# Geometric features of general differential solutions

Rosihan M. Ali     See Keong Lee     Saiful R. Mondal

**Abstract**

This papers examines the general differential equation

$$y''(z) + a(z)y'(z) + b(z)y(z) = 0$$

in the unit disk of the complex plane, and finds conditions on the analytic functions $a$ and $b$ that ensures the solutions are Janowski starlike. Also studied is Janowski convexity of solutions to

$$z(1-z)y''(z) + a(z)y'(z) + \alpha y(z) = 0,$$

where $\alpha$ is a given constant. Janowski starlikeness and Janowski convexity encompass various widely studied classes of classical starlikeness and convexity. As application, we give convexity and starlikeness geometric description of solutions to differential equations related to several important special functions.

## 1 Introduction

Let $\mathcal{A}$ denote the class of normalized analytic functions $f$ in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ satisfying $f(0) = 0 = f'(0) - 1$. Denote by $\mathcal{S}^*$ and $\mathcal{C}$ respectively the widely studied subclasses of $\mathcal{A}$ consisting of univalent (one-to-one) starlike and convex functions. Geometrically $f \in \mathcal{S}^*$ if the linear segment $tw$, $0 \leq t \leq 1$, lies completely in $f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$, while $f \in \mathcal{C}$ if $f(\mathbb{D})$ is a

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convex domain. Related to these subclasses is the Cáratheódory class $\mathcal{P}$ consisting of analytic functions $p$ satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$. Analytically, $f \in \mathcal{S}^*$ if $zf'(z)/f(z) \in \mathcal{P}$, while $f \in \mathcal{C}$ if $1 + zf''(z)/f'(z) \in \mathcal{P}$.

For two analytic functions $f$ and $g$ in $\mathbb{D}$, the function $f$ is *subordinate* to $g$, written $f \prec g$, or $f(z) \prec g(z)$, $z \in \mathbb{D}$, if there is an analytic self-map $\omega$ of $\mathbb{D}$ satisfying $\omega(0) = 0$ and $f(z) = g(\omega(z))$, $z \in \mathbb{D}$. Consider now the class $\mathcal{P}[A, B]$ of analytic functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

Here $-1 \leq B < A \leq 1$. This family $\mathcal{P}[A, B]$ has been widely studied, most notably by Janowski in [13]. Many authors in recent studies have referred to the class as the *Janowski class of functions*, which we too shall adopt in this sequel. The class contains several known classes of functions for judicious choices of $A$ and $B$. For instance, if $0 \leq \beta < 1$, then $\mathcal{P}[1 - 2\beta, -1]$ is the class of functions $p(z) = 1 + c_1 z + \cdots$ satisfying $\operatorname{Re} p(z) > \beta$ in $\mathbb{D}$. In the limiting case $\beta = 0$, the class reduces to the classical Cáratheódory class $\mathcal{P}$. It is readily shown that $p \in \mathcal{P}[A, B]$ whenever

$$p(z) = \frac{(1 - A) + (1 + A)\phi(z)}{(1 - B) + (1 + B)\phi(z)}$$

for some $\phi \in \mathcal{P}$.

The class of Janowski starlike functions $\mathcal{S}^*[A, B]$ consists of $f \in \mathcal{A}$ satisfying

$$zf'(z)/f(z) \in \mathcal{P}[A, B],$$

while the Janowski convex functions $\mathcal{C}[A, B]$ are functions $f \in \mathcal{A}$ satisfying $1 + (zf''(z)/f'(z)) \in \mathcal{P}[A, B]$. For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] := \mathcal{S}^*(\beta)$ is the classical class of *starlike functions of order* $\beta$; $\mathcal{S}^*[1 - \beta, 0] := \mathcal{S}^*_\beta = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \beta\}$, and $\mathcal{S}^*[\beta, -\beta] := \mathcal{S}^*[\beta] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta |zf'(z)/f(z) + 1|\}$. These are all classes that have been widely studied, see for example, in the works of [2, 3].

Further to recent works of [4, 14, 22, 25, 26, 30], this article treats the general second-order differential equation

$$c(z)y''(z) + a(z)y'(z) + b(z)y(z) = 0, \tag{1.1}$$

and finds sufficient conditions on the variable coefficients $c$, $a$, and $b$ so that the solution to (1.1) is either Janowski starlike or Janowski convex. With appropriate choices of the functions $a, b$, and $c$, the differential equation (1.1) gives rise to several important differential equations. These include the confluent and Gaussian hypergeometric differential equations, as well as the Bessel differential equation. The paper by Hästö *et al.* in [14] perhaps contains the best known results in the case dealing with the hypergeometric differential equation. It is this generality that piqued our interest to the present work.

In the following section, we look at the differential equation (1.1) with $c(z) = 1$. We find conditions that will ensure its solution is Janowski starlike.

Section 3 deals with Janowski convexity of solutions to (1.1) in the case $c(z) = z(1-z)$ and $b$ a complex constant function. In sections 4 and 5, examples related to several important special functions are constructed to illuminate the geometric features of the general differential solutions.

The principle of differential subordination [20, 21] provides an important tool in the investigation of various classes of analytic functions. One such useful result used in the sequel is the following lemma.

**Lemma 1.1.** *[20, 21] Let $\Omega \subset \mathbb{C}$, and $\Psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfy*

$$\Psi(i\rho, \sigma; z) \notin \Omega$$

*for $z \in \mathbb{D}$, and real $\rho, \sigma$ such that $\sigma \leq -(1+\rho^2)/2$. If $p$ is analytic in $\mathbb{D}$ with $p(0) = 1$, and $\Psi(p(z), zp'(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$.*

## 2 Janowski starlike solutions

In this section, sufficient conditions are obtained that ensure the solution of (1.1) maps $\mathbb{D}$ into a Janowski starlike domain.

**Theorem 2.1.** *Let $a$ and $b$ be two analytic functions defined in $\mathbb{D}$ for which the differential equation*

$$y''(z) + a(z)y'(z) + b(z)y(z) = 0 \tag{2.1}$$

*has a solution $F$ satisfying $F(0) = 0$, $F'(0) = 1$, and $F(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$. Suppose that $-1 \leq B < A \leq 1$, $zF'(z)/F(z) \neq (1+A)/(1+B)$ for $z \in \mathbb{D}$, and*

$$(1+B)\Big((1+A)\operatorname{Re}\{za(z)\} + (1+B)\operatorname{Re}\{z^2b(z)\}\Big) \neq -(A-B)(2+A).$$

*Further, let*

$$\begin{aligned} D_1(A,B;z) &= (1-B^2)\operatorname{Im}\{z^2b(z)\} + (1-AB)\operatorname{Im}\{za(z)\}, \\ D_2(A,B;z) &= (A-B)(2+A) \\ &\quad + (1+B)\Big((1+A)\operatorname{Re}\{za(z)\} + (1+B)\operatorname{Re}\{z^2b(z)\}\Big). \end{aligned} \tag{2.2}$$

