19. Some Metrical Theorems on Fuchsian Groups.

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1. Let E be a measurable set in $|z|<1$. We define its hyperbolic measure $\sigma(E)$ by $\sigma(E)=\iint_E \frac{dxdy}{(1-|z|^2)^2}$ $(z=x+iy)$. Let e be a linear set on a rectifiable curve C in $|z|<1$, then its hyperbolic linear measure $\lambda(e)$ is defined by $\lambda(e)=\int_e \frac{|dz|}{1-|z|^2}$.

Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and D_0 be its fundamental domain, containing $z=0$ and z_n be equivalents of $z_0=0$. For any z in $|z|<1$, we denote its equivalent in D_0 by (z). Let $E(\theta)$ be the set of points $(re^{i\theta})$ in D_0, which are equivalent to points on a radius $z=re^{i\theta}(0\leqq r<1)$ of $|z|=1$. In may formar paper[1], I have proved:

Theorem 1. (i) *If* $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, *then* $E(\theta)$ *is everywhere dense in* D_0 *for almost all* $e^{i\theta}$ *on* $|z|=1$, (ii) *If* $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, *then* $\lim_{r\to1}|(re^{i\theta})|=1$ *for almost all* $e^{i\theta}$ *on* $|z|=1$.

In this paper, we will prove the following theorem, which is a precision of Theorem 1 (i).

Theorem 2. Suppose that $\sigma(D_0)<\infty$. *Let* $\wedge$ *be a set in* D_0, *which is measurable in Jordan's sense. Let* $g: z=te^{i\theta}(0\leqq t<1)$ *be a radius of* $|z|=1$ *and* l *be a segment* $(0\leqq t\leqq r)$ *on* g *of length* r, *whose hyperbolic length be* L *and* $L(\wedge)$ *be the hyperbolic measure of the set of t-values on* $(0, r)$, *such that* $(te^{i\theta})\in\wedge$. *Then there exists a set* e_0 *of measure zero on a unit circle* $U: |z|=1$, *which does not depend on* $\wedge$, *such that if* $e^{i\theta}\in U-e_0$, *then for any* $\wedge$,

$$\lim_{L\to\infty}\frac{L(\wedge)}{L}=\frac{\sigma(\wedge)}{\sigma(D_0)}. \tag{1}$$

Proof. We consider D_0 as a Riemann manifold F of constant negative curvature with $ds=\frac{|dz|}{1-|z|^2}$ and equivalent points are considered as the same point of F. Let $z=x+iy$ be any point of D_0. We associate a direction φ at z, which makes an angle φ with the real axis. Then the line elements (z,φ) $(z\in D_0,\ 0\leqq\varphi\leqq2\pi)$ constitute a phase space Ω, which is a product space of D_0 and a unit circle U: $\Omega=D_0\times U$ and the volume element $d\mu$ in Ω is defined by $d\mu=\frac{dxdyd\varphi}{(1-|z|^2)^2}$, so that $\mu(\Omega)=2\pi\sigma(D_0)<\infty$.

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1) M. Tsuji: Theory of conformal mapping of a multiply connected domain, III. Jap. Journ. Math. **19** (1944).

Now the line element (z, φ) determines a unique geodesic $g=g(z, \varphi)$ of F, which is an arc of an orthogonal circle to $|z|=1$, which touches the direction φ at z. Let $\eta_1=e^{i\theta_1}$, $\eta_2=e^{i\theta_2}$ be the two end points of g on $|z|=1$, where η_1 is such that if we proceed on g in the direction φ, then we meet $|z|=1$ at η_1. We call η_1 the end point of g. Let z_0 be the middle point of the arc $\widehat{\eta_1\eta_2}$ on g, z be any point on g and s be the hyperbolic length of the arc $\widehat{z_0, z}$, where s is positive, if z lies on $\widehat{z_0, \eta_1}$ and negative, if z lies on $\widehat{z_0, \eta_2}$. Then we have a one-to-one correspondence between (z, φ) and (η_1, η_2, s). As Hopf proved:[1)]

$$d\mu=C.\,\frac{|d\eta_1||d\eta_2|\,ds}{|\eta_1-\eta_2|^2} \quad (C=\text{const.}). \tag{2}$$

