132. *On Simultaneous Extension of Continuous Functions.*

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1. Let Ω be a compact (=bicompact) Hausdorff space, and F a closed subset of Ω. Let $C(\Omega)$ be the normed ring of all complex-valued continuous functions on Ω, and let $C(F)$ be analogously defined. For any $x \in C(\Omega)$ and $x' \in C(F)$, their norms are defined by $\|x\|_\Omega = \max_{\omega \in \Omega} |x(\omega)|$ and $\|x'\|_F = \max_{\omega' \in F} |x'(\omega')|$, respectively.

Then, by Urysohn's extension theorem, to any $x' \in C(F)$ there corresponds an $x \in C(\Omega)$ such that $x'(\omega') = x(\omega')$ for any $\omega' \in F$. (When F is an essential subset of Ω, $x(\omega)$ is, of course, not unique). Thus a mapping $x = \varphi(x')$ of $C(F)$ into $C(\Omega)$ is defined. The purpose of this paper is to prove that we can take as φ a *linear* $\big(\varphi(x'+y') = \varphi(x') + \varphi(y')$ and $\varphi(\alpha x') = \alpha\varphi(x')$, where α is a complex number$\big)$, *multiplicative* $\big(\varphi(x'y') = \varphi(x')\varphi(y')\big)$ and *isometric* $\big(\|\varphi(x') - \varphi(y')\|_\Omega = \|x' - y'\|_F\big)$ mapping, if and only if F is a *retract* of Ω in the sense of K. Borsuk[1], i.e. if there exists a continuous mapping $\omega' = f(\omega)$ of Ω onto F such that $f(\omega') = \omega'$ on F.

2. *Lemma.*[2] *Let R be a closed subring*[3] *of $C(\Omega)$ containing the unit element of $C(\Omega)$ and satisfying the following condition:*

$$(*) \qquad x(\omega) \in R \textit{ implies } \overline{x(\omega)} \in R.^{4)}$$

Then R is equivalent[5] *to $C(\Omega^*)$, where Ω^* is a certain continuous image of Ω. Conversely, if Ω^* is a continuous image of Ω, then $C(\Omega^*)$ is equivalent to some closed subring of $C(\Omega)$ which contains the unit of $C(\Omega)$ and satisfies the condition* $(*)$.

We shall sketch the proof: To any maximal ideal[6] of R there corresponds at least one point of Ω and to any point of Ω there corresponds one maximal ideal of R. From this follows easily that the set Ω^* of all maximal ideals of R, which is topologized by the weak topology, is a continuous image of Ω : $\Omega^* = g(\Omega)$. Then R is equivalent to $C(\Omega^*)$ by the correspondence $x(\omega) \to x^*(\omega^*)$, where $x^*\big(g(\omega)\big) = x(\omega)$.

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1) K. Borsuk, Sur les rétractes, Fund. Math., **17** (1931), 152–170.

2) G. Šilov, Ideals and subrings of the ring of continuous functions, C. R. URSS, **22** (1939), 7–10.

3) If not mentioned explicitly, we do not assume that a subring of R contains the unit element of R.

4) $\bar{z}$ denotes the conjugate complex number of z.

5) Two normed rings are *equivalent* if there exists an isometric isomorphism between them.

6) Concerning these notions, see I. Gelfand and A. Kolmogoroff, On rings of continuous functions on topological spaces, C. R. URSS, **22** (1939), 11–15, and I. Gelfand and G. Šilov, Über verschiedene Methoden der Einführung der Topologie in die Menge der maximalen Ideale eines normierten Ringes, Recueil Math., **9** (1941), 25–39.