*Then $F \in \mathcal{S}^*[A,B]$ if either*

*(a)* $D_2(A,B;z) > 0$, *and*

$$\begin{aligned} &(1-A)(1-B)\operatorname{Re}\{za(z)\} + (1-B)^2\operatorname{Re}\{z^2b(z)\} \\ &\qquad - (A-B)(2-A) < -\frac{(D_1(A,B;z))^2}{D_2(A,B;z)}, \end{aligned} \tag{2.3}$$

*or*

*(b)* $D_2(A,B;z) < 0$, *and*

$$(1-A)(1-B)\operatorname{Re}\{za(z)\} + (1-B)^2\operatorname{Re}\{z^2b(z)\} - (A-B)(2-A) > -\frac{(D_1(A,B;z))^2}{D_2(A,B;z)}.$$

*Proof.* Consider the transformation

$$F(z) = \exp\left(-\frac{1}{2}\int_0^z a(t)dt\right) v(z). \tag{2.4}$$

Then

$$F'(z) = \exp\left(-\frac{1}{2}\int_0^z a(t)dt\right)\left[-\frac{1}{2}a(z)v(z) + v'(z)\right],$$

$$F''(z) = \exp\left(-\frac{1}{2}\int_0^z a(t)dt\right)\left[\left(-\frac{1}{4}a^2(z) - \frac{1}{2}a'(z) + b(z)\right)v(z) + v''(z)\right],$$

and equation (2.1) takes the form

$$v''(z) + \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right)v(z) = 0. \tag{2.5}$$

Next let

$$u(z) := \frac{zv'(z)}{v(z)} - \frac{z}{2}a(z).$$

Since $F(z) \neq 0$ for all $z \in \mathbb{D}\setminus\{0\}$, it follows from (2.4) that $v(z) \neq 0$ for all $z \in \mathbb{D}\setminus\{0\}$. Then (2.5) becomes

$$zu'(z) + (za(z)-1)u(z) + u^2(z) + z^2b(z) = 0. \tag{2.6}$$

Now define the function

$$q(z) := \frac{-(1-A)+(1-B)u(z)}{(1+A)-(1+B)u(z)}.$$

Since $\lim_{z\to 0} u(z) = 1$, it follows that $q(0) = 1$, and $q$ is analytic in $\mathbb{D}$ provided $(1+B)u(z) \neq 1+A$, or equivalently, whenever $zF'(z)/F(z) \neq (1+A)/(1+B)$. A computation gives

$$u(z) = \frac{(1-A)+(1+A)q(z)}{(1-B)+(1+B)q(z)}, \quad u'(z) = \frac{2(A-B)q'(z)}{[(1-B)+(1+B)q(z)]^2},$$

and from (2.6),

$$\begin{aligned}2(A-B)zq'(z) &+ za(z)\big(1-A+(1+A)q(z)\big)\big(1-B+(1+B)q(z)\big)\\ &+ z^2b(z)\big((1-B)+(1+B)q(z)\big)^2\\ &- (A-B)\big(1-q(z)\big)\big(1-A+(1+A)q(z)\big) = 0.\end{aligned} \tag{2.7}$$

To apply the subordination tool, let $\Omega = \{0\}$ and

$$\Psi(r,s;z) := 2(A-B)s + za(z)\big(1-A+(1+A)r\big)\big(1-B+(1+B)r\big) + z^2b(z)\big((1-B)+(1+B)r\big)^2 - (A-B)(1-r)(1-A+(1+A)r).$$

Then (2.7) shows that $\Psi(q(z), zq'(z); z) \in \Omega$. To apply Lemma 1.1, it suffices to show $\operatorname{Re}\Psi(i\rho,\sigma;z) \neq 0$ for $\rho \in \mathbb{R}$, $\sigma \leq -(1+\rho^2)/2$, and $z \in \mathbb{D}$. Now

$$\begin{aligned}
\operatorname{Re}\Psi(i\rho,\sigma;z) &= 2(A-B)\sigma + \operatorname{Re}\left\{za(z)(1-A+(1+A)i\rho)(1-B+(1+B)i\rho)\right\}\\
&\quad + \operatorname{Re}\left\{z^2b(z)(1-B+(1+B)i\rho)^2\right\} - \operatorname{Re}\left\{(A-B)(1-A+2Ai\rho+(1+A)\rho^2)\right\}\\
&< -(A-B)(1+\rho^2) + \operatorname{Re}\left\{za(z)[1-A+(1+A)i\rho][1-B+(1+B)i\rho]\right\}\\
&\quad + \operatorname{Re}\left\{z^2b(z)[1-B+(1+B)i\rho]^2\right\} - (A-B)\left[(1-A)+(1+A)\rho^2\right]\\
&= -(A-B)(2+A)\rho^2 + \operatorname{Re}\{za(z)\}\left[(1-A)(1-B)-(1+A)(1+B)\rho^2\right]\\
&\quad - 2(1-AB)\operatorname{Im}\{za(z)\}\rho + \operatorname{Re}\{z^2b(z)\}\left[(1-B)^2-(1+B)^2\rho^2\right]\\
&\quad - 2(1-B^2)\operatorname{Im}\{z^2b(z)\}\rho - (A-B)(2-A)\\
&= -D_2(A,B;z)\left[\rho + \frac{D_1(A,B;z)}{D_2(A,B;z)}\right]^2 + \frac{(D_1(A,B;z))^2}{D_2(A,B;z)}\\
&\quad + (1-A)(1-B)\operatorname{Re}\{za(z)\} + (1-B)^2\operatorname{Re}\{z^2b(z)\} - (A-B)(2-A), \qquad (2.8)
\end{aligned}$$

where $D_1(A,B;z)$ and $D_2(A,B;z)$ are given by (2.2).

For $-1 \leq B < A \leq 1$ and $z \in \mathbb{D}$, it is clear that $D_2(A,B;z) \neq 0$ unless

$$(1+B)\left((1+A)\operatorname{Re}\{za(z)\} + (1+B)\operatorname{Re}\{z^2b(z)\}\right) = -(A-B)(2+A).$$

For any $\rho \in \mathbb{R}$, it follows from (2.8) and the assumptions of the theorem that $\operatorname{Re}\Psi(i\rho,\sigma;z) \neq 0$. Thus Lemma 1.1 shows that $\operatorname{Re}(q(z)) > 0$, that is, $q(z) \in \mathcal{P}$. Consequently

$$u(z) = \frac{(1-A)+(1+A)q(z)}{(1-B)+(1+B)q(z)}$$

lies in $\mathcal{P}[A,B]$. Since $u(z) = zF'(z)/F(z)$, this implies that $F \in \mathcal{S}^*[A,B]$. ■

**Remark 2.2.** *It would be of interest to find sufficient conditions on the variable coefficients a and b that would ensure the solution to* (2.1) *vanishes only at the origin. Though this general problem seems formidable, in this sequel, we shall demonstrate several examples that show Theorem 2.1 holds true. For instance, in the case the coefficient b is the zero function, then the solution of*

$$y''(z) + a(z)y'(z) = 0$$

*satisfying $y(0) = 0$ and $y'(0) = 1$ is given by*

$$y(z) = \int_0^z exp\left(-\int_0^s a(t)dt\right) ds.$$

*Clearly there is a large class of functions a that ensure the non-vanishing solutions of* (2.1) *in the punctured unit disk.*

For $\beta \in [0,1)$, choosing $A = 1-2\beta$, and $B = -1$ in Theorem 2.1(a), leads to the following result.

