# 131. On Brownian Motions in n-Space.

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**1.** The *one-dimensional Brownian motion* $\{x(t,\omega)\mid -\infty<t<\infty, \omega\in\Omega\}$ is defined as a real-valued temporally homogeneous differential process with no moving discontinuities and having a Gaussian distribution :[1]

$$(1)\qquad Pr\{\omega\mid a<x(t,\omega)-x(s,\omega)<b\}=\frac{1}{\sqrt{2\pi(t-s)}}\int_a^b e^{-\frac{u^2}{2(t-s)}}du,$$

where $-\infty<s<t<\infty$ and $-\infty\leqq a<b\leqq\infty$. The *n-dimensional Brownian motion* (or equivalently, the *Brownian motion in n-space* $R^n$) $\{\boldsymbol{x}(t,\omega)=\{x_i(t,\omega),\ i=1,\dots,n\}\mid -\infty<t<\infty,\ \omega\in\Omega\}$ is an $n$-system of mutually independent one-dimensional Brownian motions $x_i(t,\omega)$ (with the same normalization). It is known that this definition is independent of the choice of coordinate system in $R^n$.

The mathematical theory of Brownian motions was discussed by N. Wiener[2] and P. Lévy[3], and many important results were obtained. But, for the most part, their investigations were restricted to the one- or the two-dimensional case or concerned only with the properties of Brownian motions in which the dimension number $n$ does not play an important rôle. The purpose of this paper is to discuss some new properties of Brownian motions which do not appear in the one- or the two-dimensional case. Our main results are stated in Theorems 1, 2 and 4.

It is to be noticed that in the Brownian motion in 3-space almost all paths constitute a nowhere dense set in $R^3$ (Theorem 4) and tend to $\infty$ as $t\to\infty$ (Theorem 2), while, as may be shown[4] by appealing to the theory of harmonic functions, in the two-dimensional Brownian motion almost all paths describe a curve everywhere dense in the entire plane and come back to any neighborhood of any given point infinitely many times (for infinitely large value of $t$). It is known that in the two-dimensional Brownian motion almost all paths have infinitely many double points. We have a conjecture that already in 3-space almost all paths have no double points, although thus far we could prove this only for 5-space (Theorem 1).

**2.** *Lemma 1*[5]. *If $x(t,\omega)$ is a one-dimensional Brownian motion and if $-\infty<t_0<t_1<\infty$, then*

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1) $Pr\{\omega|A\}$ denotes the probability (=measure) of the set of all $\omega\in\Omega$ with the property $A$, i.e. the probability of $A$.

2) N. Wiener, Generalized harmonic analysis, Acta Math., **54** (1932).

3) P. Lévy, Les mouvements browniens plans, Amer. Journ. of Math., **62** (1940).

4) This will be discussed in a forthcoming paper of the author.

5) Cf. P. Lévy, loc. cit. 3).

$$(2) \qquad Pr\{\omega \mid \max_{t_0 \leq t \leq t_1} \big(x(t,\omega)-x(t_0,\omega)\big) > \eta\}$$

$$= 2Pr\{\omega \mid x(t_1,\omega)-x(t_0,\omega) > \eta\} \equiv \frac{2}{\sqrt{2\pi(t_1-t_0)}} \int_\eta^\infty e^{-\frac{u^2}{2(t_1-t_0)}} du, \ \eta > 0,$$

and hence

$$(3) \qquad Pr\{\omega \mid \max_{t_0 \leq t \leq t_1} |x(t,\omega)-x(t_0,\omega)| > \eta\}$$

$$\leq \frac{4}{\sqrt{2\pi(t_1-t_0)}} \int_\eta^\infty e^{-\frac{u^2}{2(t_1-t_0)}} du, \ \eta > 0.$$

*Lemma 2.*

$$(4) \qquad \int_\eta^\infty e^{-\frac{u^2}{2}} du \leq \frac{1}{\eta} e^{-\frac{\eta^2}{2}}, \qquad \eta > 0.$$

