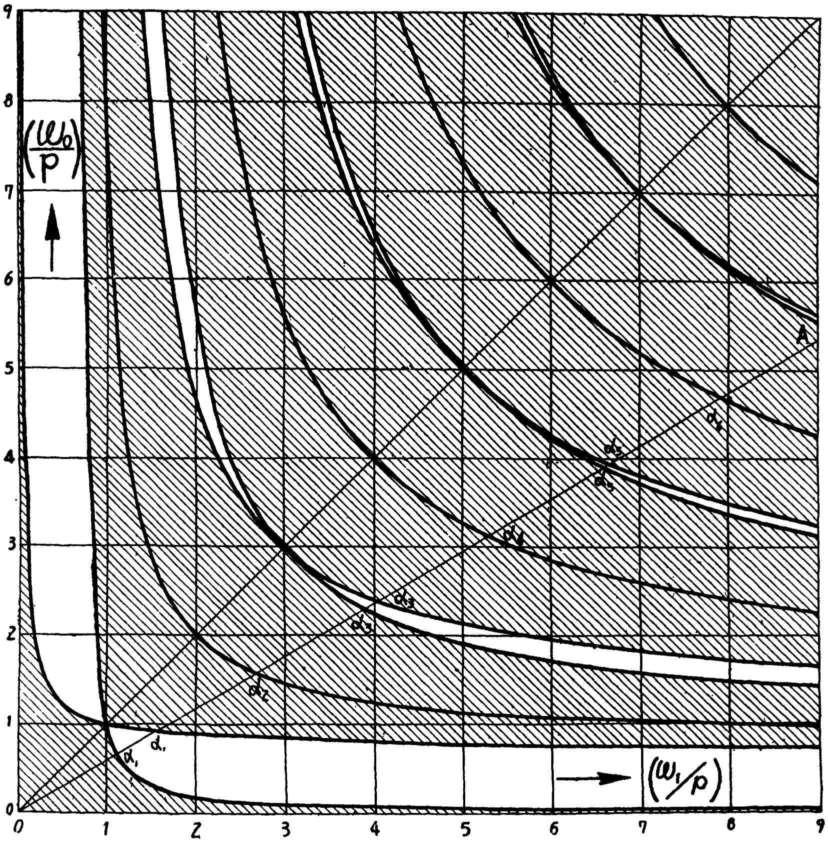

Fig. 1.

Segments a_1a_1, a_3a_3, ... represent such respective instabilities that occur when 1-, 3-, ... cycles, namely, odd numbers of cycles of ripple in periodically varying coefficient synchronize with two cycles of "reference" natural vibration. On the other hand, the critical instabilities that should belong to points a_2, a_4, ... occur when 2-, 4-, ... cycles, namely, even numbers of cycles of ripple in periodically varying coefficient synchronize with two cycles of "reference" natural vibration. Although, in the nature of things, there are innumerable natural frequencies that range from ω_0 to ω_1, yet for symplifying the problem, the reference natural vibration in the present case is specially assumed. A simple and rather advisable form of frequency of reference natural vibration would be $(\omega_0+\omega_1)/2p$, the relation between $(\omega_0+\omega_1)/2p$ and $k_1=(\omega_1-\omega_0)/(\omega_1+\omega_0)$ being calculated from data in Fig. 1, the result of which is shown in Fig. 2. In this case again each curve indicates the transition from stable to unstable conditions, the shaded areas and the blank areas representing stable and unstable regions, respectively. From this figure it will be seen that the greater the difference between ω_1 and ω_0, the more pronounced the feature of the "reference" natural frequency disagreeing with an integral multiple

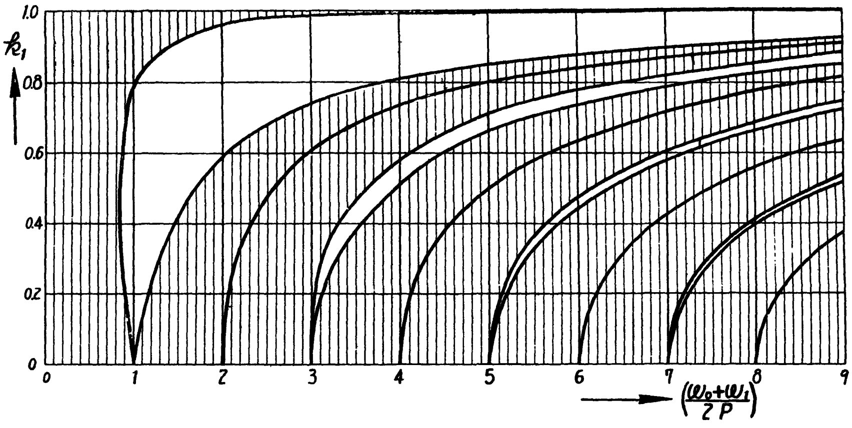

Fig. 2.

of the cycles of ripple in periodically varying coefficient. When the difference in question is zero, namely, when k_1 is zero, there exists no disagreement. Although in the case of a great difference existing, say, $k_1 \rightarrow 1$, instability does not occur within the range of a finite ratio of $(\omega_0+\omega_1)/2p$, if however the condition of instability be once attained, it is improbable for the vibration to be away from that instability for a very wide range of $(\omega_0+\omega_1)/2p$. At all events, although the above conclusion is merely mathematical, the results observed in our experiments are likely to agree with the theoretical ones, which fact will be available before long.

In the present mathematical analysis we have ascertained two cases (1 b), (1 c). As shown in the beginning of this paper, case (1 a) has already been investigated by many authors, for which reason reference to that case is omitted. Although the general case of quasi-harmonic vibration including (1 a), (1 b), (1 c) simultaneously is very difficult, yet we have hopes that it would be possible for that case to be ascertained in some tentative manner.