141. *On Student's Test.*

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1. *Introduction*[1].

Let (Ω, P_θ) be *the population of a sample* with the parameter $\theta \in \Theta$. Let H be any subset of Θ. By the *hypothesis* H, $H \leq \Theta$, we understand the hypothesis "$\theta \in H$". If H consists of only one point, it is called a simple hypothesis, and if H contains at least two points, it is called a composite hypothesis. When $\theta_1 \in \Theta - H$, the simple hypothesis θ_1 is called an *alternative hypothesis of* H.

Let K be any subset of Ω. The *test of the hypothesis* H *by the critical region* K is defined as the following rule of rejection:

(1) if the *realized sample point* ω belongs to K, H is rejected, and
(2) if $\omega \in K$, H is non-rejected.

With regard to this test we may consider two types of error. The *error of the first type* e_{I} is that which is made by rejecting H when it is true, while the *error of the second type* e_{II} is that which is made by accepting H when it is false.

e_{I} is measured by the probability $P_\theta(K)$, $\theta \in H$, say $e_{\text{I}}(K, \theta)$.

Definition 1. K is called to be *regular*, if $e_{\text{I}}(K, \theta)$, $\theta \in H$, is independent of θ, as far as θ runs over H. The common value is called the *size* of K and is denoted by $e_{\text{I}}(K)$.

e_{II} is measured by the probability $P_{\theta_1}(\Omega - K)$, $\theta_1 \in \Theta - H$, say $e_{\text{II}}(K, \theta_1)$.

Definition 2. K_0 is called to be the *most powerful against* an assigned alternative θ_1, if K_0 is regular and if, whenever K is any regular critical region with the same size as K_0, we have $e_{\text{II}}(K_0, \theta_1) \leq e_{\text{II}}(K, \theta_1)$.

The main purpose of this note is to prove that Student's method of testing the hypothesis concerning the mean of normal populations is the best one in a certain sense (Theorems 4 and 6). For the proof we make use of Theorems 3 and 5 concerning the regularity of critical regions, which are the immediate results from Theorem 1 and 2 due to Mr. K. Yosida, to whom the author owes much in this research and wishes to express his hearty thanks.

2. *Theorems of Mr. K. Yosida.*

For the later use we shall prove two theorems due to Mr. K. Yosida.

Theorem 1. Let $f(x)$, $-\infty < x < \infty$, be any real-valued bounded measurable function, and $g(x)$ be any function $\in L_1(-\infty, \infty)$ whose Fourier transform does not vanish on the real axis. Then the condition that the integral

1) Cf. Wilks: The Theory of Statistical Inference, Princeton (1937), chap. V.

$$\int_{-\infty}^{\infty} g(x-a)f(dx)$$

is independent of a implies $f(x)=\text{const.}$ almost everywhere.

Proof. Since the Fourier transform of $g(x)$ does not vanish, we make use of Wiener's theorem[1] to find, for any $h(x) \in L_1(-\infty, \infty)$ and for $\varepsilon > 0$, a linear form $\sum_{i=1}^{n} a_i g(x-a_i)$ such that

$$\int_{-\infty}^{\infty} | h(x) - \sum a_i g(x-a_i) | \, dx \leqq \varepsilon .$$

Putting $M = \sup | f(x) |$, we have

$$\left| \int_{-\infty}^{\infty} h(x-a) f(x) dx - \int_{-\infty}^{\infty} \left(\sum a_i g(x-a_i-a) \right) f(x) dx \right|$$

$$\leqq M \int_{-\infty}^{\infty} | h(x-a) - \sum a_i g(x-a_i-a) | \, dx$$

$$= M \int_{-\infty}^{\infty} | h(x) - \sum a_i g(x-a_i) | \, dx \leqq M\varepsilon ,$$

and

$$\left| \int_{-\infty}^{\infty} h(x) f(x) dx - \int_{-\infty}^{\infty} \left(\sum a_i g(x-a_i) \right) f(x) dx \right| \leqq M\varepsilon .$$