**3.** *Theorem 1. In the Brownian motion in $R^n (n \geq 5)$, almost all paths have no double points.*

*Proof.* It suffices to discuss the case $n=5$. Further it is sufficient to prove that

$$(5) \qquad \alpha \equiv Pr\{\omega \mid x_i(s,\omega)=x_i(t,\omega),\ i=1,\dots,5, \text{ for some } s \in I \text{ and } t \in J\} = 0$$

for any two disjoint closed intervals $I=(s_0, s_1)$ and $J=(t_0, t_1)$ with $s_1 < t_0$. It is easy to see that

$$(6) \qquad \alpha \leq Pr\{\omega \mid |x_i(s_1,\omega)-x_i(t_0,\omega)| < 2\eta,\ i=1,\dots,5\}$$
$$+ \sum_{i=1}^5 Pr\{\omega \mid \max_{s \in I} |x_i(s,\omega)-x_i(s_1,\omega)| > \eta\}$$
$$+ \sum_{i=1}^5 Pr\{\omega \mid \max_{t \in J} |x_i(t,\omega)-x_i(t_0,\omega)| > \eta\};$$

hence, by (1) and (3),

$$(7) \qquad \alpha \leq \left(\frac{1}{\sqrt{2\pi(t_0-s_1)}} \int_{-2\eta}^{2\eta} e^{-\frac{u^2}{2(t_0-s_1)}} du\right)^5$$
$$+ 5\frac{4}{\sqrt{2\pi(s_1-s_0)}} \int_\eta^\infty e^{-\frac{u^2}{2(s_1-s_0)}} du + 5\frac{4}{\sqrt{2\pi(t_1-t_0)}} \int_\eta^\infty e^{-\frac{u^2}{2(t_1-t_0)}} du$$
$$\leq \left(\frac{4\eta}{\sqrt{2\pi d(I,J)}}\right)^5 + \frac{20}{\sqrt{2\pi}} \int_{\frac{\eta}{\sqrt{|I|}}}^\infty e^{-\frac{u^2}{2}} du + \frac{20}{\sqrt{2\pi}} \int_{\frac{\eta}{\sqrt{|J|}}}^\infty e^{-\frac{u^2}{2}} du,$$

where $d(I,J)=t_0-s_1=$the distance of $I$ and $J$, $|I|=s_1-s_0=$the length of $I$, $|J|=t_1-t_0=$the length of $J$, and $\eta > 0$ is an arbitrary positive number.

Let us now divide $I$ and $J$ into $p$ closed subintervals $I_k$ and $J_l$ of the lengths $|I|/p$ and $|J|/p$, respectively: $I=\bigcup_{k=1}^p I_k$, $J=\bigcup_{l=1}^p J_l$. Then

$$(8) \qquad \alpha \leq \sum_{k=1}^p \sum_{l=1}^p Pr\{\omega \mid x_i(s,\omega)=x_i(t,\omega),\ i=1,\dots,5, \text{ for some } s \in I_k \text{ and } t \in J_l\}$$

$$\leqq \sum_{k=1}^{p}\sum_{l=1}^{p}\left\{\left(\frac{4\eta_p}{\sqrt{2\pi d(I_k, J_l)}}\right)^5+\frac{20}{\sqrt{2\pi}}\int_{\frac{\eta_p}{\sqrt{|I_k|}}}^{\infty} e^{-\frac{u^2}{2}}du+\frac{20}{\sqrt{2\pi}}\int_{\frac{\eta_p}{\sqrt{|J_l|}}}^{\infty} e^{-\frac{u^2}{2}}du\right\}$$

$$\leqq p^2\left\{\left(\frac{4\eta_p}{\sqrt{2\pi d(I, J)}}\right)^5+\frac{20}{\sqrt{2\pi}}\int_{\frac{\eta_p\sqrt{p}}{\sqrt{|I|}}}^{\infty} e^{-\frac{u^2}{2}}du+\frac{20}{\sqrt{2\pi}}\int_{\frac{\eta_p\sqrt{p}}{\sqrt{|J|}}}^{\infty} e^{-\frac{u^2}{2}}du\right\},$$

where $\eta_p$ is an arbitrary positive number.

Let us now put $\eta_p=p^{-\sigma}$, where $\sigma$ is a fixed real number satisfying $2/5<\sigma<1/2$. Then, by using the inequality (4), it is easy to see that the right hand side of (8) tends to 0 as $p\to\infty$. This completes the proof of Theorem 1.

**4.** *Theorem 2. In the Brownian motion in* $R^n (n\geqq 3)$ *almost all paths tend to* $\infty$ *as* $t\to\infty$, *i.e.*

$$\lim_{t\to\infty}\sum_{i=1}^{3}|x_i(t, \omega)-x_i(0, \omega)|^2=\infty \tag{9}$$

*for almost all* $\omega$.

