Discontinuity at fixed points with applications

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Abstract

In this paper, we study new contractive conditions which are strong enough to generate fixed points but which do not force the map to be continuous at fixed points. In this context, we give new results on the fixed-circle problem. We investigate some applications to complex-valued metric spaces and to discontinuous activation functions in real and complex valued neural networks.

1 Introduction

The fixed-point theory is an attractive area in mathematics. This theory has been extensively studied by many aspects (see [7, 8, 9, 17, 18] and the references therein). There are some open problems in this area. One of these problems is the following open problem raised by B. E. Rhoades in [18].

***Open Problem* D.** What are the contractive conditions which are strong enough to generate a fixed point but which do not force the map to be continuous at fixed point?

Then, in [15], R. P. Pant obtained a solution of this question using the number

$$m(u,v) = \max\{d(u,Tu), d(v,Tv)\},$$

on a complete metric space. Recently, some new solutions have been investigated using various approaches. For example, Bisht and Pant studied on this *Open Problem* D using the numbers

$$M(u,v) = \max\left\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(u,Tv)+d(v,Tu)}{2}\right\}$$

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and

$$M^*(u,v) = \max\left\{d(u,v), d(u,Tu), d(v,Tv), \frac{\alpha\left[d(u,Tv)+d(v,Tu)\right]}{2}\right\}, \alpha \in [0,1)$$

on a complete metric space (see [2, 3]). Some recent studies about this question on a metric space can be found in [4, 5, 14, 16, 20, 24]. On the other hand, some discontinuity results were applied to the fixed-circle problem and to discontinuous activation functions (see [13, 14, 20]).

Let (X,d) be a metric space and $T : X \to X$ be a self-mapping. In this paper, we consider the following number defined as

$$N(u,v) = \max\left\{\begin{array}{c} d(u,v), d(u,Tu), d(v,Tv), \frac{d(v,Tv)[1+d(u,Tu)]}{1+d(u,v)}, \\ \frac{d(u,Tu)[1+d(v,Tv)]}{1+d(Tu,Tv)} \end{array}\right\}, \quad (1.1)$$

for all $u,v \in X$ and our aim is to obtain new solutions of the *Open Problem* D using the classical technique. We prove some fixed-circle theorems related to discontinuity points.

Our paper is organized as follows: In Section 2, we give a fixed-point theorem using the number $N(u,v)$ and some related results on a complete metric space. In Section 3, we obtain a new solution of the *Open Problem* D on a complex valued metric space. In Section 4, we prove some fixed-circle theorems on metric spaces. In Section 5, we investigate some applications to discontinuous activation functions in real and complex valued neural networks.

2 Some New Results on Discontinuity at Fixed Point

At first, we give the following fixed-point theorem.

Theorem 2.1. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following conditions*:

(1) *There exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ and*

$$d(Tu,Tv) \leq \phi(N(u,v)).$$

(2) *For a given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < N(u,v) < \varepsilon + \delta$ implies $d(Tu,Tv) \leq \varepsilon$.*

Then T has a unique fixed point $u^* \in X$ and $T^n u \to u^*$ for each $u \in X$. Also, T is discontinuous at u^* if and only if $\lim\limits_{u \to u^*} N(u,u^*) \neq 0$.

Proof. Let $u_0 \in X$, $Tu_0 \neq u_0$ and the sequence $\{u_n\}$ be defined as $Tu_n = u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using the condition (1), we have

$$\begin{aligned} d(u_n,u_{n+1}) &= d(Tu_{n-1},Tu_n) \leq \phi(N(u_{n-1},u_n)) < N(u_{n-1},u_n) \\ &= \max\left\{\begin{array}{c} d(u_{n-1},u_n), d(u_{n-1},Tu_{n-1}), d(u_n,Tu_n), \\ \frac{d(u_n,Tu_n)[1+d(u_{n-1},Tu_{n-1})]}{1+d(u_{n-1},u_n)}, \frac{d(u_{n-1},Tu_{n-1})[1+d(u_n,Tu_n)]}{1+d(Tu_{n-1},Tu_n)} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d(u_{n-1},u_n), d(u_{n-1},u_n), d(u_n,u_{n+1}), \\ \frac{d(u_n,u_{n+1})[1+d(u_{n-1},u_n)]}{1+d(u_{n-1},u_n)}, \frac{d(u_{n-1},u_n)[1+d(u_n,u_{n+1})]}{1+d(u_n,u_{n+1})} \end{array}\right\} \\ &= \max\left\{d(u_{n-1},u_n), d(u_n,u_{n+1})\right\}. \end{aligned} \quad (2.1)$$

Assume that $d(u_{n-1}, u_n) < d(u_n, u_{n+1})$. Then using the inequality (2.1), we get

$$d(u_n, u_{n+1}) < d(u_n, u_{n+1}),$$

which is a contradiction. So it should be $d(u_n, u_{n+1}) < d(u_{n-1}, u_n)$. If we put $d(u_n, u_{n+1}) = s_n$ then from the inequality (2.1), we have

$$s_n < s_{n-1}, \tag{2.2}$$

that is, s_n is a strictly decreasing sequence of positive real numbers and so the sequence s_n tends to a limit $s \geq 0$. Suppose that $s > 0$. There exists a positive integer $k \in \mathbb{N}$ such that $n \geq k$ implies

$$s < s_n < s + \delta(s). \tag{2.3}$$

Using the condition (2) and the inequality (2.2), we get

$$d(Tu_{n-1}, Tu_n) = d(u_n, u_{n+1}) = s_n < s, \tag{2.4}$$

for $n \geq k$. The inequality (2.4) contradicts to the inequality (2.3). Then it should be $s = 0$.

Now we show that $\{u_n\}$ is a Cauchy sequence. Let us fix an $\varepsilon > 0$. Without loss of generality, we can assume that $\delta(\varepsilon) < \varepsilon$. There exists $k \in \mathbb{N}$ such that $\delta^2 < \varepsilon$ for $n \geq k$ since $s_n \to 0$. Following Jachymski (see [8, 9] for more details), using the mathematical induction, we prove

$$d(u_k, u_{k+n}) < \varepsilon + \delta, \tag{2.5}$$

for any $n \in \mathbb{N}$. The inequality (2.5) holds for $n = 1$ since

$$d(u_k, u_{k+1}) = s_k < \delta < \varepsilon + \delta.$$

Assume that the inequality (2.5) is true for some n. We prove it for $n+1$. Using the triangle inequality, we obtain

$$d(u_k, u_{k+n+1}) \leq d(u_k, u_{k+1}) + d(u_{k+1}, u_{k+n+1}).$$

It suffices to show $d(u_{k+1}, u_{k+n+1}) \leq \varepsilon$. To do this, we prove $N(u_k, u_{k+n}) \leq \varepsilon + \delta$, where

$$N(u_k, u_{k+n}) = \left\{ \begin{array}{c} d(u_k, u_{k+n}), d(u_k, Tu_k), d(u_{k+n}, Tu_{k+n}), \\ \frac{d(u_{k+n}, Tu_{k+n})[1+d(u_k, Tu_k)]}{1+d(u_k, u_{k+n})}, \frac{d(u_k, Tu_k)[1+d(u_{k+n}, Tu_{k+n})]}{1+d(Tu_k, Tu_{k+n})} \end{array} \right\}. \tag{2.6}$$

