133. On the Representation of Functions by Fourier Integrals.

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1. *Introduction and the theorems.* The purpose of the present note is to give the following representation theorems of complex-valued bounded continuous functions $f(t)$ on $(-\infty, \infty)$. The theorems may be applied in Fourier analysis as well as in probability theory. Since the proofs are carried through by virtue of the Plancherel's duality theorem, our results may be extended to the case of separable, locally compact abelian groups instead of the infinite line $(-\infty, \infty)$[1].

Theorem 1. $f(t)$ is positive definite[2] if and only if

$$\varphi_n(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\left(\frac{\sin\frac{t}{n}}{\frac{t}{n}}\right)^2 e^{-it\lambda}dt \geqq 0 \tag{1}$$

$$(n=1, 2, \dots),$$

and if (1) is satisfied, we have the representation[3]:

(2) $\begin{cases} f(t)=\int_{-\infty}^{\infty}e^{it\lambda}dv(\lambda) & \text{with a monotone increasing, right-con-} \\ \text{tinuous bounded function } v(\lambda). \end{cases}$

Theorem 2.[4] $f(t)$ is positive definite if and only if $f(t)$ is expressible as

(3) $\begin{cases} f(t)=\lim_{n\to\infty}\int_{-\infty}^{\infty}g_n(t+s)\overline{g_n(s)}ds & \text{uniformly in every finite inter-} \\ \text{val of } t, \text{ where} \end{cases}$

$$\sup_{n\geq 1}\int_{-\infty}^{\infty}|g_n(t)|^2dt \leqq f(0). \tag{4}$$

Theorem 3. $f(t)$ is representable in the form:

(5) $\begin{cases} f(t)=\int_{-\infty}^{\infty}e^{it\lambda}dv(\lambda) & \text{with a complex-valued right-continuous} \\ \text{function } v(\lambda) \text{ of bounded variation,} \end{cases}$

if and only if

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1) Cf. Proc. **20** (1944), 560–563.

2) $\overline{f(-t)}=f(t)$ and $\sum^{n} f(t_j-t_k)\xi_j\bar{\xi}_k \geqq 0$ for any integer n and for arbitrary complex numbers ξ.

3) S. Bochner: Vorlesungen über Fouriersche Integrale, Leipzig (1932), 76.

4) A. Khintchine: Bullt. de l'universitê d'état à Moscou, Sect. A, vol. **1**, fasc. 5, 1–3.

$$\sup_{n\geq 1}\int_{-\infty}^{\infty}\left\{\left|\int_{-\infty}^{\infty} f(t)\left(\frac{\sin\frac{t}{n}}{\frac{t}{n}}\right)^2 e^{-it\lambda}dt\right|\right\}d\lambda < \infty\,. \tag{6}$$

Theorem 4. $f(t)$ is representable in the form (5) if and only if $f(t)$ is expressible as

(7) $\begin{cases} f(t)=\lim_{n\to\infty}\int_{-\infty}^{\infty} g_n(t+s)\overline{k_n(s)}ds \quad \text{uniformly in every finite iter-} \\ \text{val of } t, \text{ where} \end{cases}$

(8) $\begin{cases} \int_{-\infty}^{\infty}|g_n(t)|^2dt=\int_{-\infty}^{\infty}|k_n(t)|^2dt \leqq a \text{ constant} < \infty \text{ independent of} \\ n. \end{cases}$ [5]

Theorem 5. Let (6) be satisfied, then, if we put,

$$\psi_{n,m}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-m}^{m} f(t)\frac{\sin\frac{t}{n}}{\frac{t}{n}}e^{-it\lambda}dt\,, \tag{9}$$

$\lim_{m\to\infty}\psi_{n,m}(\lambda)=\psi_n(\lambda)$ exists. Concerning the representation (5) we have the results: i) $v(\lambda)$ is absolutely continuous, viz. $v(\lambda)=\int_{-\infty}^{\lambda}v'(\lambda)d\lambda$ if and only if

$$\lim_{n,n'\to\infty}\int_{-\infty}^{\infty}|\psi_n(\lambda)-\psi_{n'}(\lambda)|\,d\lambda=0\,. \tag{10}$$

ii) $v(\lambda)$ is singular, viz. $v'(\lambda)=0$ almost everywhere if and only if

$$\lim_{n\to\infty}\psi_n(\lambda)=0 \qquad \text{almost everywhere.} \tag{11}$$

Theorem 6. i) Concerning the representation (5) we have the result: $v(\lambda)$ is absolutely continuous if and only if

$$\lim_{n,n'\to\infty}\int|\varphi_n(\lambda)-\varphi_{n'}(\lambda)|\,d\lambda=0\,. \tag{12}$$

ii) Concerning the representation (2) we have the result: $v(\lambda)$ is singular if

$$\lim_{n\to\infty}\varphi_n(\lambda)=0 \qquad \text{almost everywhere}\,. \tag{13}$$

2. *Proofs of the the theorems.*

Theorem 1. $f(t)e^{ist}$ is positive definite with $f(t)$ for any real s

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5) The constant may be taken as the left hand side of (6).

