Hypersurfaces of the homogeneous nearly Kähler $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$ with anticommutative structure tensors*

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Abstract

Each hypersurface of a nearly Kähler manifold is naturally equipped with two tensor fields of $(1,1)$-type, namely the shape operator A and the induced almost contact structure ϕ. In this paper, we show that, in the homogeneous nearly Kähler $\mathbb{S}^6$ a hypersurface satisfies the condition $A\phi + \phi A = 0$ if and only if it is totally geodesic; moreover, similar as for the non-flat complex space forms, the homogeneous nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ does not admit a hypersurface that satisfies the condition $A\phi + \phi A = 0$.

1 Introduction

The nearly Kähler (abbrev. NK) manifold $\mathbb{S}^3 \times \mathbb{S}^3$ is one of the only four homogeneous 6-dimensional nearly Kähler spaces (with the remaining three the NK 6-sphere $\mathbb{S}^6$, the complex projective space $\mathbb{C}P^3$ and the flag manifold $SU(3)/U(1) \times U(1)$, cf. [5, 6]). Ever since the groundbreaking research of Bolton-Dillen-Dioos-Vrancken [4], people become increasingly interested in the study of submanifolds of this homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$, and many beautiful results have been established. For details of the study, besides [4], we would refer the readers

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to [8, 12] on almost complex surfaces, to [1, 2, 9, 13, 18] on Lagrangian submanifolds, and to [11] on hypersurfaces. It is worth mentioning that Foscolo and Haskins [10] have recently constructed cohomogeneity one NK structure on both $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$. Thus, in order to avoid confusion, from now on in this paper, when we say NK $\mathbb{S}^6$ and NK $\mathbb{S}^3 \times \mathbb{S}^3$, we mean always $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$ equipped with the homogeneous NK structures that were elaborately described in [7] (cf. references therein) and [4], respectively.

In the present paper, continuing with our research starting from [11], we will focus mainly on hypersurfaces of the NK $\mathbb{S}^3 \times \mathbb{S}^3$. Recall that given a hypersurface M of an almost Hermitian manifold with almost complex structure J, it appears on M two naturally defined tensor fields of $(1,1)$-type: a submanifold structure represented by the shape operator A, and an almost contact structure ϕ induced from J. Then, it is an interesting problem to study hypersurfaces with special relations between A and ϕ. The first problem one might study is that the shape operator A and induced almost contact structure ϕ satisfy the commutativity condition $A\phi = \phi A$. Indeed, Okumura [17] and Montiel-Romero [16] considered real hypersurfaces of the non-flat complex space forms, and they obtained the classification of such real hypersurfaces satisfying $A\phi = \phi A$ for complex projective space [17] and complex hyperbolic space [16], respectively. Moreover, it was shown that hypersurfaces of the homogeneous NK $\mathbb{S}^6$ satisfy $A\phi = \phi A$ if and only if they are geodesic hyperspheres (cf. Theorem 2 of [15] and Remark 2.1 of [11]). Then following this approach, we have considered a similar situation for the NK $\mathbb{S}^3 \times \mathbb{S}^3$ [11], our result is the following classification theorem.

Theorem 1.1 (cf. [11]). *Let M be a hypersurface of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$ that satisfies the condition $A\phi = \phi A$. Then M is locally given by one of the following immersions f_1, f_2 and f_3:*

(1) $f_1: \ \mathbb{S}^3 \times \mathbb{S}^2 \to \mathbb{S}^3 \times \mathbb{S}^3 \ \ defined\ by \ \ (x,y) \mapsto (x,y);$

(2) $f_2: \ \mathbb{S}^3 \times \mathbb{S}^2 \to \mathbb{S}^3 \times \mathbb{S}^3 \ \ defined\ by \ \ (x,y) \mapsto (y,x);$

(3) $f_3: \ \mathbb{S}^3 \times \mathbb{S}^2 \to \mathbb{S}^3 \times \mathbb{S}^3 \ \ defined\ by \ \ (x,y) \mapsto (\bar{x}, y\bar{x}),$

here, $x \in \mathbb{S}^3$, $y \in \mathbb{S}^2$, and as usual $\mathbb{S}^3$ (resp. $\mathbb{S}^2$) is regarded as the set of the unit (resp. imaginary) quaternions in the quaternion space $\mathbb{H}$.