Theorem. *Let Ω be a compact Hausdorff space, and F a closed subset of Ω. In order that, for any element $x' \in C(F)$, there exists an extension $x = \varphi(x') \in C(\Omega)$ of x' such that the mapping φ is linear, multiplicative and isometric, it is necessary and sufficient that F is a retract of Ω in the sense of K. Borsuk.*

Proof. Since the sufficiency of the condition is almost evident, we shall only prove that it is necessary. We shall divide our proof into three steps:

(1) Let R be the set of all $x \in C(\Omega)$ which correspond to $x' \in C(F)$ in the required manner: $x' \to x$. Then R is a closed subring of $C(\Omega)$ and is equivalent to $C(F)$. Let e' be the unit element of $C(F)$ and $e' \to e \in R$. Then, since $e'^2 = e'$, we must have $e(\omega) = 0$ or 1 on Ω. Therefore, if we put $\Omega_0 = \{\omega \mid e(\omega) = 0, \omega \in \Omega\}$ and $\Omega_1 = \{\omega \mid e(\omega) = 1, \omega \in \Omega\}$, then $F \subseteq \Omega_1$, $\Omega_0 \frown \Omega_1 = \theta$, $\Omega_0 \smile \Omega_1 = \Omega$, and both Ω_0 and Ω_1 are compact. Hence it suffices to prove that F is a retract of Ω_1. By this reason, in the following we shall designate Ω_1 by Ω, and consider R as a subring of C(Ω) containing the unit element of $C(\Omega)$.

(2) Now we are going to prove that R satisfies the condition (*) of the Lemma. Let $x'(\omega')$ be an arbitrary element of $C(F)$ which is real-valued on F. Then the corresponding $x \in R$ is also real-valued on Ω. In fact, if at a certain point $p \in \Omega$, $x(p) = \lambda + i\mu$, $\mu \neq 0$, and if we put $y = (x - \lambda e)^2 + (\mu e)^2$, then $y(p) = 0$. On the other hand, y corresponds to $y' = (x' - \lambda e')^2 + (\mu e')^2$; and, since y' has the inverse element $y'^{-1} \in C(F)$: $y'y'^{-1} = e'$ (because $y'(\omega') \geqq \mu^2 > 0$ on F), so there must exist $y^{-1} \in R$: $y(\omega)y^{-1}(\omega) = 1$, which is a contradiction.

By the preceding argument, we can easily prove that if $\overline{y'(\omega')} = x'(\omega')$ on F, then $y(\omega) = \overline{x(\omega)}$ on Ω. For, $x'(\omega') + y'(\omega')$ and $\big(x'(\omega') - y'(\omega')\big)/i$ are both real-valued on F, hence $x(\omega) + y(\omega)$ and $\big(x(\omega) - y(\omega)\big)/i$ are also both real-valued on Ω, showing that $y(\omega) = \overline{x(\omega)}$ on Ω.

(3) By the considerations of (2) and Lemma, R is equivalent to $C(\Omega^*)$, where Ω^* is the set of all maximal ideals of R and is a continuous image of $\Omega : \Omega^* = g(\Omega)$. Since, as is easily seen, the correspondence between the points of F and the maximal ideals of R is one-to-one, F is homeomorphically embedded in $\Omega^* : F \approx g(F) \equiv F^* \subseteq \Omega^*$.

We shall now show that $F^* = \Omega^*$. Let $x^* \neq y^*$, $x^*, y^* \in C(\Omega^*)$, and let x and y be the elements of R which correspond to x^* and y^* respectively by the mapping indicated in the proof of Lemma. Then there exists a $p \in F$ such $x(p) \neq y(p)$. But, then it follows that $x^*(p^*) \neq y^*(p^*)$, where $p^* = g(p) \in F^*$. Hence F^* cannot be an essential subset of Ω^*. This completes the proof.

Remark. From Lemma and the proof of Theorem, it is easy to see that the following proposition is true:[1]

Corollary. *Let Ω_1 and Ω_2 be compact Hausdorff spaces. In order that $C(\Omega_1)$ contains a subring which is equivalent to $C(\Omega_2)$, it is necessary and sufficient that Ω_2 is a continuous image of Ω_1.*

1) For the proof of this Corollary we need the following well-known result: *If $C(\Omega_1)$ and $C(\Omega_2)$ are algebraically isomorphic, then Ω_1 and Ω_2 are homeomorphic.* Concerning this theorem, see, for example, the papers cited in 6).