Using the mathematical induction hypothesis, we find

$$d(u_k, u_{k+n}) < \varepsilon + \delta,$$

$$d(u_k, Tu_k) < \delta < \varepsilon + \delta,$$

$$d(u_{k+n}, Tu_{k+n}) < \delta < \varepsilon + \delta,$$

$$\frac{d(u_{k+n},Tu_{k+n})[1+d(u_k,Tu_k)]}{1+d(u_k,u_{k+n})} < \delta+\delta^2 < \varepsilon+\delta,$$
$$\frac{d(u_k,Tu_k)[1+d(u_{k+n},Tu_{k+n})]}{1+d(Tu_k,Tu_{k+n})} < \delta+\delta^2 < \varepsilon+\delta. \tag{2.7}$$

Using the conditions (2.6) and (2.7), we have

$$N(u_k,u_{k+n}) < \varepsilon+\delta.$$

From the condition (2), we obtain

$$d(Tu_k,Tu_{k+n}) = d(u_{k+1},u_{k+n+1}) \le \varepsilon.$$

Therefore, the inequality (2.5) implies that $\{u_n\}$ is Cauchy. Since (X,d) is a complete metric space, there exists a point $u^* \in X$ such that $u_n \to u^*$ as $n \to \infty$. Also we get $Tu_n \to u^*$.

Now we show that $Tu^* = u^*$. On the contrary, suppose that u^* is not a fixed point of T, that is, $Tu^* \neq u^*$. Then using the condition (1), we get

$$\begin{aligned} d(Tu^*,Tu_n) &\le \phi(N(u^*,u_n)) < N(u^*,u_n) \\ &= \max\left\{ \begin{array}{c} d(u^*,u_n), d(u^*,Tu^*), d(u_n,u_{n+1}), \\ \frac{d(u_n,u_{n+1})[1+d(u^*,Tu^*)]}{1+d(u^*,u_n)}, \frac{d(u^*,Tu^*)[1+d(u_n,u_{n+1})]}{1+d(Tu^*,u_{n+1})} \end{array} \right\} \end{aligned}$$

and so taking limit for $n \to \infty$ we have

$$d(Tu^*,u^*) < \frac{d(u^*,Tu^*)}{1+d(Tu^*,u^*)},$$

which is a contradiction. Thus u^* is a fixed point of T. We prove that the fixed point u^* is unique. Let v^* be another fixed point of T such that $u^* \neq v^*$. By the condition (1), we find

$$\begin{aligned} d(Tu^*,Tv^*) &= d(u^*,v^*) \le \phi(N(u^*,v^*)) < N(u^*,v^*) \\ &= \max\left\{ \begin{array}{c} d(u^*,v^*), d(u^*,Tu^*), d(v^*,Tv^*), \\ \frac{d(v^*,Tv^*)[1+d(u^*,Tu^*)]}{1+d(u^*,v^*)}, \frac{d(u^*,Tu^*)[1+d(v^*,Tv^*)]}{1+d(Tu^*,Tv^*)} \end{array} \right\} \\ &= d(u^*,v^*), \end{aligned}$$

which is a contradiction. Hence u^* is the unique fixed point of T.

Finally, we prove that T is discontinuous at u^* if and only if $\lim_{u\to u^*} N(u,u^*) \neq 0$. To do this, we show that T is continuous at u^* if and only if $\lim_{u\to u^*} N(u,u^*) = 0$. Let T be continuous at the fixed point u^* and $u_n \to u^*$. Then $Tu_n \to Tu^* = u^*$ and

$$d(u_n,Tu_n) \le d(u_n,u^*) + d(u^*,Tu_n) \to 0.$$

Hence we get $\lim_n N(u_n,u^*) = 0$. On the other hand, if $\lim_n N(u_n,u^*) = 0$ then $d(u_n,Tu_n) \to 0$ as $u_n \to u^*$. This implies $Tu_n \to u^* = Tu^*$, that is, T is continuous at u^*. ∎

Remark 2.1. (1) *In Theorem 2.1, in the cases where the condition* (2) *is satisfied, we obtain* $d(Tu,Tv) < N(u,v)$ *where* $N(u,v) > 0$. *If* $N(u,v) = 0$ *then* $d(Tu,Tv) = 0$ *and so the inequality* $d(Tu,Tv) \leq \varepsilon$ *holds for any* $u, v \in X$ *with* $\varepsilon < N(u,v) < \varepsilon + \delta$. *This shows that the conditions* (1) *and* (2) *are not independent.*

(2) *It can be also given new fixed-point results on discontinuity at the fixed point using the continuity of the self-mapping* T^2 *(resp. the continuity of the self-mapping* T^p *or the orbitally continuity of the self-mapping* T*) and the number* $N(u,v)$ *(see [2, 3]).*

As the results of Theorem 2.1, we obtain the following corollaries.

Corollary 2.1. *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a self-mapping satisfying the following conditions*:

(1) $d(Tu,Tv) < N(u,v)$ *for any* $u, v \in X$ *with* $N(u,v) > 0$,

(2) *For a given* $\varepsilon > 0$, *there exists a* $\delta(\varepsilon) > 0$ *such that* $\varepsilon < N(u,v) < \varepsilon + \delta$ *implies* $d(Tu,Tv) \leq \varepsilon$.

Then T *has a unique fixed point* $u^* \in X$ *and* $T^n u \to u^*$ *for each* $u \in X$. *Also,* T *is discontinuous at* u^* *if and only if* $\lim\limits_{u \to u^*} N(u,u^*) \neq 0$.

Corollary 2.2. *[20] Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a self-mapping satisfying the following conditions*:

(1) *There exists a function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\phi(d(u,v)) < d(u,v)$ *and* $d(Tu,Tv) \leq \phi(d(u,v))$.

(2) *For a given* $\varepsilon > 0$, *there exists a* $\delta(\varepsilon) > 0$ *such that* $\varepsilon < t < \varepsilon + \delta$ *implies* $\phi(t) \leq \varepsilon$ *for any* $t > 0$.

Then T *has a unique fixed point* $u^* \in X$ *and* $T^n u \to u^*$ *for each* $u \in X$.

We give the following illustrative example of Theorem 2.1.

