PAPERS COMMUNICATED

15. *Completely Continuous Transformations in Hilbert Spaces.*

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1. By a space of type A[1] we mean a Banach space in which there exist a linearly independent sequence $\{f_n\}$ of elements of unit norm and a double sequence $\{L_{mn}(f)\}$ of bounded linear functionals such that for every f

$$\text{(A)} \qquad \lim_{m\to\infty} \| f - \sum_{n=1}^{m_n} L_{mn}(f) f_n \| = 0 .$$

It will be seen that the conception of a space of type A is a generalization of the idea of a Banach space with a denumerable base[2].

Let $\mathfrak{T}$ denote the space of all completely continuous transformations of a Hilbert space $\mathfrak{H}$ into itself, that is, the space of all bounded linear transformations which carry every bounded set of $\mathfrak{H}$ into a compact set.

In this note we will prove that the space $\mathfrak{T}$ is a separable space of type A.

2. We prove now the following theorem:

Theorem 1. In the space $\mathfrak{T}$, there exist a linearly independent double sequence $\{T_{ij}\}$ of elements of unit norm and a double sequence $\{a_{ij}(T)\}$ of bounded linear functionals such that for any $T \in \mathfrak{T}$

$$T = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}(T) T_{ij} .$$

Proof. Let $\{\varphi_n\}$ denote the complete orthonormal set of the space $\mathfrak{H}$. We define $\{T_{ij}\}$ as follows;

$$T_{ij}(x) = (x, \varphi_j)\varphi_i \quad \text{for all} \quad x \in \mathfrak{H}, \qquad (i, j = 1, 2, \dots).$$

Then it is evident that $T_{ij} \in \mathfrak{T}$, $\| T_{ij} \| = 1$ and the sequence $\{T_{ij}\}$ is linearly independent. Let $\mathfrak{M}_j$ be the closed linear manifold determined by $\{\varphi_1, \varphi_2, \cdots, \varphi_j\}$. Then we can prove that every bounded linear transformation T with domain $\mathfrak{H}$ and with range $\mathfrak{M}_1$ is expressed in the form $T = \sum_{j=1}^{\infty} a_{1j}(T) T_{1j}$ where $a_{1j}(T)$ are bounded linear functionals.

In fact, by use of F. Riesz' theorem[3] it can be easily shown that

1) The notion of a space of type A was introduced by I. Maddaus. I. Maddaus; Completely continuous linear transformations, Bull. Amer. Math. Soc. Vol. 44 (1938), 279–282.

2) S. Banach; Théories des opérations linéaires, p. 110.

3) M. H. Stone; Linear transformations in Hilbert space and their applications to analysis, p. 62, Theorem 2. 27.

T is expressed in the form $T(x)=(x, y)\varphi_1$ where y is uniquely determined corresponding to T. Let $y=\sum_{j=1}^{\infty} a_j\varphi_j$, then we have

$$T(x)=\sum_{j=1}^{\infty} \bar{a}_j(x, \varphi_j)\varphi_1=\sum_{j=1}^{\infty} \bar{a}_j T_{1j}(x)$$

where $\bar{a}_j$ denotes the conjugate complex number of a_j.

Let $T_n(x)=\sum_{j=1}^{n} \bar{a}_j T_{1j}(x)$, then

$$\| T(x)-T_n(x) \| = \| \sum_{j=n+1}^{\infty} \bar{a}_j T_{1j}(x) \| = \| (x, y-\sum_{j=1}^{n} a_j\varphi_j)\varphi_1 \|$$
$$= | (x, y-\sum_{j=1}^{n} a_j\varphi_j) | \leqq \| x \| \cdot \| y-\sum_{j=1}^{n} a_j\varphi_j \| .$$

Therefore for every $\| x \| \leqq 1$

$$\| T(x)-T_n(x) \| \leqq \| y-\sum_{j=1}^{n} a_j\varphi_j \| ,$$

so that we have

$$\lim_{n\to\infty} \| T-T_n \| = 0 .$$

Now let $a_{1j}(T)$ denote the number $\bar{a}_j$ which corresponds to T, then we have

$$T=\sum_{j=1}^{\infty} a_{1j}(T)T_{1j} .$$

Since $a_{1j}(T)=(\varphi_j, y)$ and $\| T \| = \underset{|x|\leq 1}{\text{l.u.b.}} | (x, y) |$, we have $\| T \| \geqq | (\varphi_j, y) |$ and hence $\| T \| \geqq | a_{1j}(T) |$.

On the other hand, from the definition of $a_{1j}(T)$ it is easily seen that $a_{1j}(T)$ are additive functionals. Therefore $a_{1j}(T)$ are bounded linear functionals.

By the similar argument we can prove that every bounded linear transformation T with domain $\mathfrak{H}$ and with range $\mathfrak{M}_n$ $(n=1, 2, \dots)$ can be expressed in the form $T=\sum_{i=1}^{n}\sum_{j=1}^{\infty} a_{ij}(T)T_{ij}$ where $a_{ij}(T)$ are bounded linear functionals.