Now we consider a geodesic flow $T_t(-\infty<t<\infty)$ in Ω:

$$T_t:\ P=(\eta_1, \eta_2, s) \rightarrow P_t(\eta_1, \eta_2, s+t). \tag{3}$$

By (2), T_t is a mass-preserving transformation of Ω into itself. Hopf[1)] proved that T_t is metric transitive. Hence by Birkhoff's ergodic theorem,

$$\lim_{L\to\infty} \frac{1}{L}\int_0^L f(P_t)dt = \frac{\int_\Omega f(P)d\mu}{\mu(\Omega)}, \tag{4}$$

for almost all points $P=(z, \varphi)$ in Ω, where $f \subset L^2$ in Ω.

Let M be any set in D_0 and $S_n(M)$ $(n=0, 1, 2, \dots)$ be its equivalents and put $[M]=\sum_{n=0}^{\infty} S_n(M)$. Then $L(\wedge)$ is equal to the hyperbolic measure of the part of l contained in $[\wedge]$.

Let M be a set in D_0. We associate at every point z of M directions φ $(0\leqq\varphi\leqq 2\pi)$. Then such line elements (z, φ) $(z \in M,\ 0\leqq\varphi\leqq 2\pi)$ constitute a set E in Ω, which is a product set of M and a unit circle U: $E=M\times U$, so that $\mu(E)=2\pi\sigma(M)$.

Consider a geodesic $g=g(z, \varphi)$ and an arc $C=\widehat{z, z'}$ on g of hyperbolic length L_g. Let $L_g([M])$ be the hyperbolic measure of the part of C contained in $[M]$. If we take $f(P)$ in (4) as the characteristic function of E, then (4) becomes

$$\lim_{L_g\to\infty} \frac{L_g([M])}{L_g} = \frac{\mu(E)}{\mu(\Omega)} = \frac{\sigma(M)}{\sigma(D_0)}, \tag{5}$$

for almost all points $P=(z, \varphi)$ in Ω.

Let Δ be a polygonal domain in $|z|<1$, which has common points with D_0 and whose sides consist of segments lying on lines $x=\text{const.}=\alpha$ or $y=\text{const.}=\beta$, where α, β are rationals. If Δ contains points outside D_0, we replace such points by their equivalents in D_0. We call the so modified domain in D_0 a rational polygonal domain. Since the totality of rational polygonal domains is enumerable, let Δ_i $(i=1, 2, \dots)$ be all rational polygonal domains, then by (5),

1) E. Hopf: Fuchsian group and ergodic theory. Trans. Amer. Math. Soc. **39** (1936). Ergodentheorie Berlin (1937). M. Tsuji: On Hopf's ergodic theorem. Proc. **20** (1944).

$$\lim_{L_g\to\infty}\frac{L_g([\Delta_i])}{L_g}=\frac{\sigma(\Delta_i)}{\sigma(D_0)}\,, \tag{6}$$

if $P=(z,\varphi)\in\Omega-N_i$, where $\mu(N_i)=0$.

If D_0 extends to $|z|=1$, then let $D_0^{(r)}$ be the part of D_0 contained in $|z|\leqq r<1$. Let $0<\rho_i<1$ $(i=1,2,\dots)$ be rationals, then by (5),

$$\lim_{L_g\to\infty}\frac{L_g([D_0-D_0^{(\rho_i)}])}{L_g}=\frac{\sigma(D_0-D_0^{(\rho_i)})}{\sigma(D_0)}\,, \tag{7}$$

if $P\in\Omega-N_i'$, where $\mu(N_i')=0$.