**Corollary 2.1.** *Let $\beta \in [0,1)$.Under the assumptions of Theorem 2.1, the solution of the differential equation* (2.1) *is starlike of order $\beta$ whenever*

$$(1-\beta)\big(\operatorname{Im}\{za(z)\}\big)^2 + 2(3-2\beta)\big[\beta\operatorname{Re}\{za(z)\} + \operatorname{Re}\{z^2 b(z)\}\big] < (1-\beta)(3-2\beta)(1+2\beta).$$

As an example, consider the function $F(z) = ze^{\alpha z}$. A calculation yields

$$\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) = \operatorname{Re}(\alpha z + 1) > 1 - |\alpha|, \quad z \in \mathbb{D}.$$

Hence, $F$ is starlike of order $\beta = 1-|\alpha|$ in $\mathbb{D}$ for $0 < |\alpha| \le 1$. Since $F$ is the solution of $y''(z) - 2\alpha y'(z) + \alpha^2 y(z) = 0$ with $y(0) = 0$ and $y'(0) = 1$, the starlikeness of $F$ also follows from Corollary 2.1 by taking $a(z) = -2\alpha$, $b(z) = \alpha^2$.

The error function [1] is given by

$$erf(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\ (2n+1)}z^{2n+1}.$$

It is evident that $(\sqrt{\pi}/2)erf(z)$ is the solution of the initial-value differential equation

$$y''(z) + 2zy'(z) = 0$$

satisfying $y(0) = 0$ and $y'(0) = 1$.

The error function have widespread applications, for example, in the areas of probability theory, statistics, and partial differential equations. It relates to the confluent hypergeometric functions through $\sqrt{\pi}erf(z) = 2z{}_1F_1(1/2;3/2;-z^2)$. Functional inequalities involving the real error function can be found in [5]. Kreyszig and Todd [17] proved that the radius of univalence of $erf$ is $1.574\ldots$, while Ruscheweyh and Singh [28] showed that the error function $(\sqrt{\pi}/2)erf(z)$ is starlike of order $\beta \approx 0.4925$. In [12], Coman determined the radius of starlikeness of the error function. Later, Ponnusamy and Vuorinen [26] proved that $z{}_1F_1(1/2;3/2;-z^2)$ is close-to-convex with respect to $-\log((1+z)/(1-z))$, and starlike of order $\sqrt{2}-1 \approx 0.4142$.

Related to the error function is the Gaussian probability function [24] $A$ given by

$$A(x) := \frac{1}{\sqrt{2\pi}}\int_{-x}^{x} e^{-\frac{u^2}{2}}du = erf\left(\frac{x}{\sqrt{2}}\right). \tag{2.9}$$

The following result is a special case of Corollary 2.1 and gives starlikeness of the Gaussian probability function $A$.

**Corollary 2.2.** *Fixed $\beta \in [0,1)$, and let $\beta_0 \approx 0.386963$ be the root of*

$$6\beta^3 - 9\beta^2 + 1 = 0$$

*in $[0,1)$. Then the solution of the differential equation*

$$(1-\beta)y''(z) + \beta(3-2\beta)zy'(z) = 0, \quad y(0) = 0, y'(0) = 1, \tag{2.10}$$

*is starlike of order $\beta$ provided $\beta < \beta_0$. In particular $\sqrt{\pi/2}\,A(z)$ is starlike of order $(2-\sqrt{2})/2 \approx 0.292893$.*

*Proof.* Comparing (2.10) with (2.1), the variable coefficients are

$$a(z) = \frac{\beta(3-2\beta)}{1-\beta}z \qquad \text{and} \qquad b(z) = 0.$$

For $z = x + iy$ with $x^2 < 1 - y^2$, it follows that

$$\begin{aligned}
&(1-\beta)\big(\operatorname{Im}\{za(z)\}\big)^2 + 2(3-2\beta)[\beta \operatorname{Re}\{za(z)\} + \operatorname{Re}\{z^2 b(z)\}] \\
&= 4(1-\beta)\left(\frac{\beta(3-2\beta)}{1-\beta}\right)^2 x^2y^2 + 2(3-2\beta)\frac{\beta(3-2\beta)}{1-\beta}(x^2-y^2) \\
&< 4(1-\beta)\left(\frac{\beta(3-2\beta)}{1-\beta}\right)^2 y^2(1-y^2) + 2(3-2\beta)\frac{\beta(3-2\beta)}{1-\beta}(1-2y^2) \\
&= \frac{2\beta^2(3-2\beta)^2}{1-\beta}\left(1-2y^4\right) < \frac{2\beta^2(3-2\beta)^2}{1-\beta}.
\end{aligned}$$

Thus the condition in Corollary 2.1 holds provided

$$2\beta^2(3-2\beta)^2 \le (1-\beta)^2(3-2\beta)(1+2\beta),$$

which is satisfied for all $\beta \in [0, \beta_0)$.

As observed in the remark above, it is readily seen that the solution of the differential equation

$$y''(z) + \alpha z y'(z) = 0, \quad y(0) = 0,\ y'(0) = 1, \tag{2.11}$$

is univalent in the unit disk at least for $|\alpha| \le \pi$.
Its solution is $\sqrt{\pi/(2\alpha)}\ \mathit{erf}(\sqrt{\alpha}z/\sqrt{2})$ for $\alpha > 0$. In particular, for $\alpha = 1$, the solution of (2.11) is $\sqrt{\pi/2}\,\mathit{erf}(z/\sqrt{2}) = \sqrt{\pi/2}\,A(z)$. Since

$$\frac{\beta(3-2\beta)}{1-\beta} = 1 \implies \beta = \frac{2-\sqrt{2}}{2},$$

it follows from (2.10) that $\sqrt{\pi/2}\,A(z)$ is starlike of order $\beta = (2-\sqrt{2})/2$. ∎

## 3 Solutions which are Janowski convex

We turn to a different type of differential equation in this section, and derive, by the principle of subordination, sufficient conditions that ensure its solutions are Janowski convex.