By the assumption we have

$$\int_{-\infty}^{\infty} \left(\sum a_i g(x-a_i-a) \right) f(x) dx = \sum a_i \int_{-\infty}^{\infty} g(x) f(x) dx$$

$$= \int_{-\infty}^{\infty} \left(\sum a_i g(x-a_i) \right) f(x) dx .$$

Therefore

$$\left| \int_{-\infty}^{\infty} h(x-a) f(x) dx - \int_{-\infty}^{\infty} h(x) f(x) dx \right| \leqq 2M\varepsilon$$

ε being arbitrary, we obtain

$$\int_{-\infty}^{\infty} h(x-a) f(x) dx = \int_{-\infty}^{\infty} h(x) f(x) dx .$$

Put $L(E) = \int_E \left(f(x) + M \right) dx$. Then $L(E) = \int_{-\infty}^{\infty} C_E(x) \left(f(x) + M \right) dx$, where $C_E(x)$ is the characteristic function of a set E of finite measure. Since $C_E(x) \in L_1(-\infty, \infty)$, we see that

$$\int_{-\infty}^{\infty} C_E(x-a) \left(f(x) - M \right) dx = \int_{-\infty}^{\infty} C_E(x) \left(f(x) + M \right) dx$$

viz. that $L(E)$ is a translation-invariant set function. $L(E)$ is clearly non-negative and completely additive. Furthermore we have $L(E) \leqq 2M|E| < \infty$. Thus we have $L(E) = c|E|$, which implies $f(x) + M = c$ almost everywhere.

1) N. Wiener: Tauberian Theorems, Ann. of Math., **33** (1932), Th. II.

Theorem 2. Let $f(x)$, $0<x<\infty$, be any real-valued bounded measurable function, and $g(x)$ be any function $\in L_1(0,\infty)$ such that $\int_0^\infty g(x)x^{it}dx \neq 0$ for any real t. Then the condition that the integral $\int_0^\infty \frac{1}{a}g\left(\frac{x}{a}\right)f(x)dx$ is independent of a, implies $f(x)=\text{const.}$ almost everywhere.

Proof. Put $y=\log x$, $\beta=\log a$, $G(y)=g(e^y)$, $F(y)=f(e^y)$, and $H(y)=G(y)e^y$. Then the integral:

$$\int_{-\infty}^\infty H(y-\beta)F(y)dy$$

in independent of β by the assumption. It is clear that $H(y)\in L_1(-\infty,\infty)$. Moreover we have

$$\int_{-\infty}^\infty H(y)e^{iyt}dy=\int_{-\infty}^\infty G(y)e^{iyt}e^y dy=\int_0^\infty g(x)x^{it}dx \neq 0\,.$$

Thus we can make use of Theorem 1 to show that $f(x)=\text{const.}$ almost everywhere.

3. *Student's test (1).*

Let $(R, G_{a\beta})$ be a one-dimensional normal population with the mean a and the standard deviation β. Let $\omega=(\omega_1,\omega_2,\dots,\omega_n)$ be a random sample of size n from this population. Then the population of this sample is the product measure space $(R,G_{a\beta})^n$, say $(\Omega, P_{a\beta})$. In this case the the pair (a,β) corresponds to θ in § 1. Now we consider the hypothesis $a=a_0$, which is clearly a composite hypothesis, since β is not specified. *Student's test* is that whose critical region is determined by

$$K_+=\left\{\omega\,;\,\frac{\bar{\omega}-a_0}{s}>c\right\} \quad \text{or} \quad K_-=\left\{\omega\,;\,\frac{\bar{\omega}-a_0}{s}<-c\right\}$$

where $\bar{\omega}=\frac{1}{n}\sum\omega_i$, $s^2=\frac{1}{n}\sum(\omega_i-\bar{\omega})^2$, and c is a positive constant. This is a regular critical region. Any regular region can be obtained by the following

Theorem 3. In order that K is regular it is necessary and sufficient that $|K\cdot S(r)|=c\,|S(r)|$, where $S(r)=\{\omega\,;\,\sum(\omega_i-a_0)^2=r^2\}$ and $|\ \ |$ means the spherical area of $K\cdot S(r)$. The constant c is equal to $e_{\text{I}}(K)$.