One might realize that the next simplest relation between the shape operator A and the induced almost contact structure ϕ is the anti-commutativity condition $A\phi + \phi A = 0$. In this respect, to our knowledge only Ki-Suh have shown that (cf. Lemma 2.1 and Proposition 2.2 of [14]), by denoting $\bar{M}^n(c)$ the n-dimensional complex space form of constant holomorphic sectional curvature c, if there exists a real hypersurface M of $\bar{M}^n(c)$ that satisfies the condition $A\phi + \phi A = 0$, then $c = 0$ and M is cylindrical. To see how about other ambient spaces, in this paper, we consider the question for two important 6-dimensional homogeneous NK manifolds, namely that the homogeneous NK $\mathbb{S}^6$ and the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$. Our first result is the following

Theorem 1.2. *The totally geodesic hypersurfaces of the homogeneous NK $\mathbb{S}^6$ are the only hypersurfaces of $\mathbb{S}^6$ satisfying the condition $A\phi + \phi A = 0$.*

For the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$, however, in Theorem 1.1 of [11], we have shown that it admits neither totally umbilical hypersurfaces nor hypersurfaces having parallel second fundamental form. Now, as the second result of this paper, a further nonexistence theorem can be proved that is stated as below.

Theorem 1.3. *The homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$ does not admit a hypersurface that satisfies the condition $A\phi + \phi A = 0$.*

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2 Preliminaries

2.1 The homogeneous NK structure on $\mathbb{S}^3 \times \mathbb{S}^3$

In this subsection, we review some elementary notions and results from [4].

By the natural identification $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3$, we can write a tangent vector at $(p,q) \in \mathbb{S}^3 \times \mathbb{S}^3$ as $Z(p,q) = (U_{(p,q)}, V_{(p,q)})$ or simply $Z = (U, V)$. Then the well-known almost complex structure J on $\mathbb{S}^3 \times \mathbb{S}^3$ is given by

$$JZ(p,q) = \tfrac{1}{\sqrt{3}}(2pq^{-1}V - U, -2qp^{-1}U + V). \tag{2.1}$$

Define the Hermitian metric g on $\mathbb{S}^3 \times \mathbb{S}^3$ by

$$\begin{aligned} g(Z,Z') &= \tfrac{1}{2}(\langle Z,Z'\rangle + \langle JZ,JZ'\rangle) \\ &= \tfrac{4}{3}(\langle U,U'\rangle + \langle V,V'\rangle) - \tfrac{2}{3}(\langle p^{-1}U, q^{-1}V'\rangle + \langle p^{-1}U', q^{-1}V\rangle), \end{aligned} \tag{2.2}$$

where $Z = (U, V)$, $Z' = (U', V')$ are tangent vectors, and $\langle\cdot,\cdot\rangle$ is the standard product metric on $\mathbb{S}^3 \times \mathbb{S}^3$. Then $\{g, J\}$ gives the homogeneous NK structure on $\mathbb{S}^3 \times \mathbb{S}^3$.

As usual let G be the (1,2)-tensor field defined by $G(X,Y) := (\tilde{\nabla}_X J)Y$, where $\tilde{\nabla}$ is Levi-Civita connection of g. Then, the following further formulas hold:

$$G(X,Y) + G(Y,X) = 0, \tag{2.3}$$

$$G(X,JY) + JG(X,Y) = 0, \tag{2.4}$$

$$g(G(X,Y),Z) + g(G(X,Z),Y) = 0, \tag{2.5}$$

$$\begin{aligned} g(G(X,Y),G(Z,W)) =& \tfrac{1}{3}\big[g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\ &+ g(JX,Z)g(JW,Y) - g(JX,W)g(JZ,Y)\big]. \end{aligned} \tag{2.6}$$

An almost product structure P on $\mathbb{S}^3 \times \mathbb{S}^3$ is introduced by:

$$PZ = (pq^{-1}V, qp^{-1}U), \ \forall\, Z = (U,V) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3). \tag{2.7}$$