**Theorem 3.1.** *Consider the second order differential equation*

$$z(1-z)y''(z) + a(z)y'(z) + \alpha y(z) = 0, \tag{3.1}$$

*where $\alpha$ is a constant. Suppose that $\Phi$ is a solution of* (3.1) *satisfying $\Phi'(z) \neq 0$ for all $z \in \mathbb{D}$. Then*

$$1 + \frac{z\Phi''(z)}{\Phi'(z)} \prec \frac{1+Az}{1+Bz}$$

*provided*

$$\operatorname{Re}\left(\frac{(A-B)(1-z)}{D_2(A,B;z)}\right) > 0, \ \operatorname{Re}\left(\frac{D_1(A,B;z)}{D_2(A,B;z)}\right) \geq 0,$$
$$\text{and} \quad \operatorname{Re}\left(\frac{D_3(A,B;z)}{D_2(A,B;z)}\right) > 0.$$

*Here*

$$\begin{aligned} D_1(A,B;z) &= (A-B)(1+B)(a(z)-z) + z(1+B)^2(a'(z)+\alpha), \\ D_2(A,B;z) &= -2B(A-B)(a(z)-z) - 2(A-B)^2(1-z) + \\ &\qquad 2z(1-B^2)(a'(z)+\alpha) \neq 0, \\ D_3(A,B;z) &= (1-B)(A-B)(a(z)-z) - (1-B)^2 z(a'(z)+\alpha) \\ &\qquad - (A-B-1)(A-B)(1-z). \end{aligned} \tag{3.2}$$

*Further, if for some $a, \alpha$, $A$ and $B$, the expression $D_2(A,B;z) = 0$ for all $z \in \mathbb{D}$, then the conclusion also holds provided*

$$\operatorname{Re} D_1(A,B;z) \geq 0 \quad \text{and} \quad \operatorname{Re} D_3(A,B;z) \geq 0.$$

*Proof.* Let the analytic function $p$ be given by

$$p(z) := \frac{(A-B)\Phi'(z) + (1-B)z\Phi''(z)}{(A-B)\Phi'(z) - (1+B)z\Phi''(z)},$$

or equivalently,

$$\frac{z\Phi''(z)}{\Phi'(z)} = \frac{(A-B)(p(z)-1)}{(1-B)+(1+B)p(z)}. \tag{3.3}$$

Differentiating, (3.3) yields

$$\frac{z^2\Phi'''(z)}{\Phi'(z)} + \frac{z\Phi''(z)}{\Phi'(z)} - \left(\frac{z\Phi''(z)}{\Phi'(z)}\right)^2 = \frac{(A-B)zp'(z)}{(1-B)+(1+B)p(z)}$$
$$- \frac{(A-B)(1+B)(p(z)-1)zp'(z)}{((1-B)+(1+B)p(z))^2},$$

that is,

$$\frac{z^2\Phi'''(z)}{\Phi'(z)} = \frac{(A-B)\,(zp'(z)-p(z)+1)}{(1-B)+(1+B)p(z)} + \frac{(A-B)\big(p(z)-1\big)\left[(A-B)\big(p(z)-1\big)-(1+B)zp'(z)\right]}{\left[(1-B)+(1+B)p(z)\right]^2}. \tag{3.4}$$

As a solution of (3.1), the function $\Phi$ satisfies the differential equation

$$z(1-z)\Phi''(z)+a(z)\Phi'(z)+\alpha\Phi(z)=0.$$

Thus

$$z(1-z)\Phi'''(z)+(1-2z+a(z))\Phi''(z)+(a'(z)+\alpha)\Phi'(z)=0,$$

and consequently,

$$(1-z)\frac{z^2\Phi'''(z)}{\Phi'(z)}+(1-2z+a(z))\frac{z\Phi''(z)}{\Phi'(z)}+(a'(z)+\alpha)z=0.$$

From (3.3) and (3.4), the above equation takes the form

$$(1-z)\left(\frac{2(A-B)zp'(z)}{(1-B+(1+B)p(z))^2}-\frac{(A-B)(p(z)-1)}{1-B+(1+B)p(z)}+\frac{(A-B)^2(p(z)-1)^2}{\big(1-B+(1+B)p(z)\big)^2}\right) + \frac{(A-B)\big(1-2z+a(z)\big)(p(z)-1)}{1-B+(1+B)p(z)}+(a'(z)+\alpha)z=0.$$

Equivalently,

$$2(1-z)(A-B)zp'(z)+(A-B)(a(z)-z)(p(z)-1)(1-B+(1+B)p(z)) + (A-B)^2(1-z)(p(z)-1)^2+z(a'(z)+\alpha)(1-B+(1+B)p(z))^2=0.$$

A further simplification leads to

$$(A-B)(1-z)\left(2zp'(z)+(A-B)p^2(z)+1\right)+D_1(A,B;z)p^2(z) + D_2(A,B;z)p(z)-D_3(A,B;z)=0,$$

where $D_1(A,B;z), D_2(A,B;z)$, and $D_3(A,B;z)$ are given by (3.2).

Next suppose that $D_2(A,B;z)\neq 0$, and let $\Psi:\mathbb{C}^2\times\mathbb{D}\to\mathbb{C}$ be given by

$$\Psi\big(p(z),zp'(z);z\big)=\frac{(A-B)(1-z)}{D_2(A,B;z)}\big(2zp'(z)+(A-B)p^2(z)+1\big) + \frac{D_1(A,B;z)}{D_2(A,B;z)}p^2(z)+p(z)-\frac{D_3(A,B;z)}{D_2(A,B;z)}. \tag{3.5}$$

Let $\Omega=\{0\}$. We shall show that $\Psi(i\rho,\sigma;z)\notin\Omega$ for $\sigma\leq-(1+\rho^2)/2$ and $\rho\in\mathbb{R}$. For this purpose, it suffices to show that $\operatorname{Re}\Psi(i\rho,\sigma;z)<0$. From (3.5), it is clear that

$$\begin{aligned}
&\operatorname{Re}\Psi\big(p(z), zp'(z); z\big)\\
&= \operatorname{Re}\frac{(A-B)(1-z)}{D_2(A,B;z)}\left(2\sigma-(A-B)\rho^2+1\right)-\frac{D_1(A,B;z)}{D_2(A,B;z)}\rho^2-\operatorname{Re}\frac{D_3(A,B;z)}{D_2(A,B;z)}\\
&\leq -\operatorname{Re}\left(\frac{(A-B)(1-z)}{D_2(A,B;z)}\right)(1+A-B)\rho^2\\
&\qquad -\operatorname{Re}\left(\frac{D_1(A,B;z)}{D_2(A,B;z)}\right)\rho^2-\operatorname{Re}\left(\frac{D_3(A,B;z)}{D_2(A,B;z)}\right)<0.
\end{aligned}$$

Accordingly, Lemma 1.1 yields $\operatorname{Re} p(z) > 0$, which readily implies

$$1+\frac{z\Phi''(z)}{\Phi'(z)} \prec \frac{1+Az}{1+Bz}. \qquad \blacksquare$$

With $B = -1$ and $A = 1 - 2\beta$, Theorem 3.1 describes the convexity of order $\beta \in [0, 1)$ of solutions to the differential equation (3.1).

**Corollary 3.1.** *Under the assumptions of Theorem 3.1, the solution $\Phi$ to the differential equation* (3.1) *satisfying $\Phi(0) = 1$ is convex of order $\beta \in [0, 1)$ provided*

$$\operatorname{Re}\frac{1-z}{a(z)+(1-2\beta)z-2(1-\beta)}>0,$$

$$\textit{and}\quad \operatorname{Re}\left(\frac{a(z)-\frac{1}{2}+\beta-\left(\frac{1}{2}+\beta+\frac{a'(z)+\alpha}{1-\beta}\right)z}{a(z)+(1-2\beta)z-2(1-\beta)}\right)>0.$$

## 4 Special examples of Janowski starlikeness

This section considers applications of Theorem 2.1 to deduce Janowski starlikeness of solutions to several widely studied differential equations.