Proof. Since the sufficiency is obvious, we shall prove only the necessity. The regularity of K implies that $e_{\text{I}}(K,a_0,\beta)$ is independent of β, viz.

$$(1)\qquad \int\cdots\int_K\left(\frac{1}{\sqrt{2\pi}\,\beta}\right)^n\exp\left\{-\frac{\sum(\omega_i-a_0)^2}{2\beta^2}\right\}d\omega_1\dots d\omega_n=e_{\text{I}}(K)\,.$$

We put

$$(2)\qquad r=\left(\sum(\omega_i-a_0)^2\right)^{\frac{1}{2}}.$$

Then the above integral will be written as

$$(3) \qquad e_{\mathrm{I}}(K)=\Big(\frac{1}{\sqrt{2\pi}}\Big)^n\int_0^\infty\frac{1}{\beta}\Big(\frac{r}{\beta}\Big)^{n-1}e^{-\frac{1}{2}\left(\frac{r}{\beta}\right)^2}\frac{1}{r^{n-1}}\,|KS(r)|\,dr\,.$$

Since we have

$$\int_0^\infty r^{n-1}e^{-\frac{r^2}{2}}r^{it}dr=2^{\frac{n-2}{2}+i\frac{t}{2}}\Gamma\Big(\frac{n}{2}+i\frac{t}{2}\Big)\neq 0\,,$$

We obtain by Theorem 2,

$$(4) \qquad \frac{1}{r^{n-1}}\,|K\cdot S(r)|=\text{const.}$$

almost everywhere, from which follows $|K\cdot S(r)|=e_{\mathrm{I}}(K)\,|S(r)|$ on account of (3).

Theorem 4. K_+ is the most powerful against any alternative hypothesis (α,β) such that $\alpha>\alpha_0$ and K_- is the most powerful against any alternative hypothesis (α,β) such that $\alpha<\alpha_0$.

Proof. We prove only the first part, since the second part can be proved in the same way. Let K be any regular critical region of the same size as K_+. Then we have by Theorem 3

$$(5) \qquad |K\cdot S(r)|=|K_+\cdot S(r)|\,.$$

Let $\bar{x}=\frac{1}{n}\sum x_i$. Since $r^2=\sum(x_i-\bar{x})^2+n(\bar{x}-\alpha_0)^2$, K_+ is determined by the condition of the form:

$$(6) \qquad \frac{\bar{x}-\alpha_0}{r}>c_1\,.$$

We have, by a simple calculation,

$$1-e_{\mathrm{II}}(K,\alpha,\beta)=P_{\alpha\beta}(K)=\Big(\frac{1}{\sqrt{2\pi}\,\beta}\Big)^n\int_0^\infty\exp\Big\{-\frac{r^2+n(\alpha-\alpha_0)^2}{2\beta^2}\Big\}$$

$$\int_{K\cdot S(r)}\exp\Big\{\frac{n(\alpha-\alpha_0)(\bar{x}-\alpha_0)}{\beta^2}\Big\}d\sigma dr\,,$$

$$1-e_{\mathrm{II}}(K_+,\alpha,\beta)=P_{\alpha\beta}(K_+)=\Big(\frac{1}{\sqrt{2\pi}\,\alpha}\Big)^n\int_0^\infty\exp\Big\{-\frac{r^2+n(\alpha-\alpha_0)^2}{2\beta^2}\Big\}$$

$$\int_{K_+\cdot S(r)}\exp\Big\{\frac{n(\alpha-\alpha_0)(\bar{x}-\alpha_0)}{\beta^2}\Big\}d\sigma dr\,,$$

where σ represents the coordinate on the sphere $S(r)$ and $d\sigma$ is the surface element of $S(r)$.