Now let T be an arbitrary element of the space $\mathfrak{T}$, then

$$T(x)=(x, y_1)\varphi_1+(x, y_2)\varphi_2+\cdots$$
$$=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}(T)T_{ij}(x) .$$

Form the proof of I. Maddaus' theorem[1], that is, *every completely continuous transformation of a Banach space into a space of type A is the strong limit of a sequence of singular transformations*[2], we can prove that $T=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}(T)T_{ij}$ and every $a_{ij}(T)$ is a bounded linear

1) I. Maddaus; loc. cit.

2) A Singular transformation is, by definition, a bounded linear transformation which transforms its domain into a space of a finite number of dimensions.

functional. Thus the proof of the theorem is completed.

Remark. Since $\sum_{i=1}^{n}\sum_{j=1}^{\infty} a_{ij}(T)T_{ij}$ is contained in $\mathfrak{T}$ for each n, it follows from Theorem 1 that elements T of $\mathfrak{T}$ are characterized by being expressed in the form $T=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}(T)T_{ij}$.

Theorem 2. The space $\mathfrak{T}$ is a separable space of type A.

Proof. Let T be an arbitrary element of the space $\mathfrak{T}$ and let $\varepsilon > 0$ any prescribed number. In view of Theorem 1, there exist positive integers m, n such that

$$\| T-\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}(T)T_{ij} \| < \frac{\varepsilon}{2}.$$

On the other hand, there exists a sequence $\{\mathfrak{r}_{ij}\}$ of complex numbers, each with rational real and imaginary parts, such that

$$\| \sum_{i=1}^{m}\sum_{j=1}^{n} \{a_{ij}(T)-\mathfrak{r}_{ij}\}T_{ij} \| < \frac{\varepsilon}{2}.$$

Therefore $\| T-\sum_{i=1}^{m}\sum_{j=1}^{n} \mathfrak{r}_{ij}T_{ij} \| < \varepsilon$, hence the space $\mathfrak{T}$ is a separable space.

Let $\{T_k\}$ $(k=1, 2, \dots)$ denote the denumerable set which is everywhere dense in $\mathfrak{T}$ and let $\{\varepsilon_l\}$ be a decreasing sequence of positive numbers such that $\lim_{l\to\infty} \varepsilon_l=0$. Then it can be shown that there exists an increasing[1] sequence $\{(m_l^{(k)}, n_l^{(k)})\}$ of pairs of positive integers such that

$$\| T_k-\sum_{i=1}^{m_l^{(k)}}\sum_{j=1}^{n_l^{(k)}} a_{ij}(T_k)T_{ij} \| < \varepsilon_l \quad \text{for} \quad k, l=1, 2, \dots,$$

and

$$m_l^{(k+1)} \geqq m_l^{(k)}, \ n_l^{(k+1)} \geqq n_l^{(k)} \quad \text{for} \quad k, l=1, 2, \dots.$$

Let $\{(m_l, n_l)\}$ be a sequence such that $m_l=m_l^{(l)}$, $n_l=n_l^{(l)}$. Then

$$\lim_{l\to\infty} \| T_k-\sum_{i=1}^{m_l}\sum_{j=1}^{n_l} a_{ij}(T_k)T_{ij} \| = 0$$

for every k.

Since $\{T_k\}$ is everywhere dense in $\mathfrak{T}$, for any $T \in \mathfrak{T}$

$$\lim_{l\to\infty} \| T-\sum_{i=1}^{m_l}\sum_{j=1}^{n_l} a_{ij}(T)T_{ij} \| = 0. \tag{1}$$

By the method of diagonal process we renumber the double sequence $\{a_{ij}(T)T_{ij}\}$ into a simple sequence $\{a_n(T)T_n\}$. We express each of the expression (1) in the form with terms of $\{a_n(T)T_n\}$ and denote by (1)* the new expressions. Let l_n be the greatest integer in the expression (1)* for each $l=1, 2, \dots$ When the term $a_\alpha(T)T_\alpha$ $(\alpha < l_n)$ is not contained in (1)* for each l, we define $a_\alpha(T)=0$ for all $T \in \mathfrak{T}$. Let $L_{ln}(T)=a_n(T)$ in the expression (1)* which corresponds to l.

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1) A sequence $\{(m_l^{(k)}, n_l^{(k)})\}$ is said to be increasing if $m_{l+1}^{(k)} > m_l^{(k)}$ and $n_{l+1}^{(k)} > n_l^{(k)}$ for $l=1, 2, \dots$

Then we have

$$\sum_{i=1}^{m_l}\sum_{j=1}^{n_l} a_{ij}(T)T_{ij}=\sum_{n=1}^{l_n} L_{ln}(T)T_n$$

for each l.

Therefore for every $T\in\mathfrak{T}$

$$\lim_{l\to\infty}\| T-\sum_{n=1}^{l_n} L_{ln}(T)T_n\|=0\,.$$

Thus the space $\mathfrak{T}$ satisfies the condition (A).

Since the fact that the space $\mathfrak{T}$ is a Banach space is easily shown by means of the property that the space $\mathfrak{H}$ is complete, we omit the proof. Thus we have established the theorem.

As an immediate consequence of Theorem 2 and I. Maddaus' theorem we get the following theorem:

Theorem 3. Every completely continuous transformation of the space $\mathfrak{T}$ into itself is the strong limit of a sequence of singular transformations.