Example 2.1. *Let* $X = [0,2]$ *be the metric space with the usual metric* $d(u,v) = |u - v|$. *Let us define the self-mapping* $T : X \to X$ *be defined as*

$$Tu = \begin{cases} 1 & \text{if } u \leq 1 \\ 0 & \text{if } u > 1 \end{cases},$$

for all $u \in X$. *Then* T *satisfies the conditions of Theorem 2.1 and has a unique fixed point* $u = 1$. *Indeed, we have*

$$d(Tu,Tv) = 0 \text{ and } 0 < N(u,v) \leq 2 \text{ when } u, v \leq 1,$$

$$d(Tu,Tv) = 0 \text{ and } 2 < N(u,v) \leq 6 \text{ when } u, v > 1,$$

$$d(Tu,Tv) = 1 \text{ and } 1 < N(u,v) \leq 2 \text{ when } u \leq 1, v > 1$$

and

$$d(Tu,Tv) = 1 \text{ and } 1 < N(u,v) \leq 2 \text{ when } u > 1, v \geq 1.$$

Then T *satisfies the condition* (1) *given in Theorem 2.1 with*

$$\phi(t) = \begin{cases} 1 & \text{if } t > 1 \\ \frac{t}{2} & \text{if } t \leq 1 \end{cases}$$

and also T satisfies the condition (2) *given in Theorem 2.1 with*

$$\delta(\varepsilon) = \begin{cases} 5 & \text{if} \quad \varepsilon \geq 1 \\ 5-\varepsilon & \text{if} \quad \varepsilon < 1 \end{cases} .$$

It can be easily seen that $\lim\limits_{u\to 1} N(u,1) \neq 0$ *and so T is discontinuous at the fixed point* $u = 1$.

Now we see that the power contraction of the type $N(u,v)$ allows the possibility of discontinuity at the fixed point with the number

$$N^*(u,v) = \max\left\{ \begin{array}{c} d(u,v), d(u,T^m u), d(v,T^m v), \\ \frac{d(v,T^m v)[1+d(u,T^m u)]}{1+d(u,v)}, \frac{d(u,T^m u)[1+d(v,T^m v)]}{1+d(T^m u,T^m v)} \end{array} \right\}.$$

Theorem 2.2. *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a self-mapping satisfying the following conditions*:

(1) *There exists a function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\phi(t) < t$ *for each* $t > 0$ *and*

$$d(T^m u, T^m v) \leq \phi(N^*(u,v)).$$

(2) *For a given* $\varepsilon > 0$, *there exists a* $\delta(\varepsilon) > 0$ *such that* $\varepsilon < N^*(u,v) < \varepsilon + \delta$ *implies* $d(T^m u, T^m v) \leq \varepsilon$.

Then T has a unique fixed point $u^* \in X$. *Also, T is discontinuous at* u^* *if and only if* $\lim\limits_{u\to u^*} N^*(u,u^*) \neq 0$.

Proof. Using Theorem 2.1, we see that the function T^m has a unique fixed point u^*, that is, $T^m u^* = u^*$. Hence we get

$$Tu^* = TT^m u^* = T^m T u^*$$

and so Tu^* is a fixed point of T^m. From the uniqueness of the fixed point, we obtain $Tu^* = u^*$. Consequently, T has a unique fixed point. ■

3 Some New Results on Discontinuity at Fixed Point on Complex Valued Metric Spaces

In this section, we give a new solution of the *Open Question* D on a complex valued metric space. At first, we recall the following background.

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \Leftrightarrow Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(*i*) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
(*ii*) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
(*iii*) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
(*iv*) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

It is written $z_1 \precsim\!\!\!\!\!\not\;\; z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and it is written $z_1 \prec z_2$ if only (iii) is satisfied. Also,

$$0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2, z_2 \prec z_3 \Longrightarrow z_1 \prec z_3.$$

Definition 3.1. *[1] Let X be a nonempty set and $d_C : X \times X \to \mathbb{C}$ a mapping satisfying the following conditions*:

(1) $0 \precsim d_C(u,v)$ *for all* $u,v \in X$ *and* $d_C(u,v) = 0$ *if and only if* $u = v$,

(2) $d_C(u,v) = d_C(v,u)$ *for all* $u,v \in X$,

(3) $d_C(u,v) \precsim d_C(u,w) + d_C(w,v)$ *for all* $u,v,w \in X$.

Then d_C is called a complex valued metric on X and (X,d_C) is called a complex valued metric space.

Definition 3.2. *[1] Let (X,d_C) be a complex valued metric space, $\{u_n\}$ be a sequence in X and $u \in X$.*

(1) *If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_C(u_n,u) \prec c$, then $\{u_n\}$ is said to be convergent and $\{u_n\}$ converges to u. It is denoted by $\lim_n u_n = u$ or $u_n \to u$ as $n \to \infty$.*

(2) *If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_C(u_n,u_{n+m}) \prec c$, then $\{u_n\}$ is called a Cauchy sequence in (X,d_C).*

(3) *If every Cauchy sequence is convergent in (X,d_C) then (X,d_C) is called a complete complex valued metric space.*

Lemma 3.1. *[1] Let (X,d_C) be a complex valued metric space and $\{u_n\}$ be a sequence in X.*

(1) *$\{u_n\}$ converges to u if and only if $|d_C(u_n,u)| \to 0$ as $n \to \infty$.*

(2) *$\{u_n\}$ is a Cauchy sequence if and only if $|d_C(u_n,u_{n+m})| \to 0$ as $n \to \infty$.*

Definition 3.3. *[21] The "max" function is defined for the partial order relation $\precsim$ as follow*:

(1) $\max\{z_1,z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.

(2) $z_1 \precsim \max\{z_2,z_3\} \Rightarrow z_1 \precsim z_2$ *or* $z_1 \precsim z_3$.

(3) $\max\{z_1,z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ *or* $|z_1| < |z_2|$.

Lemma 3.2. *[21] Let $z_1,z_2,z_3,\ldots \in \mathbb{C}$ and the partial order relation $\precsim$ be defined on $\mathbb{C}$. Then the following statements are satisfied*:

(1) *If* $z_1 \precsim \max\{z_2,z_3\}$ *then* $z_1 \precsim z_2$ *if* $z_3 \precsim z_2$,

(2) *If* $z_1 \precsim \max\{z_2,z_3,z_4\}$ *then* $z_1 \precsim z_2$ *if* $\max\{z_3,z_4\} \precsim z_2$,

(3) *If* $z_1 \precsim \max\{z_2,z_3,z_4,z_5\}$ *then* $z_1 \precsim z_2$ *if* $\max\{z_3,z_4,z_5\} \precsim z_2$, *and so on.*

Now we give the following theorem.