and hence $f(t)\dfrac{\sin\frac{t}{n}}{\frac{t}{n}}=\dfrac{n}{2}\displaystyle\int_{-\frac{1}{n}}^{\frac{1}{n}} f(t)e^{ist}ds$ is positive definite. In the same manner we see that

$$f_n(t)=f(t)\left(\frac{\sin\frac{t}{n}}{\frac{t}{n}}\right)^2 \tag{14}$$

is also positive definite and hence

$$|f_n(t)|\leqq f_n(0)=f(0)\,. \tag{15}$$

Thus $f_n(t)\in L_1(-\infty,\infty), \in L_2(-\infty,\infty)$. The continuous function

$$\varphi_n(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_n(t)e^{-it\lambda}dt \tag{16}$$

is non-negative, since for any $\alpha<\beta$

$$\int_\alpha^\beta \varphi_n(\lambda)d\lambda=\int_{-\infty}^{\infty} f_n(t)dt\left\{\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-it\lambda}d\lambda\right\} \tag{17}$$

$$=\int_{-\infty}^{\infty} f_n(t)h(t)dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_n(t)h(t+s)\overline{h(s)}dtds\geqq 0$$

by the positive definite character of $f_n(t)$. That $h(t)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_\alpha^\beta e^{-it\lambda}d\lambda$ $=\displaystyle\int_{-\infty}^{\infty} h(t+s)\overline{h(s)}ds$ follows from the Plancherel's theorem. Next we will prove that the non-negative function $\varphi_n(\lambda)$ is $\in L_1(-\infty,\infty)$. If we put

$$v_n(\lambda)=\frac{1}{\sqrt{2\pi}}\int_0^\lambda \varphi_n(\lambda)d\lambda\,, \tag{18}$$

then we have from (16)

$$\int_{-\delta}^{\delta} e^{is\lambda}dv_n(\lambda)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\delta(s-t)}{s-t}f_n(t)dt$$

and hence

$$\frac{1}{\gamma}\int_0^\gamma d\delta\left\{\int_{-\delta}^{\delta} e^{is\lambda}dv_n(\lambda)\right\}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos\gamma(s-t)}{\gamma(s-t)^2}f_n(t)dt \tag{18$'$}$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty} f_n\left(s+\frac{\tau}{\gamma}\right)\frac{1-\cos\tau}{\tau^2}d\tau\,,$$

in particular (upon putting $s=0$), by (15).

$$\frac{1}{\gamma}\int_0^\gamma\{v_n(\delta)-v_n(-\delta)\}d\delta\leqq \sup_t|f_n(t)|=f(0)\,.$$

Therefore the monotone increasing function $v_n(\lambda)$ satisfies

(19) $$v_n(\infty)-v_n(-\infty)\leqq f(0),$$

and hence $\varphi_n(\lambda)\in L_1(-\infty,\infty)$.

Thus, by (16) and the Plancherel's theorem, we have

(2)′ $$f_n(t)=\int_{-\infty}^{\infty}e^{it\lambda}dv_n(\lambda).$$

By (19) and the Helly's selection theorem, there exists a monotone increasing right-continuous function $v(\lambda)$ with $v(\infty)-v(-\infty)\leqq f(0)$ and a subsequence $\{v_{n_p}(\lambda)\}$ such that

$$\lim_{p\to\infty}v_{n_p}(\lambda)=v(\lambda) \text{ at the continuity points } \lambda \text{ of } v(\lambda).$$

Therefore we have, by taking $\lim_{\gamma\to\infty}\lim_{n\to\infty}$ of (18)′,

$$f(t)=\int_{-\infty}^{\infty}e^{it\lambda}dv(\lambda).$$

Theorem 2. Put

$$g_n(t)=\underset{m\to\infty}{\text{l. i. m.}}\frac{1}{\sqrt{2\pi}}\int_{-m}^{m}\sqrt{u_n(\lambda)}\,e^{it\lambda}d\lambda$$

(l. i. m. = limit in the mean)

and apply the Parseval form of the Plancherel's theorem to (2)′. That $f(t)$ of the form (3) is positive definite may easily be verified.

Theorem 3 and 4 will be clear from the above proofs of theorem 1 and 2.

Theorem 5. We have, from (5) and (9),

$$\psi_{n,m}(\lambda)=\int_{-\infty}^{\infty}dv(\lambda')\left\{\frac{1}{\sqrt{2\pi}}\int_{-m}^{m}e^{it(\lambda'-\lambda)}\frac{\sin\frac{t}{n}}{\frac{t}{n}}dt\right\},$$

and hence, if $v(\lambda')$ is continuous at $\lambda+\frac{1}{n}$, $\lambda-\frac{1}{n}$,