**Example 1.** *Consider the function*

$$f_1(z)=\sqrt{2}\,\sin\left(\frac{z}{\sqrt{2}}\right).$$

*It is easily seen that $f_1$ is a solution of*

$$2W''(z)+W(z)=0.$$

*Choosing $A = 1$, $B = -1$, $b(z) = 1/2$, and $a(z) = 0$ in* (2.3)*, the condition is satisfied whenever* $\operatorname{Re}(z^2) \leq 1$*, which trivially holds for $|z| < 1$. Thus the solution $f_1$ is starlike.*

**Example 2.** *The function*

$$f_2(z)=2\sqrt{\pi}e^{z^2/16}erf\left(\frac{z}{4}\right)$$

*is easily seen to be a solution of the differential equation*

$$8y''(z) - zy'(z) - y(z) = 0.$$

*Since* $\operatorname{erf}(0) = 0$, *it follows that* $f_2(0) = 0 = f_2'(0) - 1$. *Let* $B = -1$, $a(z) = -z/8$, *and* $b(z) = -1/8$ *in* (2.3). *For* $z = x + iy$ *with* $x, y \in (-1, 1)$ *and* $x^2 + y^2 < 1$, *it follows that*

$$\begin{aligned}
&(1+A)(\operatorname{Im}(za(z)))^2 + 2(1-A)(2+A)\operatorname{Re}(za(z)) + 4(2+A)\operatorname{Re}(z^2 b(z))\\
&= (1+A)\left(\operatorname{Im}\left(-\frac{z^2}{8}\right)\right)^2 + 2(1-A)(2+A)\operatorname{Re}\left(-\frac{z^2}{8}\right) + 4(2+A)\operatorname{Re}\left(-\frac{z^2}{8}\right)\\
&= \frac{1}{16}(1+A)x^2y^2 - \frac{1}{4}(2+A)(3-A)(x^2-y^2)\\
&< \frac{1}{16}(1+A)x^2(1-x^2) - \frac{1}{4}(2+A)(3-A)(2x^2-1) \le \frac{1}{4}(2+A)(3-A).
\end{aligned}$$

*Thus, condition* (2.3) *holds whenever* $4(1+A)(4-A^2) \ge (2+A)(3-A)$. *Consequently, Theorem 2.1 implies that* $f_2 \in \mathcal{S}^*[A, -1]$ *for* $A \ge A_0 \simeq -0.655869$, *where* $A_0$ *is the root of the equation* $4A^3 + 3A^2 - 15A - 10 = 0$ *in* $(-1, 1]$. *In particular,* $f_2$ *is univalently starlike in* $\mathbb{D}$.

**Example 3.** *The Bessel function* $J_\nu$ *of order* $\nu$ *is the solution of the differential equation*

$$z^2y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0. \tag{4.1}$$

*Several earlier works on the geometric properties of the Bessel function and its generalizations can be found in [6–11, 15, 16, 23, 29, 31].*

*Here we consider the function*

$$f_3(z) = \frac{\pi}{\sin(\pi\nu)}\left(J_{-\nu}(1)J_\nu(\sqrt{e^z}) - J_\nu(1)J_{-\nu}(\sqrt{e^z})\right), \quad \nu \notin \mathbb{Z}.$$

*Clearly,*

$$f_3(0) = \frac{\pi}{\sin(\pi\nu)}\left(J_{-\nu}(1)J_\nu(1) - J_\nu(1)J_{-\nu}(1)\right) = 0,$$

*and*

$$f_3'(z) = \frac{\pi\sqrt{e^z}}{2\sin(\pi\nu)}\left(J_{-\nu}(1)J_\nu'(\sqrt{e^z}) - J_\nu(1)J_{-\nu}'(\sqrt{e^z})\right).$$

*From the recurrence relation*

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z),$$

*it follows that*

$$\begin{aligned}
f_3'(z) = \frac{\pi\sqrt{e^z}}{4\sin(\pi\nu)}\Big(&J_{-\nu}(1)J_{\nu-1}\left(\sqrt{e^z}\right) - J_{-\nu}(1)J_{\nu+1}\left(\sqrt{e^z}\right)\\
&- J_\nu(1)J_{-\nu-1}\left(\sqrt{e^z}\right) + J_\nu(1)J_{1-\nu}\left(\sqrt{e^z}\right)\Big).
\end{aligned}$$

*It is known [1, p.360] that the Wronskian is*

$$\mathcal{W}\left(J_\nu(z), J_{-\nu}(z)\right) = J_{\nu+1}(z)J_{-\nu}(z) + J_\nu(z)J_{-\nu-1}(z) = -\frac{2\sin(\pi\nu)}{\pi z}.$$

*Thus,*

$$\begin{aligned} f_3'(0) &= \frac{\pi}{4\sin(\pi\nu)}\Big(J_{-\nu}(1)J_{\nu-1}(1) - J_{-\nu}(1)J_{\nu+1}(1) - J_\nu(1)J_{-\nu-1}(1) \\ &\qquad + J_\nu(1)J_{1-\nu}(1)\Big) \\ &= \frac{\pi}{4\sin(\pi\nu)}\Big(\Big(J_{-\nu}(1)J_{\nu-1}(1) + J_\nu(1)J_{1-\nu}(1)\Big) - \Big(J_{-\nu}(1)J_{\nu+1}(1) \\ &\qquad + J_\nu(1)J_{-\nu-1}(1)\Big)\Big) \\ &= \frac{\pi}{4\sin(\pi\nu)}\left(-\frac{2\sin(-\pi\nu)}{\pi} + \frac{2\sin(\pi\nu)}{\pi}\right) = 1. \end{aligned}$$

*The second order derivative of $f_3$ can now be expressed as*

$$\begin{aligned} f_3''(z) &= \frac{\pi e^z}{4\sin(\pi\nu)}\left(J_{-\nu}(1)J_\nu''(\sqrt{e^z}) - J_\nu(1)J_{-\nu}''(\sqrt{e^z})\right) \\ &\quad + \frac{\pi\sqrt{e^z}}{4\sin(\pi\nu)}\left(J_{-\nu}(1)J_\nu'(\sqrt{e^z}) - J_\nu(1)J_{-\nu}'(\sqrt{e^z})\right) \\ &= \frac{\pi J_{-\nu}(1)}{4\sin(\pi\nu)}\left(e^z J_\nu''(\sqrt{e^z}) + \sqrt{e^z}J_\nu'(\sqrt{e^z})\right) \\ &\quad - \frac{\pi J_\nu(1)}{4\sin(\pi\nu)}\left(e^z J_{-\nu}''(\sqrt{e^z}) + \sqrt{e^z}J_{-\nu}'(\sqrt{e^z})\right). \end{aligned}$$

*Recalling the Bessel differential equation* (4.1), *it follows that*

$$\begin{aligned} 4f_3''(z) + (e^z - \nu^2)f_3(z) &= \frac{\pi J_{-\nu}(1)}{\sin(\pi\nu)}\left(e^z J_\nu''(\sqrt{e^z}) + \sqrt{e^z}J_\nu'(\sqrt{e^z}) + (e^z - \nu^2)J_\nu(\sqrt{e^z})\right) \\ - \frac{\pi J_\nu(1)}{\sin(\pi\nu)}&\left(e^z J_{-\nu}''(\sqrt{e^z}) + \sqrt{e^z}J_{-\nu}'(\sqrt{e^z}) + (e^z - (-\nu)^2)J_{-\nu}(\sqrt{e^z})\right) = 0, \end{aligned}$$

*and hence $f_3$ is a solution of the differential equation*

$$4F''(z) + (e^z - \nu^2)F(z) = 0.$$

*Thus, an application of Theorem 2.1 yields*

*(i) The function $f_3 \in S^*[A, -1]$ for $-1 < A \leq 1$ if*

$$\mathrm{Re}\left(z^2(e^z - \nu^2)\right) < (1 + A)(2 - A).$$

*In particular, $f_3$ is starlike whenever* $\mathrm{Re}(z^2(e^z - \nu^2)) < 2$.