Theorem 3.1. *Let (X,d_C) be a complete complex valued metric space and $T : X \to X$ be a self-mapping satisfying the following conditions*:

(1) *There exists a function $\chi : \mathbb{C} \to \mathbb{C}$ such that $\chi(t) \prec t$ for each $0 \prec t$ and*

$$d_C(Tu,Tv) \precsim \chi(N_C(u,v)),$$

where

$$N_C(u,v)=\max\left\{\begin{array}{c} d_C(u,v), d_C(u,Tu), d_C(v,Tv), \\ \frac{d_C(v,Tv)[1+d_C(u,Tu)]}{1+d_C(u,v)}, \frac{d_C(u,Tu)[1+d_C(v,Tv)]}{1+d_C(Tu,Tv)} \end{array}\right\},$$

for all $u,v\in X$.

(2) *For a given* $0\prec\varepsilon$, *there exists a* $0\prec\delta(\varepsilon)$ *such that* $\varepsilon\prec N_C(u,v)\prec\varepsilon+\delta$ *implies* $d_C(Tu,Tv)\precsim\varepsilon$.

Then T *has a unique fixed point* $u^*\in X$ *and* $|d_C(T^n u,u^*)|\to 0$ *for each* $u\in X$. *Also,* T *is discontinuous at* u^* *if and only if* $\lim_{u\to u^*}|N_C(u,u^*)|\neq 0$.

Proof. Let $u_0\in X$, $Tu_0\neq u_0$ and the sequence $\{u_n\}$ be defined as $Tu_n=u_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$. Using the condition (1), we have

$$\begin{aligned} d_C(u_n,u_{n+1}) &= d_C(Tu_{n-1},Tu_n)\precsim\chi(N_C(u_{n-1},u_n))\prec N_C(u_{n-1},u_n)\\ &= \max\left\{\begin{array}{c} d_C(u_{n-1},u_n), d_C(u_{n-1},Tu_{n-1}), d_C(u_n,Tu_n), \\ \frac{d_C(u_n,Tu_n)[1+d_C(u_{n-1},Tu_{n-1})]}{1+d_C(u_{n-1},u_n)}, \frac{d_C(u_{n-1},Tu_{n-1})[1+d_C(u_n,Tu_n)]}{1+d_C(Tu_{n-1},Tu_n)} \end{array}\right\}\\ &= \max\left\{\begin{array}{c} d_C(u_{n-1},u_n), d_C(u_{n-1},u_n), d_C(u_n,u_{n+1}), \\ \frac{d_C(u_n,u_{n+1})[1+d_C(u_{n-1},u_n)]}{1+d_C(u_{n-1},u_n)}, \frac{d_C(u_{n-1},u_n)[1+d_C(u_n,u_{n+1})]}{1+d_C(u_n,u_{n+1})} \end{array}\right\}\\ &= \max\left\{d_C(u_{n-1},u_n), d_C(u_n,u_{n+1})\right\}. \end{aligned} \tag{3.1}$$

Assume that $d_C(u_{n-1},u_n)\prec d_C(u_n,u_{n+1})$. Then using the inequality (3.1) and Definition 3.3, we have

$$d_C(u_n,u_{n+1})\prec d_C(u_n,u_{n+1})$$

and so

$$|d_C(u_n,u_{n+1})|<|d_C(u_n,u_{n+1})|,$$

which is a contradiction. Hence it should be $d_C(u_n,u_{n+1})\prec d_C(u_{n-1},u_n)$. If we put $d_C(u_n,u_{n+1})=c_n$ then from the inequality (3.1), we get

$$c_n\prec c_{n-1}, \tag{3.2}$$

that is,

$$|c_n|<|c_{n-1}|.$$

So the sequence c_n tends to a limit $0\precsim c$. Suppose that $0\prec c$. There exists a positive integer $k\in\mathbb{N}$ such that $n\geq k$ implies

$$c\prec c_n\prec c+\delta(c). \tag{3.3}$$

Using the condition (2) and the inequality (3.2), we get

$$d_C(Tu_{n-1},Tu_n)=d_C(u_n,u_{n+1})=c_n\prec c, \tag{3.4}$$

for $n\geq k$. The inequality (3.4) contradicts to the inequality (3.3). Then it should be $c=0$.

Now we show that $\{u_n\}$ is a Cauchy sequence. Let us fix an $0\prec\varepsilon$. Without loss of generality, we can assume that $\delta(\varepsilon)\prec\varepsilon$. There exists $k\in\mathbb{N}$ such that

$$d_C(u_n,u_{n+1})=c_n\prec\delta$$

and $\delta^2 < \varepsilon$ for $n \geq k$ since $c_n \to 0$. Following Jachymski (see [8, 9] for more details), using the mathematical induction, we prove

$$d_C(u_k, u_{k+n}) \prec \varepsilon + \delta, \tag{3.5}$$

for any $n \in \mathbb{N}$. The inequality (3.5) holds for $n = 1$ since

$$d_C(u_k, u_{k+1}) = c_k \prec \delta \prec \varepsilon + \delta.$$

Assume that the inequality (3.5) is true for some n. We prove it for $n + 1$. Using the triangle inequality for the complex valued metric, we obtain

$$d_C(u_k, u_{k+n+1}) \precsim d_C(u_k, u_{k+1}) + d_C(u_{k+1}, u_{k+n+1}).$$

It suffices to show $d_C(u_{k+1}, u_{k+n+1}) \precsim \varepsilon$. To do this, we prove $N_C(u_k, u_{k+n}) \precsim \varepsilon + \delta$, where

$$N_C(u_k, u_{k+n}) = \left\{ \begin{array}{c} d_C(u_k, u_{k+n}), d_C(u_k, Tu_k), d_C(u_{k+n}, Tu_{k+n}), \\ \frac{d_C(u_{k+n}, Tu_{k+n})[1+d_C(u_k, Tu_k)]}{1+d_C(u_k, u_{k+n})}, \frac{d_C(u_k, Tu_k)[1+d_C(u_{k+n}, Tu_{k+n})]}{1+d_C(Tu_k, Tu_{k+n})} \end{array} \right\}. \tag{3.6}$$

Using the mathematical induction hypothesis, we find

$$\begin{aligned} d_C(u_k, u_{k+n}) &\prec \varepsilon + \delta, \\ d_C(u_k, Tu_k) &\prec \delta \prec \varepsilon + \delta, \\ d_C(u_{k+n}, Tu_{k+n}) &\prec \delta \prec \varepsilon + \delta, \\ \frac{d_C(u_{k+n}, Tu_{k+n})[1 + d_C(u_k, Tu_k)]}{1 + d_C(u_k, u_{k+n})} &\prec \delta + \delta^2 \prec \varepsilon + \delta, \\ \frac{d_C(u_k, Tu_k)[1 + d_C(u_{k+n}, Tu_{k+n})]}{1 + d_C(Tu_k, Tu_{k+n})} &\prec \delta + \delta^2 \prec \varepsilon + \delta. \end{aligned} \tag{3.7}$$

Using the conditions (3.6) and (3.7), we have

$$N_C(u_k, u_{k+n}) \prec \varepsilon + \delta.$$

From the condition (2), we obtain

$$d_C(Tu_k, Tu_{k+n}) = d_C(u_{k+1}, u_{k+n+1}) \precsim \varepsilon.$$

Therefore, the inequality (3.5) implies that $\{u_n\}$ is Cauchy. Since (X, d_C) is a complete complex valued metric space, there exists a point $u^* \in X$ such that $|d_C(u_n, u^*)| \to 0$ as $n \to \infty$. Also we get $|d_C(Tu_n, u^*)| \to 0$.