Now we have, by (6) and $\alpha>\alpha_0$,

$$\int_{K\cdot S(r)}\exp\Big\{\frac{n(\alpha-\alpha_0)(\bar{x}-\alpha_0)}{\beta^2}\Big\}d\sigma-\int_{K_+\cdot S(r)}\exp\Big\{\frac{n(\alpha-\alpha_0)(\bar{x}-\alpha_0)}{\beta^2}\Big\}d\sigma$$

$$=\int_{(K-K_+)\cdot S(r)}\exp\Big\{\frac{n(\alpha-\alpha_0)(\bar{x}-\alpha_0)}{\beta^2}\Big\}d\sigma$$

$$-\int_{(K_+-K)\cdot S(r)} \exp\left\{\frac{n(a-a_0)(\bar{x}-a_0)}{\beta^2}\right\}d\sigma$$

$$\leqq \exp\left\{\frac{n(a-a_0)rc_1}{a^2}\right\}\cdot\left(|(K-K_+)\cdot S(r)|-|(K_+-K)\cdot S(r)|\right)$$

$$=0 \text{ (by (5))}.$$

Thus we obtain $e_{\text{II}}(K_+, a, \beta) \leqq e_{\text{II}}(K, a, \beta)$.

4. *Student's test (2).*

Let $(R, G_{a\gamma})$ and $(R, G_{\beta\gamma})$ be one-dimensconal normal populations with the mean a and β respectively and with the same standard deviation γ. We choose a random sample of size m: $(x_1, x_2 \cdots, x_m)$ from $(R, G_{a\gamma})$ and a random sample of size n: $(y_1, y_2, \cdots, y_n)$ from $(R, G_{\beta\gamma})$. Then the sample point is $(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n)$.

The population of the sample is the product measure space $(R, G_{a\gamma})^m \otimes (R, G_{\beta\gamma})^n$, say $(\Omega, P_{a\beta\gamma})$. In this case the triple (a, β, γ) correspond to θ in § 1.

We consider a composite hypothesis "$a=\beta$". *Student's test* for this hypothesis is that whose critical region is

$$K_+=\{(x_1, x_2, \dots, x_m, y_1, y_2, \cdots, y_n);\ \bar{x}-\bar{y}>c\sqrt{ms_x^2+ns_y^2}\}$$

or

$$K_-=\{(x_1, x_2, \dots, x_m, y_1, y_2, \cdots, y_n);\ \bar{x}-\bar{y}<c\sqrt{ms_x^2+ns_y^2}\}$$

where $\bar{x}=\frac{1}{m}\sum x_i$, $\bar{y}=\frac{1}{n}\sum y_i$, $s_x=\sqrt{\frac{1}{m}\sum(x_i-\bar{x})^2}$, $s_y=\sqrt{\frac{1}{n}\sum(y_i-\bar{y})^2}$, and $c>0$.

Theorem 5. Let $S(u, r)=\left\{(x_1, x_2, \dots, x_m, y_1, y_2, \cdots, y_n);\ \frac{m\bar{x}+n\bar{y}}{m+n}=u, \sum(x_i-u)^2+\sum(y_i-u)^2=r^2\right\}$. Then, in order that K is a regular critical region for "$a=\beta$", it is necessary sufficient that the spherical area of $KS(u, r)\left(=|KS(u, r)|\right)$ is equal to $c\,|S(u, r)|$. c is equal to $e_{\text{I}}(K)$.

Proof. We shall prove only the necessity, since the sufficiency is evident. By the assumption we have

$$(1)\quad e_{\text{I}}(K)=\left(\frac{1}{\sqrt{2\pi}\gamma}\right)^n\int\cdots\int_K \exp\left\{-\frac{1}{2r^2}\left(\sum(x_i-a)^2+\sum(y_i-\beta)^2\right)\right\}$$

$$dx_1\dots dx_m dy_1\cdots dy_n.$$

Now we consider an orthogonal transformation

$$T: (x_1, \dots x_m, y_1, \dots y_n)\rightarrow(u_0, u_1, \dots u_{m+n-1}),$$

where $u_0=\sqrt{m+n}\,u=\frac{m\bar{x}+n\bar{y}}{\sqrt{m+n}}$. Then we have

$$(2)\qquad \sum(x_i-a)^2+\sum(y_i-a)^2=(u_0-\sqrt{m+n}\,a)^2+\sum_{i=1}^{m+n-1}u_i^2.$$