If we put $N=\sum_{i=1}^{\infty}N_i+\sum_{i=1}^{\infty}N_i'$, then $\mu(N)=0$ and if $P\in\Omega-N$, then (6) and (7) hold for $i=1,2,\dots$

By Fubini's theorem, there exists a set M_0 in D_0, such that $\sigma(M_0)=0$ and for any $z\in D_0-M_0$, (6) and (7) $(i=1,2,\dots)$ hold for geodesics $g=g(z_0,\varphi)$ for almost all φ. Let $z_0\in D_0-M_0$ and e_0 be the set of points on a unit circle U, which are the end points of the exceptional geodesics $g=g(z_0,\varphi)$, then $me_0=0$ and if $e^{i\theta}\in U-e_0$ and and $\eta=e^{i\theta}$ be the end point of a geodesic $g=g(z_0,\varphi)$, then (6) ahd (7) $(i=1,2,\dots)$ hold for such a geodesic. Let $e^{i\theta}\in U-e_0$ and consider a radius $g_0: z=re^{i\theta}$ $(0\leqq r<1)$ of $|z|=1$, which is a geodesic $g_0=g(0,\theta)$ touching $g_0=g(z_0,\varphi)$ at η.

We will prove that (1) holds for such a radius $z=re^{i\theta}$ $(0\leqq r<1)$.

Let z',z and ζ',ζ be points on g_0 and g respectively, such that $|z'|=|\zeta'|$, $|z|=|\zeta|$, $(|z'|<|z)$ and $L_{g_0}(z',z)$, $L_g(\zeta',\zeta)$ be the hyperbolic lengths of the arc $\widehat{z',z}$ on g_0 and $\widehat{\zeta',\zeta}$ on g, then

$$L_{g_0}(z',z)=\int_{z'}^{z}\frac{dr}{1-r^2}\,,\qquad L_g(\zeta',\zeta)=\int_{\zeta'}^{\zeta}\frac{|dz|}{1-r^2}\qquad(|z|=r)\,.$$

Since g_0 touches g at η, we have $(1-\varepsilon)dr\leqq|dz|\leqq(1+\varepsilon)dr$ for $r_0\leqq r<1$, so that

$$(1-\varepsilon)L_{g_0}(z',z)\leqq L_g(\zeta',\zeta)\leqq(1+\varepsilon)L_{g_0}(z',z)\quad(r_0\leqq r<1)\,. \tag{8}$$

Let z,ζ be points on g_0 and g respectively, such that $|z|=|\zeta|=r$ and $\sigma(z,\zeta)$ be the hyperbolic distance between z and ζ, then $\sigma(z,\zeta)\leqq\dfrac{\widehat{z,\zeta}}{1-r^2}$, where $\widehat{z,\zeta}$ is the arc length of the arc $\widehat{z,\zeta}$ on $|z|=r$. Since g_0 touches g at η, we have

$$\sigma(z,\zeta)\to0\qquad\text{for}\qquad r\to1\,. \tag{9}$$

(i) First we suppose that $\wedge$ is contained in $|z|\leqq r<1$.

Since $\wedge$ is measurable in Jordan's sense, we can find two polygonal domains Δ_1,Δ_2' in $|z|<1$, such that $\Delta_1\subset\wedge\subset\Delta_2'$, $\sigma(\Delta_2')-\sigma(\Delta_1)<\varepsilon$, where Δ_1 consists of only inner points of $\wedge$ and the boundary of Δ_2' consists of only outer points of $\wedge$ and the sides of Δ_1,Δ_2' consists of segments on lines $x=\text{const.}=\alpha$ or $y=\text{const.}=\beta$, where α,β are rationals. If Δ_2' contains points outside D_0, we replace such points by their equivalents in D_0 and let the so modified domain in D_0 be Δ_2, then we have two rational polygonal domains Δ_1, Δ_2 in D_0, such that $\Delta_1 \subset \wedge \subset \Delta_2$, $\sigma(\Delta_2)-\sigma(\Delta_1)<\varepsilon$. Then by (6),

$$\lim_{L_g\to\infty}\frac{L_g([\Delta_i])}{L_g}=\frac{\sigma(\Delta_i)}{\sigma(D_0)} \qquad (i=1,2). \tag{10}$$

By (9), there exists $\rho<1$, such that if a point z ($|z|=r\geqq\rho$) on g_0 lies in $[\wedge]$, then the corresponding ζ ($|\zeta|=|z|$) on g lies in $[\Delta_2]$ and if ζ lies in $[\Delta_1]$, then z lies in $[\wedge]$, so that by (8),

$$-\text{const.}+\frac{1}{1+\varepsilon}L_g([\Delta_1])\leqq L_{g_0}([\wedge])\leqq \text{const.}+\frac{1}{1+\varepsilon}L_g([\Delta_2]), \tag{11}$$

$$-\text{const.}+\frac{1}{1+\varepsilon}L_g\leqq L_{g_0}\leqq \text{const.}+\frac{1}{1-\varepsilon}L_g, \tag{12}$$

where L_{g_0}, L_g are hyperbolic lengths of the arc $\widehat{0,z}$ on g_0 and $\widehat{z_0,\zeta}$ on g respectively, where $|z|=|\zeta|$.