*(ii) The function $f_3 \in S^*[A, B]$ for $-1 < B < A \leq 1$ provided*

$$|e^z - \nu^2| \leq 4\min\left\{\frac{(A-B)(2-A)}{(1-B)^2}, \frac{(A-B)(2+A)}{(1+B)^2}\right\}.$$

## 5 Special examples of Janowski convexity

In this final section, Theorem 3.1 is applied to deduce Janowski convexity of solutions to certain differential equations. The first example looks at the case when $a(z) = c - (a+b+1)z$ and $\alpha = -ab$, where $a$, $b$ are real, and $c \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}$. Equation (3.1) then takes the form of the widely studied Gaussian hypergeometric differential equation given by

$$z(1-z)y''(z) + \big(c - (a+b+1)z\big)y'(z) - aby(z) = 0. \tag{5.1}$$

The equation (5.1) also holds for $a, b \in \mathbb{C}$, and $c \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$. The solution of (5.1) is known as the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$. In series form, the Gaussian function is given by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n.$$

More on the hypergeometric functions can be found, for example, in the book [32].

Geometric properties of the Gaussian hypergeometric functions have also been widely investigated. For instance, using continued fractions, starlikeness of the hypergeometric function was studied in [18], while its order of starlikeness in the work [28]. In [19], Miller and Mocanu applied the tool of differential subordination to obtain sufficient conditions on $a$, $b$, and $c$ for which ${}_2F_1(a,b;c;z)$ is either univalent, starlike, or convex. Their results were later generalized in [27].

We next apply Theorem 3.1 to determine convexity of ${}_2F_1(a,b;c;z)$. In this case,

$$\begin{aligned}
D_1(A,B;z) &= D_1(a,b,c,A,B,z) \\
&= (A-B)(1+B)(c-(a+b+2)z) - (1+B)^2 z(a+1)(b+1), \\
D_2(A,B;z) &= D_2(a,b,c,A,B,z) \\
&= -2B(A-B)(c-(a+b+2)z) \\
&\qquad -2(A-B)^2(1-z) - 2(1-B^2)z(a+1)(b+1), \\
D_3(A,B;z) &= D_3(a,b,c,A,B,z) \\
&= (1-B)(A-B)(c-(a+b+2)z) \\
&\qquad +(1-B)^2 z(a+1)(b+1) - (A-B-1)(A-B)(1-z).
\end{aligned} \tag{5.2}$$

The following result is immediately deduced from Theorem 3.1.

**Corollary 5.1.** *Let $a, b, c$ be real, and $-1 \le B < A \le 1$. Suppose ${}_2F_1'(a,b;c;z) \neq 0$ in $\mathbb{D}$. If*

$$\operatorname{Re}\left(\frac{(A-B)(1-z)}{D_2(a,b,c,A,B,z)}\right) > 0, \ \operatorname{Re}\left(\frac{D_1(a,b,c,A,B,z)}{D_2(a,b,c,A,B,z)}\right) \ge 0,$$
$$\textit{and} \quad \operatorname{Re}\left(\frac{D_3(a,b,c,A,B,z)}{D_2(a,b,c,A,B,z)}\right) > 0,$$

*then ${}_2F_1(a,b;c;z) \in \mathcal{C}[A,B]$, that is,*

$$1 + \frac{z\,{}_2F_1''(a,b;c;z)}{{}_2F_1'(a,b;c;z)} \prec \frac{1+Az}{1+Bz}.$$

The relation $c\ {}_2F_1'(a,b;c;z) = ab\ {}_2F_1(a+1,b+1;c+1;z)$ yields

$$\frac{z\left(z{}_2F_1(a,b;c;z)\right)'}{z{}_2F_1(a,b;c;z)} = 1 + \frac{z{}_2F_1''(a-1,b-1;c-1;z)}{{}_2F_1'(a-1,b-1;c-1;z)}. \tag{5.3}$$

Along with Corollary 5.1, and with $D_i(a,b,c,A,B,z)$, $i = 1,2,3$ as given in (5.2), the above relation leads to Janowski starlikeness of $z{}_2F_1(a,b;c;z)$ as stated below.

**Corollary 5.2.** *Let $a,b,c$ be real, and $-1 \leq B < A \leq 1$. Suppose ${}_2F_1'(a,b;c;z) \neq 0$ in $\mathbb{D}$. If*

$$\operatorname{Re}\left(\frac{2(A-B)(1-z)}{D_2(a-1,b-1,c-1,A,B,z)}\right) > 0, \ \operatorname{Re}\left(\frac{D_1(a-1,b-1,c-1,A,B,z)}{D_2(a-1,b-1,c-1,A,B,z)}\right) > 0,$$
$$\text{and} \quad \operatorname{Re}\left(\frac{D_3(a-1,b-1,c-1,A,B,z)}{D_2(a-1,b-1,c-1,A,B,z)}\right) > 0,$$

*then $z{}_2F_1(a,b;c;z) \in S^*[A,B]$.*

For specific values of $A$ and $B$, for instance, $B = -1$ and $A = 1-2\beta$, $\beta \in [0,1)$, Corollary 5.1 leads to an earlier result of Ponnusamy and Vuorinen [27] on the order of convexity of ${}_2F_1(a,b;c;z)$.

**Corollary 5.3.** *[27, Theorem 5.1, p. 342] Let a, b, and c be real. Suppose that $(a+1)(b+1)\beta \leq 0$ for $\beta \in [0,1)$, and ${}_2F_1'(a,b;c;z) \neq 0$ in $\mathbb{D}$. If*

$$c \geq \max\left\{2(1-\beta) + |a+b+2\beta|, 1 - ab - \frac{(a+1)(b+1)\beta}{1-\beta}\right\},$$

*then ${}_2F_1(a,b;c;z)$ is convex of order $\beta$.*

The next result gives the convexity of ${}_2F_1$ which cannot be obtained from Corollary 5.3.