Now we show that $Tu^* = u^*$. On the contrary, suppose that u^* is not a fixed point of T, that is, $Tu^* \neq u^*$. Then using the condition (1), we get

$$\begin{aligned} d_C(Tu^*, Tu_n) &\precsim \chi(N_C(u^*, u_n)) \prec N_C(u^*, u_n) \\ &= \max \left\{ \begin{array}{c} d_C(u^*, u_n), d_C(u^*, Tu^*), d_C(u_n, u_{n+1}), \\ \frac{d_C(u_n, u_{n+1})[1+d_C(u^*, Tu^*)]}{1+d_C(u^*, u_n)}, \frac{d_C(u^*, Tu^*)[1+d_C(u_n, u_{n+1})]}{1+d_C(Tu^*, u_{n+1})} \end{array} \right\} \end{aligned}$$

and so taking limit for $n \to \infty$ we have

$$d_C(Tu^*, u^*) \prec \frac{d_C(u^*, Tu^*)}{1 + d_C(Tu^*, u^*)},$$

that is

$$|d_C(Tu^*, u^*)| < \frac{|d_C(u^*, Tu^*)|}{|1 + d_C(Tu^*, u^*)|},$$

which is a contradiction. Thus u^* is a fixed point of T. We prove that the fixed point u^* is unique. Let v^* be another fixed point of T such that $u^* \neq v^*$. By the condition (1), we find

$$\begin{aligned} d_C(Tu^*, Tv^*) &= d_C(u^*, v^*) \precsim \chi(N_C(u^*, v^*)) \prec N_C(u^*, v^*) \\ &= \max \left\{ \begin{array}{c} d_C(u^*, v^*), d_C(u^*, Tu^*), d_C(v^*, Tv^*), \\ \frac{d_C(v^*, Tv^*)[1 + d_C(u^*, Tu^*)]}{1 + d_C(u^*, v^*)}, \frac{d_C(u^*, Tu^*)[1 + d_C(v^*, Tv^*)]}{1 + d_C(Tu^*, Tv^*)} \end{array} \right\} \\ &= d_C(u^*, v^*), \end{aligned}$$

which is a contradiction. Hence u^* is the unique fixed point of T.

Finally, we prove that T is discontinuous at u^* if and only if $\lim_{u \to u^*} |N_C(u, u^*)| \neq 0$. To do this, we show that T is continuous at u^* if and only if $\lim_{u \to u^*} |N_C(u, u^*)| = 0$. Let T be continuous at the fixed point u^* and $u_n \to u^*$. Then $Tu_n \to Tu^* = u^*$ and

$$d_C(u_n, Tu_n) \precsim d_C(u_n, u^*) + d_C(u^*, Tu_n),$$

that is

$$|d_C(u_n, Tu_n)| \leq |d_C(u_n, u^*)| + |d_C(u^*, Tu_n)| \to 0.$$

Hence we get $\lim_n |N_C(u_n, u^*)| = 0$. On the other hand, if $\lim_n |N_C(u_n, u^*)| = 0$ then $|d_C(u_n, Tu_n)| \to 0$ as $u_n \to u^*$. This implies $Tu_n \to u^* = Tu^*$, that is, T is continuous at u^*. ■

Now we give the following example.

Example 3.1. *If we consider the self-mapping $T : X \to X$ defined in Example 2.1, then T satisfies the conditions of Theorem 3.1. Consequently, T has a unique fixed point $u = 1$ and T discontinuous at the fixed point $u = 1$ since $\lim_{u \to 1} |N_C(u, 1)| \neq 0$.*

By the similar arguments used in the proof of Theorem 2.2 and the number

$$N_C^*(u, v) = \max \left\{ \begin{array}{c} d_C(u, v), d_C(u, T^m u), d_C(v, T^m v), \\ \frac{d_C(v, T^m v)[1 + d_C(u, T^m u)]}{1 + d_C(u, v)}, \frac{d_C(u, T^m u)[1 + d_C(v, T^m v)]}{1 + d_C(T^m u, T^m v)} \end{array} \right\},$$

we obtain the following theorem.

Theorem 3.2. *Let (X, d_C) be a complete complex valued metric space and $T : X \to X$ a self-mapping satisfying the following conditions*:

(1) *There exists a function* $\chi : \mathbb{C} \to \mathbb{C}$ *such that* $\chi(t) \prec t$ *for each* $0 \prec t$ *and*

$$d_C(T^m u, T^m v) \precsim \chi(N_C^*(u,v)).$$

(2) *For a given* $0 \prec \varepsilon$*, there exists a* $0 \prec \delta(\varepsilon)$ *such that* $\varepsilon \prec N_C^*(u,v) \prec \varepsilon + \delta$ *implies* $d_C(T^m u, T^m v) \precsim \varepsilon$.

Then T *has a unique fixed point* $u^* \in X$*. Also,* T *is discontinuous at* u^* *if and only if* $\lim\limits_{u \to u^*} \left| N_C^*(u,u^*) \right| \neq 0$.

We note that every complex valued metric space (X, d_C) is metrizable by the real valued metric defined as $d_* (u,v) = \max \{Re(d_C (u,v)), Im(d_C (u,v))\}$ such that the metrics d_C and d_* induce the same topology on X (see [19] for the necessary background). However, the classes of contractive mappings with respect to two metrics need not to be same. On the other hand, complex valued functions have many applications in various areas such as activation functions in neural networks, signal analysis, control theory, geometry, fractals etc.

4 Some Fixed-Circle Results using the number $\mathbf{N}(\mathbf{u}, \mathbf{v})$

In recent years, the fixed-circle problem has been considered as a new direction of extension of the fixed-point results (see [13, 14]). In this section, we obtain new fixed-circle results using the number $N(u,v)$. At first, we recall some necessary notions.

Let (X, d) be a metric space. Then a circle and a disc are defined on a metric space as follows, respectively:

$$C_{u_0,r} = \{u \in X : d(u,u_0) = r\}$$

and

$$D_{u_0,r} = \{u \in X : d(u,u_0) \leq r\}.$$

Definition 4.1. *[13] Let* (X, d) *be a metric space,* $C_{u_0,r}$ *be a circle and* $T : X \to X$ *be a self-mapping. If* $Tu = u$ *for every* $u \in C_{u_0,r}$ *then the circle* $C_{u_0,r}$ *is called as the fixed circle of* T.