Hence by (10), (11), (12),

$$\frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{\sigma(\Delta_1)}{\sigma(D_0)}\leqq \varliminf_{L_{g_0}\to\infty}\frac{L_{g_0}([\wedge])}{L_{g_0}}\leqq \varlimsup_{L_{g_0}\to\infty}\frac{L_{g_0}([\wedge])}{L_{g_0}}\leqq\frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{\sigma(\Delta_2)}{\sigma(D_0)}.$$

Making $\varepsilon\to 0$, $\sigma(\Delta_1)\to\sigma(\wedge)$, $\sigma(\Delta_2)\to\sigma(\wedge)$, we have

$$\lim_{L_{g_0}\to\infty}\frac{L_{g_0}([\wedge])}{L_{g_0}}=\frac{\sigma(\wedge)}{\sigma(D_0)}. \tag{13}$$

(ii) Next suppose that $\wedge$ contains points tending to $|z|=1$.

Let $\wedge^{(r)}$ be the part of $\wedge$ contained in $|z|\leqq r<1$. Then $\wedge^{(r)}$ is measurable in Jordan's sense, hence by (13),

$$\lim_{L_{g_0}\to\infty}\frac{L_{g_0}([\wedge^{(r)}])}{L_g}=\frac{\sigma(\wedge^{(r)})}{\sigma(D_0)}. \tag{14}$$

Since $L_{g_0}([\wedge])\geqq L_{g_0}([\wedge^{(r)}])$, we have for $r\to 1$.

$$\varliminf_{L_{g_0}\to\infty}\frac{L_{g_0}([\wedge])}{L_{g_0}}\geqq\lim_{L_{g_0}\to\infty}\frac{L_{g_0}([\wedge^{(r)}])}{L_{g_0}}=\frac{\sigma(\wedge^{(r)})}{\sigma(D_0)}\to\frac{\sigma(\wedge)}{\sigma(D_0)}. \tag{15}$$

By (9), there exists a rational $0<\rho<1$, such that if z ($|z|\geqq r_0$) on g_0 lies in $[D_0-D_0^{(r)}]$, then the corresponding ζ ($|\zeta|=|z|$) on g lies in $[D_0-D_0^{(\rho)}]$, where $\rho\to 1$ with $r\to 1$. By (7),

$$\lim_{L_g\to\infty}\frac{L_g([D_0-D_0^{(\rho)}])}{L_g}=\frac{\sigma(D_0-D_0^{(\rho)})}{\sigma(D_0)}<\delta, \quad\text{if}\quad \rho_0\leqq\rho<1.$$

Since $[\wedge-\wedge^{(r)}]\subset[D_0-D_0^{(r)}]$, we have from (8),

$$\varlimsup_{L_{g_0}\to\infty}\frac{L_{g_0}([\wedge-\wedge^{(r)}])}{L_{g_0}}\leqq\varlimsup_{L_{g_0}\to\infty}\frac{L_{g_0}([D_0-D_0^{(r)}])}{L_{g_0}}$$

$$\leqq\frac{1+\varepsilon}{1-\varepsilon}\cdot\lim_{L_g\to\infty}\frac{L_g([D_0-D_0^{(\rho)}])}{L_g}=\frac{1+\varepsilon}{1-\varepsilon}\,\frac{\sigma(D_0-D_0^{(\rho)})}{\sigma(D_0)}<\frac{1+\varepsilon}{1-\varepsilon}\delta.$$