*Proof.* It suffices to discuss the case $n=3$. Let $M$ be an arbitrary positive number. Let us put $t_k=k^{\frac{3}{4}}$, $k=1, 2, \dots$. Then $t_k<t_{k+1}$, $k=1, 2, \dots$, $t_k\to\infty$ and $t_{k+1}-t_k\to 0$. It is easy to see that

(10) $\beta_k\equiv Pr\{\omega \,|\, |x_i(t, \omega)-x_i(0, \omega)|<M,\ i=1, 2, 3$ for some $t$ with $t_k\leqq t\leqq t_{k+1}\}$

$$\leqq Pr\{\omega\,|\,|x_i(t_k, \omega)-x_i(0, \omega)|<2M,\ i=1, 2, 3\}$$
$$+\sum_{i=1}^{3} Pr\{\omega\,|\max_{t_k\leqq t\leqq t_{k+1}}|x_i(t, \omega)-x_i(t_k, \omega)|>M\}$$

$$\leqq\left(\frac{1}{\sqrt{2\pi t_k}}\int_{-2M}^{2M} e^{-\frac{u^2}{2t_k}}du\right)^3+3\frac{4}{\sqrt{2\pi(t_{k+1}-t_k)}}\int_M^{\infty} e^{-\frac{u^2}{2(t_{k+1}-t_k)}}du$$

$$\leqq\left(\frac{4M}{\sqrt{2\pi t_k}}\right)^3+\frac{12}{\sqrt{2\pi}}\int_{\frac{M}{\sqrt{t_{k+1}-t_k}}}^{\infty} e^{-\frac{u^2}{2}}du$$

and further that $\sum_{k=1}^{\infty}\beta_k<\infty$. The last fact follows from (10) by using the inequality (4) and the relation $t_{k+1}-t_k=O(k^{-\frac{1}{4}})$. From this follows, by Borel-Cantelli's theorem, that for almost all $\omega$ there exists an integer $k_0=k_0(\omega)$ such that $\max_{i=1,2,3}|x_i(t, \omega)-x_i(0, \omega)|>M$ for $t>t_{k_0}=k_0^{\frac{3}{4}}$. Since $M$ is arbitrary, this proves our theorem.

**5.** *Theorem 3. Let* $S=S(\boldsymbol{x}^0, r)$ *be a sphere in* $R^3$, *with a center* $\boldsymbol{x}^0=(x_1^0, x_2^0, x_3^0)$ *and a radius* $r$. *Denote by* $\rho=(\sum_{i=1}^{3}|x_i^0|^2)^{\frac{1}{2}}$ *the distance of* $\boldsymbol{x}^0$ *from the origin of* $R^3$. *Then the probability*

(11) $\gamma\equiv Pr\{\omega\,|\,\boldsymbol{x}(t, \omega)-\boldsymbol{x}(0, \omega)\in S(\boldsymbol{x}^0, r)$ for some $t>0\}$

$$\equiv Pr\{\omega\,|\sum_{i=1}^{3}|x_i(t, \omega)-x_i(0, \omega)-x_i^0|^2<r^2 \text{ for some } t>0\},$$

*which is clearly*[1] *a function of* $\rho/r$, *tends to* 0 *as* $\rho/r\to\infty$.

*Remark.* By appealing to the theory of potential functions in $R^3$, it may be shown[2] that $\gamma=1$ if $\rho\leqq r$ and $\gamma=r/\rho$ if $\rho>r$.

*Proof of Theorem 3.* Without the loss of generality we may assume that $x_1^0=\rho$, $x_2^0=x_3^0=0$ and $r=1$. Put $t_k=k^{\frac{3}{4}}$, $k=1,2,\dots$ as in the proof of Theorem 2. Then in the same way as in above