Definition 4.2. *[23] Let* $\mathbb{F}$ *be the family of all functions* $F : (0, \infty) \to \mathbb{R}$ *such that*

(F_1) *F is strictly increasing,*

(F_2) *For each sequence* $\{\alpha_n\}$ *in* $(0, \infty)$ *the following holds*

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty,$$

(F_3) *There exists* $k \in (0,1)$ *such that* $\lim\limits_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

Some functions satisfying the conditions (F_1), (F_2) and (F_3) of Definition 4.2 are $F(x) = \ln(x)$, $F(x) = \ln(x) + x$, $F(x) = -\frac{1}{\sqrt{x}}$ and $F(x) = \ln(x^2 + x)$ (see [23]).

Now we define a new type contraction which generates fixed circles.

Definition 4.3. *Let (X,d) be a metric space and $N(u,v)$ be defined as in (1.1). A self-mapping T on X is said to be $F_{C-N_{u_0}}$-contraction on X if there exist $F \in \mathbb{F}$, $t > 0$ and $u_0 \in X$ such that for all $u \in X$ the following holds:*

$$d(Tu,u) > 0 \Longrightarrow t + F(d(Tu,u)) \leq F(N(u,u_0)).$$

Using these types contractions, we prove the following fixed-circle theorem.

Theorem 4.1. *Let (X,d) be a metric space, T be an $F_{C-N_{u_0}}$-contractive self-mapping with $u_0 \in X$ and $r = \inf\{d(Tu,u) : Tu \neq u\}$. If $Tu_0 = u_0$ then $C_{u_0,r}$ is a fixed circle of T.*

Proof. Let $u \in C_{u_0,r}$. Assume that $Tu \neq u$. By the definition of r, we have $d(Tu,u) \geq r$. Then using the $F_{C-N_{u_0}}$-contractive property, the hypothesis $Tu_0 = u_0$ and the fact that F is increasing, we have

$$\begin{aligned} F(r) &\leq F(d(Tu,u)) \leq F(N(u,u_0)) - t < F(N(u,u_0)) \\ &= F\left(\max\left\{\begin{array}{c} d(u,u_0), d(u,Tu), d(u_0,Tu_0) \\ \frac{d(u_0,Tu_0)[1+d(u,Tu)]}{1+d(u,u_0)}, \frac{d(u,Tu)[1+d(u_0,Tu_0)]}{1+d(Tu,Tu_0)} \end{array}\right\}\right) \\ &= F\left(\max\left\{r, d(u,Tu), 0, 0, \frac{d(u,Tu)}{1+d(Tu,u_0)}\right\}\right) \\ &= F(d(u,Tu)), \end{aligned} \tag{4.1}$$

which is a contradiction. Consequently, it should be $Tu = u$ and $C_{u_0,r}$ is a fixed circle of T. ∎

Proposition 4.1. *Let (X,d) be a metric space, T be an $F_{C-N_{u_0}}$-contractive self-mapping with $u_0 \in X$ and $r = \inf\{d(Tu,u) : Tu \neq u\}$. If $Tu_0 = u_0$ then T fixes every circle $C_{u_0,\rho}$ with $\rho < r$.*

Proof. Let $u \in C_{u_0,\rho}$ and $d(Tu,u) > 0$. By the $F_{C-N_{u_0}}$-contractive property, the hypothesis $Tu_0 = u_0$ and the fact that F is increasing, we have

$$\begin{aligned} F(d(Tu,u)) &\leq F(N(u,u_0)) - t < F(N(u,u_0)) \\ &= F\left(\max\left\{\rho, d(u,Tu), 0, 0, \frac{d(u,Tu)}{1+d(Tu,u_0)}\right\}\right) \\ &= F(d(u,Tu)), \end{aligned} \tag{4.2}$$

which is a contradiction since $d(u,Tu) \geq r > \rho$. Consequently, it should be $Tu = u$ and T fixes every circle $C_{u_0,\rho}$ with $\rho < r$. ∎

As an immediate result of Theorem 4.1 and Proposition 4.1, we obtain the following corollary.

Corollary 4.1. *Let (X,d) be a metric space, T be an $F_{C-N_{u_0}}$-contractive self-mapping with $u_0 \in X$ and $r = \inf\{d(Tu,u) : Tu \neq u\}$. If $Tu_0 = u_0$ then T fixes the disc $D_{u_0,r}$.*

Proof. It follows by Theorem 4.1 and Proposition 4.1. ∎

In the following example we see that the converse statement of Theorem 4.1 is not always true.

Example 4.1. *Let $X = \mathbb{R}$ be the metric space with the usual metric and the self-mapping $T : X \to X$ be defined as*

$$Tu = \begin{cases} u & \text{if} \quad |u-3| \leq r \\ 3 & \text{if} \quad |u-3| > r \end{cases},$$

for all $u \in X$ with any $r > 0$. Then T is not an $F_{C-N_{u_0}}$-contractive self-mapping for the point $u_0 = 3$ but T fixes every circle $C_{3,\rho}$ where $\rho \leq r$.

We give the following example.

Example 4.2. *Let $X = \mathbb{R}$ be the metric space with the usual metric. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Tu = \begin{cases} u & \text{if} \quad |u+1| < 2 \\ u + \frac{1}{2} & \text{if} \quad |u+1| \geq 2 \end{cases},$$

for all $u \in \mathbb{R}$. The self-mapping T is an $F_{C-N_{u_0}}$-contractive self-mapping with $F = \ln u$, $t = \ln 4$ and $u_0 = -1$. Indeed, we get

$$d(Tu, u) = \left| u + \frac{1}{2} - u \right| = \frac{1}{2} \neq 0,$$

for all $u \in \mathbb{R}$ such that $|u+1| \geq 2$. Then we have

$$\begin{aligned} \ln 4 + \ln\left(\frac{1}{2}\right) &\leq \ln\left(|u+1|\right) \\ &= \ln\left(\max\left\{\begin{array}{c} |u+1|, \frac{1}{2}, 0, \\ \frac{|-1+1|\left[1+\left|u-u-\frac{1}{2}\right|\right]}{1+|u+1|}, \frac{\left|u-u-\frac{1}{2}\right|[1+|-1+1|]}{1+\left|u+\frac{1}{2}+1\right|} \end{array}\right\}\right) \\ &= \ln\left(N\left(u,-1\right)\right) \\ \implies & t + F(d(Tu,u)) \leq F\left(N\left(u,-1\right)\right). \end{aligned}$$

Clearly, we have

$$r = \min\left\{d(Tu, u) : Tu \neq u\right\} = \frac{1}{2}$$

and the circle $C_{-1,\frac{1}{2}} = \{-\frac{3}{2}, -\frac{1}{2}\}$ is a fixed circle of T.