**Corollary 5.4.** *Let $\beta \in [1/4, 1/2]$, and $\alpha \in \mathbb{R}$. Further, let*

$$a_{\alpha,\beta} := \frac{\beta-1}{2} \quad \text{and} \quad b_{\alpha,\beta} := \frac{1}{2}\sqrt{1+4\alpha-2\beta+\beta^2}.$$

*Then the Gaussian hypergeometric function $f_1(z) = {}_2F_1\left(a_{\alpha,\beta} - b_{\alpha,\beta}, a_{\alpha,\beta} + b_{\alpha,\beta}; 2-\beta; z\right)$ is convex of order $\beta$ provided $2\beta^2 + \beta - 2 \leq \alpha \leq 2\beta^2 - 2\beta + 1$.*

*Proof.* From its definition, it is clear that $f_1(z) = {}_2F_1\left(a_{\alpha,\beta} - b_{\alpha,\beta}, a_{\alpha,\beta} + b_{\alpha,\beta}; 2-\beta; z\right)$ is a solution of the differential equation

$$z(1-z)y''(z) + (2-\beta-(2a_{\alpha,\beta}+1)z)y'(z) + (a_{\alpha,\beta}^2 - b_{\alpha,\beta}^2)y(z) = 0.$$

It is easily shown that $a_{\alpha,\beta}^2 - b_{\alpha,\beta}^2 = -\alpha$ and $2a_{\alpha,\beta} + 1 = \beta$.

Now choose

$$a(z) = 2 - \beta(1+z)$$

in the differential equation (3.1). Then,

$$\mathrm{Re}\left(\frac{1-z}{a(z)+(1-2\beta)z-2(1-\beta)}\right)=\mathrm{Re}\left(\frac{1-z}{\beta+(1-3\beta)z}\right)>0$$

whenever $|1-3\beta|\leq\beta$; in particular for $\beta\in[1/4,1/2]$. Similarly,

$$\mathrm{Re}\left(\frac{a(z)-\frac{1}{2}+\beta-\left(\frac{1}{2}+\beta+\frac{a'(z)+\alpha}{1-\beta}\right)z}{a(z)+(1-2\beta)z-2(1-\beta)}\right)=\frac{3}{2\beta}\,\mathrm{Re}\left(\frac{1+\frac{2\beta-2\alpha-(1+4\beta)(1-\beta)}{3(1-\beta)}z}{1+\frac{(1-3\beta)}{\beta}z}\right).$$

For $\beta\in[1/4,1/2]$, define

$$\omega(z):=\frac{1+Az}{1+Bz},$$

where

$$A=\frac{2\beta-2\alpha-(1+4\beta)(1-\beta)}{3(1-\beta)},\quad \text{and}\quad B=\frac{1-3\beta}{\beta}.$$

To complete the proof, it suffices to find conditions for which $\mathrm{Re}\,\omega(z)\geq 0$ for $z\in\mathbb{D}$.

For $\beta\in[1/4,1/2]$, it immediately follows that $|B|\leq 1$. A computation yields $|A|\leq 1$ if and only if $2\beta^2+\beta-2\leq\alpha\leq 2\beta^2-2\beta+1$. The remainder of the proof is divided into three cases.

But first, let us recall certain mapping properties of the unit disk by the function $\omega(z)=(1+Mz)/(1+Nz)$, $-1\leq M,N\leq 1$. If $N\neq\pm1$, the function $\omega$ maps the unit disk $\mathbb{D}$ conformally onto the disk

$$\left|\omega-\frac{1-MN}{1-N^2}\right|<\frac{|N-M|}{1-N^2},$$

from which it follows that

$$\mathrm{Re}(\omega(z))>\begin{cases}\dfrac{1+M}{1+N}, & N>M\\[2ex] \dfrac{1-M}{1-N}, & M>N\end{cases}\tag{5.4}$$

On the other hand, when $N=\pm1$ , then

$$\mathrm{Re}(\omega(z))>\begin{cases}\dfrac{1+M}{2}, & N=1\\[2ex] \dfrac{1-M}{2}, & N=-1\end{cases}\tag{5.5}$$

The three cases to be considered are as follows:

(i) Let $\beta=1/2$. Then $B=-1$, $A=-(4\alpha+1)/3$ and $-1\leq\alpha\leq 1/2$. In this case, (5.5) implies

$$\mathrm{Re}\,\omega(z)>\frac{1-A}{2}=\frac{2}{3}(1+\alpha)\geq 0.$$

(ii) Let $\beta = 1/4$. Then $B = 1$, $A = -4(1+2\alpha)/9$, and $-13/8 \leq \alpha \leq 5/8$. Then (5.5) implies

$$\operatorname{Re}\omega(z) > \frac{1+A}{2} = \frac{5-8\alpha}{18} \geq 0.$$

(iii) Finally consider the case when $\beta \in (1/4, 1/2)$. For

$$2\beta^2 + \beta - 2 \leq \alpha < \frac{4\beta^3 - 10\beta^2 + 11\beta - 3}{2\beta},$$

we have $B < A$, and (5.4) implies

$$\operatorname{Re}\omega(z) > \frac{1-A}{1-B} = \frac{2\beta(\alpha + 2 - \beta - 2\beta^2)}{3(1-\beta)(4\beta - 1)} \geq 0.$$

On the other hand, suppose

$$\frac{4\beta^3 - 10\beta^2 + 11\beta - 3}{2\beta} < \alpha \leq 2\beta^2 - 2\beta + 1.$$

Then $A < B$, and (5.4) implies that

$$\operatorname{Re}\omega(z) > \frac{1+A}{1+B} = \frac{2\beta(2\beta^2 - 2\beta + 1 - \alpha)}{3(1-\beta)(1-2\beta)} \geq 0.$$

From Corollary 3.1, we deduce that $f_1$ is convex of order $\beta \in [1/4, 1/2]$. ∎

**Corollary 5.5.** *Suppose that $\beta \in [0,1)$, and $c > 2(1-\beta)$. Then the Gaussian hypergeometric function $f_2(z) = {}_2F_1(-\beta - \sqrt{\beta^2 + 2\beta - 1}, -\beta + \sqrt{\beta^2 + 2\beta - 1}; c; z)$ is convex of order $\beta$ in the unit disk .*

*Proof.* It is easy to show that $f_2$ is a solution of the differential equation

$$z(1-z)y''(z) + (c - (1-2\beta)z)y'(z) + (1-2\beta)y(z) = 0.$$

It suffices to show the conditions of Corollary 3.1 hold with $a(z) = c - (1-2\beta)z$, and $\alpha = 1 - 2\beta$. In this case,

$$\operatorname{Re}\left(\frac{1-z}{a(z) + (1-2\beta)z - 2(1-\beta)}\right) = \operatorname{Re}\left(\frac{1-z}{c - 2(1-\beta)}\right) > 0$$

for all $\beta \in [0,1)$, and $c > 2(1-\beta)$. Also

$$\begin{aligned}
&\operatorname{Re}\left(\frac{a(z) - \frac{1}{2} + \beta - \left(\frac{1}{2} + \beta + \frac{a'(z)+\alpha}{1-\beta}\right)z}{a(z) + (1-2\beta)z - 2(1-\beta)}\right)\\
&\qquad = \operatorname{Re}\left(\frac{c - \frac{1}{2} + \beta - (1-2\beta)z - \left(\frac{1}{2} + \beta\right)z}{c - 2(1-\beta)}\right)\\
&\qquad = \frac{1}{c - 2(1-\beta)}\operatorname{Re}\left(c - \frac{1}{2} + \beta - \left(\frac{3}{2} - \beta\right)z\right)\\
&\qquad > \frac{c - 2(1-\beta)}{c - 2(1-\beta)} = 1.
\end{aligned}$$