Therefore by (1) we have

$$(3)\qquad e_{\mathrm{I}}(K)=\frac{1}{\sqrt{2\pi}\,\gamma}\int_{-\infty}^{\infty}\exp\left\{-\frac{(u_0-\sqrt{m+n}\,\alpha)^2}{2\gamma^2}\right\}du_0\left(\frac{1}{\sqrt{2\pi}\,\gamma}\right)^{m+n-1}$$
$$\int\cdots\int_{K\cdot H(u_0)}\exp\left\{-\frac{1}{2\gamma^2}\sum_{i=1}^{m+n-1}u_i^2\right\}du_1du_2\cdots du_{m+n-1},$$

where $H(u_0)$ is the hyperplane: $u_0=\text{const.}$

From (3) we deduce, by Theorem 1,

$$(4)\qquad \left(\frac{1}{\sqrt{2\pi}\,\gamma}\right)^{m+n-1}\int\cdots\int_{K\cdot H(u_0)}\exp\left\{\frac{1}{2\gamma^2}\sum_{i=1}^{m+n-1}u_i^2\right\}du_1\cdots du_{m+n-1}=e_{\mathrm{I}}(K),$$

because

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,\gamma}\exp\left\{-\frac{u_0^2}{2\gamma^2}\right\}e^{iu_0t}du_0=\exp\left\{-\frac{\gamma^2t^2}{2}\right\}\neq 0.$$

In the same way as in the proof of Theorem 3, we obtain from (4)

$$(5)\qquad |K\cdot H(u_0)\{(u_0,u_1,\ldots,u_{m+n-1});\ \sum_{i=1}^{m+n-1}u_i^2=r^2\}|$$
$$=e_{\mathrm{I}}(K)\,|\{(u_0,\ldots,u_{m+n-1});\ \sum_{1}^{m+n-1}u_i^2=r^2\}|,$$

where $|\ \ |$ denotes the $(m+n-2)$-dimensional spherical area.

On the other hand we have

$$\sum_{i=1}^{m+n-1}u_i^2=\sum(x_i-\alpha)^2+\sum(y_i-\alpha)^2-(u_0-\sqrt{m+n}\,\alpha)^2$$
$$=\sum(x_i-\alpha)^2+\sum(y_i-\alpha)^2-(m+n)(u-\alpha)^2$$
$$=\sum(x_i-u)^2+\sum(y_i-u)^2.$$

Therefore the set in the left hand side of (5) is nothing but $S(u,r)$.

Theorem 6. K_+ is the most powerful against any alternative hypothesis (α,β,γ) such that $\alpha>\beta$ and K_- is the most powerful against any alternative hypothesis (α,β,γ) such that $\alpha<\beta$.

Proof. We prove only the first part, since the second part can be proved in the same way. We use the same notations as in the proof of Theorem 5. A simple calculation shows that

$$1-e_{\mathrm{II}}(K,\alpha,\beta)=\left(\frac{1}{\sqrt{2\pi}\,\gamma}\right)^n\int_{-\infty}^{\infty}\exp\left\{-\frac{m+n}{2\gamma^2}\left(u-\frac{m\alpha+n\beta}{m+n}\right)^2\right\}\frac{du}{\sqrt{m+n}}$$
$$\int_0^{\infty}\exp\left(-\frac{r^2}{2\gamma^2}\right)dr\times\int_{K\cdot S(u,r)}\exp\left\{-\frac{mn(\alpha-\beta)^2}{2(m+n)\gamma^2}\right\}\exp\left\{\frac{mn(\bar{x}-\bar{y})(\alpha-\beta)}{(m+n)\gamma^2}\right\}d\sigma$$

where σ represents the coordinate on $S(u,r)$ and $d\sigma$ is the surface element on $S(u,r)$. Since we have

$$ms_x^2+ns_y^2=r^2-\frac{mn}{m+n}(\bar{x}-\bar{y})^2$$

K_+ is expressible in the form

$$K_+=\{(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n)\ ;\ \bar{x}-\bar{y}\geqq c_1 r\}\ .$$

Therefore in the same manner as in the proof of Thorem 4 we can prove $e_{\text{II}}(K_+, \alpha, \beta)\leqq e_{\text{II}}(K, \alpha, \beta)$ for any regular critical region K.