Then we have the following formula for $\tilde{\nabla}P$:

$$2(\tilde{\nabla}_X P)Y = JG(X, PY) + JPG(X, Y). \tag{2.8}$$

The curvature tensor $\tilde{R}$ of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$ is given by:

$$\begin{aligned}\tilde{R}(X,Y)Z =& \tfrac{5}{12}\big[g(Y,Z)X - g(X,Z)Y\big] \\ &+ \tfrac{1}{12}\big[g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ\big] \\ &+ \tfrac{1}{3}\Big[g(PY,Z)PX - g(PX,Z)PY \\ &\quad + g(JPY,Z)JPX - g(JPX,Z)JPY\Big].\end{aligned} \tag{2.9}$$

2.2 Hypersurfaces of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$

Let M be a hypersurface of the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$ with ξ its unit normal vector field. For any vector field X tangent to M, we have the decomposition

$$JX = \phi X + f(X)\xi, \tag{2.10}$$

where ϕX and $f(X)\xi$ are, respectively, the tangent and normal parts of JX. Then ϕ is a tensor field of type (1,1), and f is a 1-form on M. By definition, ϕ and f satisfy the following relations:

$$\begin{cases} f(X) = g(X,U), \;\; f(\phi X) = 0, \;\; \phi^2 X = -X + f(X)U, \\ g(\phi X, Y) = -g(X, \phi Y), \;\; g(\phi X, \phi Y) = g(X,Y) - f(X)f(Y), \end{cases} \tag{2.11}$$

where $U := -J\xi$, which is called the *structure vector field* of M. Equation (2.11) shows that (ϕ, U, f) determines an *almost contact structure* over M.

Let ∇ be the induced connection on M with R its Riemannian curvature tensor. The formulas of Gauss and Weingarten state that

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \tilde{\nabla}_X \xi = -AX, \;\; \forall\, X, Y \in TM, \tag{2.12}$$

where h is the second fundamental form, and it is related to the shape operator A by $h(X,Y) = g(AX,Y)\xi$. Here, using the formulas of Gauss and Weingarten, we have

$$\nabla_X U = \phi AX - G(X, \xi). \tag{2.13}$$

The Gauss and Codazzi equations of M are given by

$$\begin{aligned}R(X,Y)Z =& \tfrac{5}{12}\big[g(Y,Z)X - g(X,Z)Y\big] \\ &+ \tfrac{1}{12}\big[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\big] \\ &+ \tfrac{1}{3}\Big[g(PY,Z)(PX)^{\top} - g(PX,Z)(PY)^{\top} \\ &\quad + g(JPY,Z)(JPX)^{\top} - g(JPX,Z)(JPY)^{\top}\Big] \\ &+ g(AZ,Y)AX - g(AZ,X)AY,\end{aligned} \tag{2.14}$$

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X =& \tfrac{1}{12}\big[g(X,U)\phi Y - g(Y,U)\phi X - 2g(\phi X,Y)U\big]\\ &+ \tfrac{1}{3}\Big[g(PX,\xi)(PY)^\top - g(PY,\xi)(PX)^\top\\ &\quad + g(PX,U)(JPY)^\top - g(PY,U)(JPX)^\top\Big],\end{aligned} \tag{2.15}$$

where $\cdot^\top$ denotes the tangential part.

Following the usual terminology, we call a hypersurface M of the NK $\mathbb{S}^3 \times \mathbb{S}^3$ the *Hopf hypersurface* if the integral curves of the structure vector field U are geodesics of M, that is $\nabla_U U = 0$. It is also equivalent that the structure vector field U is a principal direction, with principal curvature function denoted by μ. A basic lemma for Hopf hypersurfaces of the NK $\mathbb{S}^3 \times \mathbb{S}^3$ is stated as follows:

Lemma 2.1. *Let M be a Hopf hypersurface in the homogeneous NK $\mathbb{S}^3 \times \mathbb{S}^3$. Then we have*

$$\begin{aligned}&\tfrac{1}{6}g(\phi X,Y) - \tfrac{2}{3}\big[g(PX,\xi)g(PY,U) - g(PX,U)g(PY,\xi)\big]\\ &= g((\mu I - A)G(X,\xi),Y) + g(G((\mu I - A)X,\xi),Y)\\ &\quad - \mu g((A\phi + \phi A)X,Y) + 2g(A\phi AX,Y),\quad X,Y \in \{U\}^\perp,\end{aligned} \tag{2.16}$$

where $\{U\}^\perp$ denotes a distribution of TM that is orthogonal to U, and I denotes the identity transformation.