(20) $$\psi_n(\lambda)=\lim_{m\to\infty}\psi_{n,m}(\lambda)=\int_{\lambda-\frac{1}{n}}^{\lambda+\frac{1}{n}}\frac{n}{2}\,dv(\lambda')$$

$$=\frac{n}{2}\left\{v\left(\lambda+\frac{1}{n}\right)-v\left(\lambda-\frac{1}{n}\right)\right\}=v(\lambda,n)\text{ say}.$$

ii) The condition (11) thus becomes

(11)′ $$\lim_{n\to\infty}\frac{n}{2}\left\{v\left(\lambda+\frac{1}{n}\right)-v\left(\lambda-\frac{1}{n}\right)\right\}=0$$ almost everywhere, and hence ii) is proved

i) The condition (10) becomes

(10)′ $$\lim_{n,n'\to\infty}\int_{-\infty}^{\infty}|v(\lambda,n)-v(\lambda,n')|\,d\lambda=0.$$

Since $v(\lambda)$ is of bounded variation, we have

$$\lim_{n\to\infty} v(\lambda, n)=v'(\lambda)=\frac{dv(\lambda)}{d\lambda} \qquad \text{almost everywhere,}$$

and hence (10)′ is equivalent to

$$(10)'' \qquad \lim_{n\to\infty}\int_{-\infty}^{\infty} |v(\lambda, n)-v'(\lambda)|\, d\lambda=0 .$$

Let (10)″ be satisfied, then for any $\alpha<\beta$

$$(21) \qquad \lim_{n\to\infty}\int_{\alpha}^{\beta} v(\lambda, n)d\lambda=\int_{\alpha}^{\beta} v'(\lambda)d\lambda .$$

On the other hand,

$$\int_{\alpha}^{\beta} v(\lambda, n)d\lambda=\frac{n}{2}\int_{\alpha}^{\beta}\left(v\left(\lambda+\frac{1}{n}\right)-v\left(\lambda-\frac{1}{n}\right)\right)d\lambda$$

$$=\frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}}\big(v(\beta+\lambda)-v(\alpha+\lambda)\big)d\lambda .$$

Therefore, if α and β are continuity points of $v(\lambda)$, we have from (21)

$$v(\beta)-v(\alpha)=\int_{\alpha}^{\beta} v'(\lambda)d\lambda .$$

Thus the condition (10) is sufficient.

The necessity of (10)″ may be proved as follows From $v(\lambda)=\int_{-\infty}^{\lambda} v'(\lambda)d\lambda$ we have

$$v(\lambda, n)=\frac{n}{2}\int_{\lambda-\frac{1}{n}}^{\lambda+\frac{1}{n}} v'(\lambda')d\lambda' .$$

Since $v'(\lambda)\in L_1(-\infty, \infty)$, we must have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\left|\frac{n}{2}\int_{\lambda-\frac{1}{n}}^{\lambda+\frac{1}{n}} v'(\lambda')d\lambda'-v'(\lambda)\right| d\lambda=0 ,$$

which is (10)″.

Theorem 6. We have, from (1) and (5),

$$(22) \qquad \frac{1}{\sqrt{2\pi}}\varphi_n(\lambda)=\int_{-\infty}^{\infty} dv(\lambda')\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\lambda'-\lambda)}\left(\frac{\sin\frac{t}{n}}{\frac{t}{n}}\right)^2 dt\right.$$

$$=\int_{|\lambda'-\lambda|\le\frac{2}{n}} dv(\lambda')\left\{\frac{n}{2}\left(1-\frac{n}{2}\left|\lambda'-\lambda\right|\right)\right\}$$

$$=\frac{1}{2}\int_{-1}^{1}\left(v\left(\lambda+\frac{2}{n}\sigma+\frac{2}{n}\right)-v\left(\lambda+\frac{2}{n}\sigma-\frac{2}{n}\right)\right)\Big/\frac{4}{n}\,d\sigma .$$

ii) Since $v(\lambda)$ is monoton increasing we have, from (22) and Fatou's theorem

$$\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\varphi_n(\lambda)\geqq\frac{1}{2}\int_{-1}^{1}v'(\lambda)d\sigma=v'(\lambda)\,.$$

This proves ii).

i) Let $v(\lambda)=\int_{-\infty}^{\lambda}v'(\lambda)d\lambda$, then we have, from (22),

$$\frac{1}{\sqrt{2\pi}}\varphi_n(\lambda)=\frac{1}{2}\int_{-1}^{1}d\sigma\cdot\frac{n}{4}\int_{\lambda+\frac{2}{n}\sigma-\frac{2}{n}}^{\lambda+\frac{2}{n}\sigma-\frac{2}{n}}v'(\lambda')d\lambda'\,.$$

Hence, by $v'(\lambda')\in L_1(-\infty,\infty)$, we must have (12).

Next let (12) be satisfied, then the indefinite integrals

$$\frac{1}{\sqrt{2\pi}}\int_M\varphi_n(\lambda)d\lambda\qquad(n=1,2\ldots)$$

converges at every measurable set M on $(-\infty,\infty)$. Hence, by Vitali-Hahn-Saks' theorem, the limit $v(\lambda)=\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_0^{\lambda}\varphi_n(\lambda)d\lambda$ must be absolutely continuous.