Now we construct a new technique to obtain new fixed-circle results. We give the following definition.

Definition 4.4. *Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. Then T is called N_{u_0}-type contraction if there exists an $u_0 \in X$ and a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ satisfying*

$$d(Tu, u) \leq \phi(N(u, u_0)),$$

for all $u \in X$.

Using the N_{u_0}-type contractive property, we get the following fixed-circle theorem.

Theorem 4.2. *Let (X,d) be a metric space, $T : X \to X$ be a self-mapping and $r = \inf\{d(Tu,u) : Tu \neq u\}$. If T is an N_{u_0}-type contraction with $u_0 \in X$ and $Tu_0 = u_0$ then T fixes the circle $C_{u_0,r}$.*

Proof. Let $u \in C_{u_0,r}$. Suppose that $Tu \neq u$. Using the N_{u_0}-type contractive condition with $Tu_0 = u_0$, we get

$$\begin{aligned} d(Tu,u) &\leq \phi(N(u,u_0)) < N(u,u_0) \\ &= \max\left\{r, d(u,Tu), 0, 0, \frac{d(u,Tu)}{1+d(Tu,u_0)}\right\} \\ &= d(u,Tu), \end{aligned} \tag{4.3}$$

which is a contradiction since $r = \inf\{d(Tu,u) : Tu \neq u\}$. Consequently, it should be $Tu = u$ and $C_{u_0,r}$ is a fixed circle of T. ∎

As a result of Theorem 4.2, we obtain the following corollary.

Corollary 4.2. *Let (X,d) be a metric space, T be an N_{u_0}-type contraction with $u_0 \in X$ and $r = \inf\{d(Tu,u) : Tu \neq u\}$. If $Tu_0 = u_0$ then T fixes the disc $D_{u_0,r}$.*

Proof. Using similar arguments as used in the proofs of Theorem 4.2 and Proposition 4.1, it can be easily checked that T fixes the disc $D_{u_0,r}$. ∎

We give the following example.

Example 4.3. *Let $X = \mathbb{C}$ be the metric space with the usual metric. Let us define the self-mapping $T : \mathbb{C} \to \mathbb{C}$ as*

$$Tu = \begin{cases} u & \text{if } |u| < 8 \\ u+3 & \text{if } |u| \geq 8 \end{cases},$$

for all $u \in \mathbb{C}$. The self-mapping T is an N_{u_0}-type contractive self-mapping with $\phi(t) = \frac{t}{2}$ and $u_0 = 0$. Indeed, we get

$$d(Tu,u) = |u-u| = 0, \tag{4.4}$$

for all $u \in \mathbb{R}$ such that $|u| < 8$ and

$$d(Tu,u) = |u+3-u| = 3, \tag{4.5}$$

for all $u \in \mathbb{R}$ such that $|u| \geq 8$. Then using the equality (4.4), we have

$$\begin{aligned} 0 &\leq \phi(N(u,0)) = \phi\left(\max\left\{\begin{array}{c} d(u,0), d(u,Tu), d(0,T0), \\ \frac{d(0,T0)[1+d(u,Tu)]}{1+d(u,0)}, \frac{d(u,Tu)[1+d(0,T0)]}{1+d(Tu,T0)} \end{array}\right\}\right) \\ &= \phi(|u|) = \frac{|u|}{2} \end{aligned}$$

and using the equality (4.5), we get

$$\begin{aligned} 3 &\leq \phi(N(u,0)) = \phi\left(\max\left\{\begin{array}{c} d(u,0), d(u,Tu), d(0,T0), \\ \frac{d(0,T0)[1+d(u,Tu)]}{1+d(u,0)}, \frac{d(u,Tu)[1+d(0,T0)]}{1+d(Tu,T0)} \end{array}\right\}\right) \\ &= \phi\left(\max\left\{|u|, 3, 0, 0, \frac{3}{1+|u+3|}\right\}\right) = \phi(|u|) = \frac{|u|}{2}. \end{aligned}$$

Clearly, we have

$$r = \min\{d(Tu,u) : Tu \neq u\} = 3$$

and the circle $C_{0,3}$ is a fixed circle of T.

We note that discontinuity of any self-mapping T on its fixed circle can be determined using the number $N(u,v)$ defined in (1.1). We give the following proposition.

Proposition 4.2. *Let (X,d) be a metric space, T a self-mapping on X and $C_{u_0,r}$ a fixed circle of T. Then T is discontinuous at any $u \in C_{u_0,r}$ if and only if $\lim\limits_{z \to u} N(z,u) \neq 0$.*

Proof. Let T be a continuous self-mapping at $u \in C_{u_0,r}$ and $u_n \to u$. Then $Tu_n \to Tu = u$ and $d(u_n, Tu_n) \to 0$. Hence we get

$$\lim_n N(u_n,u) = \lim_n \left(\max\left\{d(u,u_n), d(u_n,Tu_n), \frac{d(u_n,Tu_n)[1+d(u,Tu)]}{1+d(u,u_n)}\right\}\right) = 0.$$

Conversely, if $\lim\limits_{u_n \to u} N(u_n,u) = 0$ then $d(u_n,Tu_n) \to 0$ as $u_n \to u$. This implies $Tu_n \to u = Tu$, that is, T is continuous at u. ■

Example 4.4. *If we consider the function T defined in Example 4.2 then it is easy to check that T satisfies the conditions of Theorem 4.1 for the circle $C_{-1,\frac{1}{2}} = \{-\frac{3}{2}, -\frac{1}{2}\}$. By the above proposition, it can be easily deduced that the function T is continuous on its fixed circle.*

5 An Application to Complex-Valued Activation Functions

In the past decades, real and complex-valued neural networks with discontinuous activation functions have emerged as an important area of research. For example, in [6], global convergence of neural networks with discontinuous neuron activations was studied. In [12], the problem of multistability was examined for competitive neural networks associated with discontinuous non-monotonic piecewise linear activation functions. In [22], some theoretical results were presented on dynamical behavior of complex-valued neural networks with discontinuous neuron activations. In [11], the multistability issue is considered for the complex-valued neural networks with discontinuous activation functions and time-varying delays using geometrical properties of the discontinuous activation functions and the Brouwer's fixed point theory. Recently, some theoretical results on the fixed-point (resp. the fixed-circle) problem have been applied to real-valued discontinuous activation functions (see [13, 14, 20] for more details).

By these motivations, we investigate some applications of our obtained results to real or complex-valued discontinuous activation functions.