Hence by (14),

$$\overline{\lim_{L_{g_0}\to\infty}} \frac{L_{g_0}([\wedge])}{L_{g_0}} \leq \lim_{L_{g_0}\to\infty} \frac{L_{g_0}([\wedge^{(r)}])}{L_{g_0}} + \overline{\lim_{L_{g_0}\to\infty}} \frac{L_{g_0}([\wedge - \wedge^{(r)}])}{L_{g_0}} < \frac{\sigma(\wedge^{(r)})}{\sigma(D_0)} + \frac{1+\varepsilon}{1-\varepsilon}\delta .$$

Making $r\to 1$, $\delta\to 0$, $\varepsilon\to 0$, we have

$$\overline{\lim_{L_{g_0}\to\infty}} \frac{L_{g_0}([\wedge])}{L_{g_0}} \leqq \frac{\sigma(\wedge)}{\sigma(D_0)} . \tag{16}$$

From (15), (16),

$$\lim_{L_{g_0}\to\infty} \frac{L_{g_0}([\wedge])}{L_{g_0}} = \frac{\sigma(\wedge)}{\sigma(D_0)} .$$

Since $L(\wedge)=L_{g_0}([\wedge])$, we have (1), q.e.d.

Remark. The same result holds, if G contains anti-analytic transformations: $z'=\dfrac{a\bar{z}+b}{c\bar{z}+d}$, where $\bar{z}$ is the conjugate complex of z.

As a special case, consider a domain D_0 in $|z|<1$, bounded by n circles: $C_1, \dots, C_n$, which are orthogonal to $|z|=1$ and touch each other externally as for a modular figure. Let G be the group generated by inversions on C_i $(i=1,2,\dots,n)$, then D_0 is its fundamental domain and $\sigma(D_0)<\infty$. The set in D_0, which is equivalent to a radius $g: z=re^{i\theta}(0\leqq r<1)$ of $|z|=1$ is obtained as follows. We start from $z=0$ and proceed on g till we meet the boundary of D_0, say C_1, at z_1, then reflect g on C_1 and proceed on the reflected line till we meet the boundary of D_0 and so on. Let $L(\wedge)$ be the hyperbolic measure of the part of such a path contained in $\wedge$ and L be the total hyperbolic length of the path, then (1) holds for almost all starting directions for any $\wedge$.

2. Let $\sigma(D_0)<\infty$ and Ω be defined as before. We consider a product space $\Omega^n=\overbrace{\Omega\times\cdots\times\Omega}^{n}$, where $\Pi=(P^{(1)},\dots,P^{(n)})$ $(P^{(i)}\in\Omega)$ is considered as a point of Ω^n and consider the product flow $\Pi=(P^{(1)},\dots, P^{(n)})\to \Pi_t=(P_t^{(1)},\dots,P_t^{(n)})$ in Ω^n. Then we can prove easily that the flow is metric transitive. From this we proceed similarly as the proof of Theorem 2 and can prove the following extension of Theorem 2.

Theorem 3. Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and D_0 be its fundamental domain, containing $z=0$ and $\sigma(D_0)<\infty$. Let $\wedge_1,\dots,\wedge_n$ be n sets in D_0, which are measurable in Jordan's sense. Let $g_k: z=te^{i\theta_k}$ $(0\leqq t<1)$ $(k=1, 2,\dots,n)$ be n radii of $|z|=1$ and l_k be segments $(0\leqq t\leqq r)$ on g_k of the same length r, whose hyperbolic length be L. Let $L(\wedge_1\times\cdots\times\wedge_n)$ be the hyperbolic measurs of the set of t- values on $(0,r)$, such that $(te^{i\theta_1})\in\wedge,\dots,(te^{i\theta_n})\in\wedge_n$. Then

$$\lim_{L\to\infty}\frac{L(\wedge_1\times\cdots\times\wedge_n)}{L}=\frac{\sigma(\wedge_1)\cdots\sigma(\wedge_n)}{[\sigma(D_0)]^n},$$

when $(\theta_1, \dots, \theta_n)$ *does not belong to a certain set* e_0 *of measure zerp on an n-dimensional torus* Θ $(0\leqq\theta_k\leqq 2\pi,\ k=1,2,\dots,n)$, *where* e_0 *does not depend on* $\wedge_1, \dots, \wedge_n$.