Hence $f_2$ is convex of order $\beta \in [0,1)$. ∎

**Remark 5.1.** *It is interesting to note here that for certain judicious choices of $\alpha$, the example given in Corollary 5.4 cannot be obtained from Corollary 5.3. Evidently,*

$$\begin{aligned}&(1+a_{\alpha,\beta}+b_{\alpha,\beta})(1+a_{\alpha,\beta}-b_{\alpha,\beta})\\&=\frac{1}{4}\left(\beta+1+\sqrt{1+4\alpha-2\beta+\beta^2}\right)\left(\beta+1-\sqrt{1+4\alpha-2\beta+\beta^2}\right)\\&=\beta-\alpha>0\end{aligned}$$

*for $2\beta^2+\beta-2\leq\alpha<\beta$. Also, for any value of $\alpha<(-1+2\beta-\beta^2)/4$, the expression $a_{\alpha,\beta}+b_{\alpha,\beta}$ and $a_{\alpha,\beta}-b_{\alpha,\beta}$ are complex conjugate numbers.*

*For $\beta\in[0,1/2)$ in Corollary 5.5, it follows that*

$$(1-\beta-\sqrt{\beta^2+2\beta-1})(1-\beta+\sqrt{\beta^2+2\beta-1})=2-4\beta>0.$$

*Thus, in either case, it does not satisfy the hypothesis of Corollary 5.3.*

As a final comparison, we recall the following result by Hästö *et. al*:

**Corollary 5.6.** *[14, Theorem 1.4, page 175] Let $a$, $b$ and $c$ be nonzero real numbers such that ${}_2F_1(a,b;c;z)$ has no zeros in $\mathbb{D}$. Then $z{}_2F_1(a,b;c;z)$ is starlike of order $\beta\in[0,1)$ if*

$$(i)\ \mathbf{C}\geq 0\quad(ii)\ \mathbf{C}+(1-\beta)\geq 2\mathbf{A}\quad(iii)\ (1-\beta+2(1-\beta)^2)\mathbf{C}+2\mathbf{BD}+\mathbf{D}^2\geq 0,$$

*where $\tilde{c}=c-1-(a+b)$, $\mathbf{A}=(1-\beta)^2-(1-\beta)(a+b)+ab$, $\mathbf{B}=(1-\beta)(a+b)-2(1-\beta)^2$, $\mathbf{C}=(1-\beta)\tilde{c}+ab$, and $\mathbf{D}=(1-\beta)\tilde{c}$.*

**Remark 5.2.** *Consider the function $f_3(z)=z\,{}_2F_1\left(1+a_{\alpha,\beta}-b_{\alpha,\beta},1+a_{\alpha,\beta}+b_{\alpha,\beta};\right.$ $\left.3-\beta;z\right)$, where $a_{\alpha,\beta}$ and $b_{\alpha,\beta}$ are defined in Corollary 5.4. Further, let*

$$a=1+a_{\alpha,\beta}-b_{\alpha,\beta};\quad b=1+a_{\alpha,\beta}+b_{\alpha,\beta}\quad\text{and}\quad c=3-\beta.$$

*A series of computation yields*

$$\begin{aligned}a+b&=2+2a_{\alpha,\beta}=2+\beta-1=1+\beta,\\ab&=(1+a_{\alpha,\beta}+b_{\alpha,\beta})(1+a_{\alpha,\beta}-b_{\alpha,\beta})=\beta-\alpha,\\\tilde{c}&=c-1-(a+b)=3-\beta-1-(1+\beta)=1-2\beta,\\\mathbf{A}&=(1-\beta)^2-(1-\beta)(a+b)+ab\\&=(1-\beta)^2-(1-\beta)(1+\beta)+\beta-\alpha=2\beta^2-\beta-\alpha,\\\mathbf{B}&=(1-\beta)(a+b)-2(1-\beta)^2=-(1-\beta)(1-3\beta),\\\mathbf{C}&=(1-\beta)\tilde{c}+ab=(1-\beta)(1-2\beta)+\beta-\alpha,\\\mathbf{D}&=(1-\beta)\tilde{c}=(1-\beta)(1-2\beta)\end{aligned}$$

*Now for $\beta\in[1/4,1/2]$, computations show that*

*(i)* $\mathbf{C}\geq 0$ *if* $\alpha\leq(1-\beta)(1-2\beta)+\beta=1-2\beta+2\beta^2$;

*(ii)* $\mathbf{C}+(1-\beta)\geq 2\mathbf{A}$ *is equivalent to* $(1-\beta)(1-2\beta)+\beta-\alpha+(1-\beta)\geq 4\beta^2-2\beta-2\alpha$, *which holds if* $\alpha\geq 2\beta^2+\beta-2$;

*(iii)* *clearly* $2\mathbf{B}+\mathbf{D} = -2(1-\beta)(1-3\beta)+(1-\beta)(1-2\beta) = (1-\beta)(-1+4\beta) \geq 0$. *Since* $\mathbf{C} \geq 0$ *and* $\mathbf{D} \geq 0$, *it follows that* $(1-\beta+(1-\beta)^2)\mathbf{C}+2\mathbf{BD}+\mathbf{D}^2 \geq 0$.

*We deduce from Corollary 5.6 that the function* $f_3$ *is starlike of order* $\beta \in [1/4, 1/2]$ *if* $2\beta^2+\beta-2 \leq \alpha \leq 1-2\beta+2\beta^2$. *This result also follows from relation* (5.3) *and Corollary 5.4.*

**Remark 5.3.** *From relation* (5.3) *and Corollary 5.5, it can be shown that* $f_4(z) = z\,{}_2F_1(1-\beta-\sqrt{\beta^2+2\beta-1}, 1-\beta+\sqrt{\beta^2+2\beta-1}; c+1; z)$ *is starlike of order* $\beta \in [0,1)$ *whenever* $c > 2(1-\beta)$. *Proceeding similarly as described in Remark 5.2, it follows from Corollary 5.6 that* $f_4$ *is starlike of order* $\beta \in [0,1)$ *provided*

$$c > \max_{\beta\in[0,1)} \left\{ \frac{2\beta^2}{1-\beta}, \frac{2(1-2\beta)}{1-\beta} \right\} = \begin{cases} \dfrac{2(1-2\beta)}{1-\beta}, & 0 \leq \beta \leq \sqrt{2}-1, \\[2ex] \dfrac{2\beta^2}{1-\beta}, & \sqrt{2}-1 \leq \beta < 1. \end{cases}$$

*Clearly, Corollary 5.5 gives a better range for c whenever* $\beta > 1/2$; *on the other hand, Corollary 5.6 is better for* $\beta < 1/2$. *At* $\beta = 1/2$, *both results give the range* $c > 1$.

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School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 USM Penang, Malaysia
emails : rosihan@usm.my, sklee@usm.my

Department of Mathematics and Statistics,
College of Science, King Faisal University,
Al-Hasa 31982, Saudi Arabia
email :smondal@kfu.edu.sa