Proof. A direct calculation of $(\nabla_X A)U$, with using $AU = \mu U$, (2.13), we have

$$(\nabla_X A)U = X(\mu)U + (\mu I - A)(-G(X,\xi) + \phi AX). \tag{2.17}$$

It follows that, for $\forall\, X,Y \in \{U\}^\perp$,

$$g((\nabla_X A)Y,U) = g((\nabla_X A)U,Y) = g((\mu I - A)(-G(X,\xi) + \phi AX),Y). \tag{2.18}$$

Thus, we have

$$\begin{aligned}g((\nabla_X A)Y - (\nabla_Y A)X,U) =& - g((\mu I - A)G(X,\xi),Y) - 2g(A\phi AX,Y)\\ &- g(G((\mu I - A)X,\xi),Y) + \mu g((A\phi + \phi A)X,Y).\end{aligned} \tag{2.19}$$

On the other hand, by using the Codazzi equation (2.15), we get

$$\begin{aligned}&g((\nabla_X A)Y - (\nabla_Y A)X,U)\\ &= -\tfrac{1}{6}g(\phi X,Y) + \tfrac{2}{3}(g(PX,\xi)g(PY,U) - g(PX,U)g(PY,\xi)).\end{aligned} \tag{2.20}$$

From (2.19) and (2.20), we immediately get (2.16). ∎

Before concluding this section, following that in [11] we introduce the distribution $\mathfrak{D}$. When we study hypersurfaces of the NK $\mathbb{S}^3 \times \mathbb{S}^3$, the consideration of $\mathfrak{D}$ is very helpful for the choice of a canonical frame. Precisely, for each point $p \in M$, we define

$$\mathfrak{D}(p) := \mathrm{Span}\,\{\xi(p), U(p), P\xi(p), PU(p)\}.$$

Since P is anti-commutative with J, it is clear that $\mathfrak{D}$ defines a distribution on M with dimension 2 or 4, and that it is invariant under the action of both J and

P. Along M, let $\mathfrak{D}^{\perp}$ denote the distribution in $T(\mathbb{S}^3 \times \mathbb{S}^3)$ that is orthogonal to $\mathfrak{D}$ at each $p \in M$.

If $\dim \mathfrak{D} = 4$ holds in an open set, then there exists a unit tangent vector field $e_1 \in \mathfrak{D}$ and functions a, b, c with $c > 0$ such that

$$P\xi = a\xi + bU + ce_1, \;\; a^2 + b^2 + c^2 = 1. \tag{2.21}$$

Put $e_2 = Je_1$. From the fact $\dim \mathfrak{D}^{\perp} = 2$ and that $\mathfrak{D}^{\perp}$ is invariant under the action of both J and P, we can choose a local unit vector field $e_3 \in \mathfrak{D}^{\perp}$ such that $Pe_3 = e_3$. Put $e_4 = Je_3$ and $e_5 = U$. Then $\{e_i\}_{i=1}^5$ is a well-defined orthonormal basis of TM and, acting by P, it has the following properties:

$$\begin{cases} P\xi = a\xi + ce_1 + be_5, \;\; Pe_1 = c\xi - ae_1 - be_2, \\ Pe_2 = ce_5 - be_1 + ae_2, \;\; Pe_3 = e_3, \\ Pe_4 = -e_4, \;\; Pe_5 = b\xi + ce_2 - ae_5. \end{cases} \tag{2.22}$$

If $\dim \mathfrak{D} = 2$ holds in an open set, then we can write

$$P\xi = a\xi + bU, \;\; a^2 + b^2 = 1. \tag{2.23}$$

Now, $\mathfrak{D}^{\perp}$ is a 4-dimensional distribution that is invariant under the action of both J and P. Hence, we can choose unit vector fields e_1, $e_3 \in \mathfrak{D}^{\perp}$ such that $Pe_1 = e_1$, $Pe_3 = e_3$. Put $e_2 = Je_1$, $e_4 = Je_3$ and $e_5 = U$. In this way, we obtain an orthonormal basis $\{e_i\}_{i=1}^5$ of TM. However, we would remark that such choice of $\{e_1, e_3\}$ (resp. $\{e_2, e_4\}$) is unique up to an orthogonal transformation.