In [10], the authors considered some partitioned activation functions for real numbers. For example, the typical form of these activation functions is

$$f(x)=\begin{cases} f_0(x) & , \quad x<0 \\ f_1(x) & , \quad x\geq 0 \end{cases},$$

where f_0 and f_1 are local functions. Also this typical form was generalized as follows:

$$f(x)=\begin{cases} f_0(x) & , \quad x<x_0 \\ f_1(x) & , \quad x_0<x\leq x_1 \\ & \vdots \\ f_{n-1}(x) & , \quad x_{n-2}<x\leq x_{n-1} \\ f_n(x) & , \quad x_{n-1}<x \end{cases}. \tag{5.1}$$

If we consider the following example of a partitioned activation function defined as

$$f(x)=\begin{cases} 0 & , \quad x<0 \\ x^2-27x+192 & , \quad x\geq 0 \end{cases},$$

for all $x\in\mathbb{R}$, then the function f fixes the points $x_1=12$, $x_2=16$. The function f is continuous at the fixed points $x_1=12$, $x_2=16$. This follows easily by calculating the following equation

$$\lim_{u\to x} N(u,x)=0.$$

These fixed points can be also considered on a circle. Using the usual metric, we deduce that the circle $C_{14,2}=\{12,16\}$ is the fixed circle of f and f is continuous on its fixed circle.

If we use a generalized form of the typical activation functions defined as in (5.1), then our discontinuity and fixed-circle results will important for determining the fixed points and discontinuity points.

The usage of a complex-valued neural network can be lead many advantages. For example, from [11], we know that it would be better to choose the complex-valued networks instead of the real-valued ones for the high-capacity associative memory tasks.

Now we consider the complex function $f_k(v)$ defined in [11] as

$$f_k(v)=f_k^R(\widetilde{v})+if_k^I(\widehat{v}),$$

where $v=\widetilde{v}+i\widehat{v}$ with $\widetilde{v},\widehat{v}\in\mathbb{R}$ and $f_k^R(.)$, $f_k^I(.):\mathbb{R}\to\mathbb{R}$ are discontinuous functions defined as follows:

$$f_k^R(\widetilde{v})=\begin{cases} \mu_k & , \quad -\infty<\widetilde{v}<r_k \\ f_{k,1}^R(\widetilde{v}) & , \quad r_k\leq\widetilde{v}\leq s_k \\ f_{k,2}^R(\widetilde{v}) & , \quad s_k<\widetilde{v}\leq p_k \\ \omega_k & , \quad p_k<\widetilde{v}<+\infty \end{cases}$$

and

$$f_k^I(\widehat{v}) = \begin{cases} \overline{\mu_k} & , \quad -\infty < \widehat{v} < \overline{r_k} \\ f_{k,1}^I(\widehat{v}) & , \quad \overline{r_k} \le \widehat{v} \le \overline{s_k} \\ f_{k,2}^I(\widehat{v}) & , \quad \overline{s_k} < \widehat{v} \le \overline{p_k} \\ \overline{\omega_k} & , \quad \overline{p_k} < \widehat{v} < +\infty \end{cases} ,$$

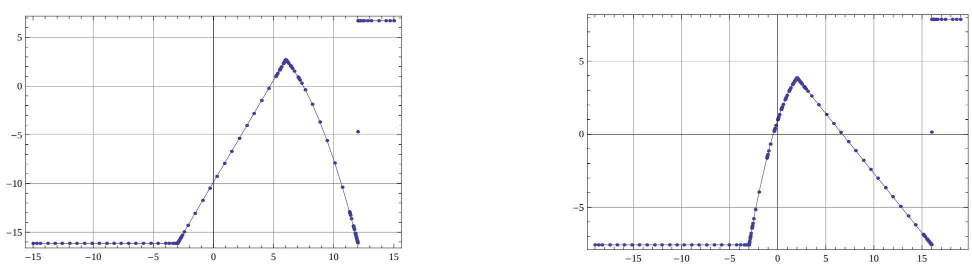
in which $f_k^R(s_k) = f_{k,2}^R(s_k)$, $f_k^I(\overline{s_k}) = f_{k,2}^I(\overline{s_k})$, $f_{k,1}^R(r_k) = f_{k,2}^R(p_k) = \mu_k$, $f_{k,1}^I(\overline{r_k}) = f_{k,2}^I(\overline{p_k}) = \overline{\mu_k}$, $\omega_k \neq \mu_k, \overline{\omega_k} \neq \overline{\mu_k}$. Then the real and imaginary parts of the function $f_k(v)$, that is, the functions $f_k^R(.)$ and $f_k^I(.)$ are discontinuous at the points p_k and $\overline{p_k}$, respectively. In [11], an example of a two-neuron complex-valued neural network was given using the following activation functions defined as:

$$f_1^R(\eta) = f_2^I(\eta) = \begin{cases} -\frac{113}{7} & , \quad -\infty < \eta < -3 \\ \frac{132}{63}\eta - \frac{621}{63} & , \quad -3 \le \eta \le 6 \\ -\frac{2}{7}\eta^2 + 2\eta + 1 & , \quad 6 < \eta \le 12 \\ \frac{47}{7} & , \quad 12 < \eta < +\infty \end{cases} \tag{5.2}$$

and

$$f_2^R(\eta) = f_1^I(\eta) = \begin{cases} -\frac{53}{7} & , \quad -\infty < \eta < -3 \\ -\frac{2}{7}\eta^2 + 2\eta + 1 & , \quad -3 \le \eta \le 2 \\ -\frac{40}{49}\eta + \frac{269}{49} & , \quad 2 < \eta \le 16 \\ \frac{55}{7} & , \quad 16 < \eta < +\infty \end{cases} , \tag{5.3}$$

whose images are seen in the following figure.



(a) $f_1^R(\eta), f_2^I(\eta)$. (b) $f_2^R(\eta), f_1^I(\eta)$.

Figure 1: The graphs of the activation functions for $k = 1, 2$.

The functions $f_1^R(\eta)$, $f_2^I(\eta)$ defined in (5.2) are discontinuous at the point $\eta = 12$, but this point is not fixed by these functions. Also, the point $\eta = -\frac{113}{7}$ is the fixed point of these functions and they are continuous at this point. Indeed, if we use the number of $N(u, v)$ defined in (1.1), then we have

$$\lim_{u \to \eta} N(u, \eta) = 0,$$

that is, the functions $f_1^R(\eta)$, $f_2^I(\eta)$ are continuous at the fixed point $\eta = -\frac{113}{7}$.

By the similar approaches, the functions $f_2^R(\eta)$ and $f_1^I(\eta)$ defined in (5.3) are discontinuous at the point $\eta = 16$, but this point is not a fixed point of these functions. These functions fix the points $\eta_1 = \frac{269}{89}$ and $\eta_2 = -\frac{53}{7}$ and they are continuous at these points. Alternatively, we can say that the functions $f_2^R(\eta)$,

$f_1^I(\eta)$ have a fixed circle. That is, the circle $C_{-\frac{1417}{623},\frac{3300}{623}} = \left\{-\frac{53}{7}, \frac{269}{89}\right\}$ is the fixed circle both of the functions $f_2^R(\eta)$ and $f_1^I(\eta)$.

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