$$
\begin{aligned}
(12)\quad \gamma_k &\equiv Pr\{\omega\,|\,\boldsymbol{x}(t,\omega)-\boldsymbol{x}(0,\omega)\in S(\boldsymbol{x}^0,1) \text{ for some } t \text{ with } t_k\leqq t\leqq t_{k+1}\}\\
&\leqq Pr\{\omega\,|\,|x_1(t,\omega)-x_1(0,\omega)-\rho|<1,\ |x_2(t,\omega)-x_2(0,\omega)|<1,\\
&\qquad |x_3(t,\omega)-x_3(0,\omega)|<1 \quad\text{for some } t \text{ with } t_k\leqq t\leqq t_{k+1}\}\\
&\leqq Pr\{\omega\,|\,|x_1(t_k,\omega)-x_1(0,\omega)-\rho|<2,\ |x_2(t_k,\omega)-x_2(0,\omega)|<2,\\
&\qquad |x_3(t_k,\omega)-x_3(0,\omega)|<2\}\\
&\quad+\textstyle\sum_{i=1}^3 Pr\{\omega\,|\max\limits_{t_k\leqq t\leqq t_{k+1}}|x_i(t,\omega)-x_i(t_k,\omega)|>1\}\\
&\leqq\left(\frac{1}{\sqrt{2\pi t_k}}\int_{\rho-2}^{\rho+2}e^{-\frac{u^2}{2t_k}}du\right)\left(\frac{1}{\sqrt{2\pi t_k}}\int_{-2}^{2}e^{-\frac{u^2}{2t_k}}du\right)^2\\
&\quad+3\frac{4}{\sqrt{2\pi(t_{k+1}-t_k)}}\int_1^\infty e^{-\frac{u^2}{2(t_{k+1}-t_k)}}du\\
&\leqq\left(\frac{4}{\sqrt{2\pi t_k}}\right)^3+\frac{12}{\sqrt{2\pi}}\int_{\frac{1}{\sqrt{t_{k+1}-t_k}}}^\infty e^{-\frac{u^2}{2}}du,
\end{aligned}
$$

and consequently

$$
\begin{aligned}
(13)\quad \gamma &\leqq Pr\Big\{\omega\,\Big|\max_{0\leqq t\leqq t_{k_0}}\big(x_1(t,\omega)-x_1(0,\omega)\big)>\rho-1\Big\}+\textstyle\sum_{k=k_0}^\infty\gamma_k\\
&\leqq\frac{2}{\sqrt{2\pi}}\int_{\frac{\rho-1}{\sqrt{t_{k_0}}}}^\infty e^{-\frac{u^2}{2}}du+\sum_{k=k_0}^\infty\left\{\left(\frac{4}{\sqrt{2\pi t_k}}\right)^3+\frac{12}{\sqrt{2\pi}}\int_{\frac{1}{\sqrt{t_{k+1}-t_k}}}^\infty e^{-\frac{u^2}{2}}du\right\}
\end{aligned}
$$

for any positive integer $k_0$.

Let now $\rho>n^4$, and take $k_0=n^8$. Then it is easy to see that the right hand side of (13) tends to 0 as $n\to\infty$. This shows that $\gamma\to 0$ as $\rho\to\infty$, completing the proof of Theorem 3.

**6.** *Theorem 4. In the Brownian motion in* $R^n (n\geqq 3)$, *almost all paths constitute a nowhere dense set in* $R^n$.

*Proof.* It suffices to show this for $n=3$. Let $\boldsymbol{x}^0=(x_1^0, x_2^0, x_3^0)$ be an arbitrary point in $R^3$ different from the origin of $R^3$. It suffices to show that for almost all $\omega$ there exists an $r=r(\omega)$ such that

1) This follows from the homogeneity property of the Brownian motion.

2) This will be discussed in a forthcoming paper. Here we shall give a direct proof.

(14) $$\boldsymbol{x}(t, \omega)-\boldsymbol{x}(0, \omega) \in S\big(\boldsymbol{x}^0, r(\omega)\big)$$

for no $t > 0$, or equivalently

(15) $$\sum_{i=1}^{3} | x_i(t, \omega)-x_i(0, \omega)-x_i^0 |^2 > \big(r(\omega)\big)^2$$

for any $t > 0$.

In order to show this, let $r_n > 0$ be a sufficiently small number such that

(16) $$Pr\{\omega \mid \boldsymbol{x}(t, \omega)-\boldsymbol{x}(0, \omega) \in S(\boldsymbol{x}^0, r_n) \text{ for some } t > 0\} < 2^{-n}.$$

The existence of such an $r_n$ follows from Theorem 3. Our proposition then follows from this immediately by using Borel-Cantelli's theorem.