3 Proof of Theorem 1.2

For basic results of the well-known NK $\mathbb{S}^6$, i.e., the six-dimensional unit sphere $\mathbb{S}^6$ equipped with a homogeneous NK structure (J, g), of which J is the almost complex structure defined by using the vector cross product of purely imaginary Cayley numbers $\mathbb{R}^7$ and g is the metric induced from the Euclidean space $\mathbb{R}^7$, we refer to [7] and the references therein.

Let M be an orientable hypersurface of the NK $\mathbb{S}^6$ with ξ its unit normal vector field. Then, the equations from (2.10) up to (2.13) in subsection 2.2 also hold, so that M admits an almost contact metric structure (ϕ, U, f, g) induced from the NK structure of $\mathbb{S}^6$, whereas the Codazzi equation becomes

$$(\nabla_X A)Y = (\nabla_Y A)X, \;\; \forall\, X, Y \in TM. \tag{3.1}$$

For the NK $\mathbb{S}^6$, totally geodesic hypersurfaces do exist and they trivially satisfy the relation $A\phi + \phi A = 0$.

Now, we assume that M is an orientable hypersurface of the NK $\mathbb{S}^6$ that satisfies the condition $A\phi + \phi A = 0$. Then, by definition $\phi U = 0$, we have $AU = \mu U$, i.e., M is a Hopf hypersurface and, μ is the principal curvature function corresponding to the structure vector field U. Moreover, if $X \in \{U\}^{\perp}$ is a principal vector field with principal curvature function λ, then $A\phi X = -\phi AX = -\lambda\phi X$

implies that ϕX is also a principal vector field with principal curvature function $-\lambda$.

Recall that Berndt-Bolton-Woodward (Theorem 2 of [3]) proved that a connected Hopf hypersurface of the NK $\mathbb{S}^6$ is an open part of either a geodesic hypersphere of $\mathbb{S}^6$ or a tube around an almost complex curve in the NK $\mathbb{S}^6$, and the principal curvature function μ is constant (Lemma 2 of [3]).

Similar to the proof of Lemma 2.1, for Hopf hypersurfaces of the NK $\mathbb{S}^6$, we can easily show that, by using (2.13), the following basic equation holds:

$$\begin{aligned} &g((\mu I - A)G(X,\xi),Y) + g(G((\mu I - A)X,\xi),Y) \\ &\quad - \mu g((A\phi + \phi A)X,Y) + 2g(A\phi AX,Y) = 0, \quad X,Y \in TM. \end{aligned} \tag{3.2}$$

If M is a geodesic hypersphere, then M is totally umbilical and we have a function λ on M such that $AX = \lambda X, \forall\, X \in TM$. This together with $A\phi + \phi A = 0$ implies that $\lambda = 0$. Hence, M is a totally geodesic hypersurface.

If M is a tube around an almost complex curve Γ with radius r in $\mathbb{S}^6$, then, according to the proof of Proposition 2 and subsequent Remark in [3], we have $AU = -\cot r\, U$, and the remaining principal curvatures on the distribution $\{U\}^{\perp}$ are $\tan(\theta + r)$, $\tan(\theta - r)$ and $-\cot r$ for $\theta \in [0, \frac{\pi}{2})$ which is a function on M. Moreover, as [3] has pointed out, the hypersurface M has exactly two or three distinct principal curvatures at each point. We denote by ν, $2 \le \nu \le 3$, the maximum number of distinct principal curvatures on M, then the set $M_\nu = \{x \in M | M$ has exactly ν distinct principal curvatures at $x\}$ is a non-empty open subset of M. By the continuity of the principal curvature function, each connected component of M_ν is an open subset, and the multiplicities of distinct principal curvatures remain unchanged on each connected component of M_ν, so we can find a local smooth frame field with respect to the principal curvatures. The following discussion will be divided into two cases, depending on the value of ν.

Case I. $\nu = 3$.

In this case, on each connected component of M_3, the multiplicities of the distinct principal curvatures, namely $\tan(\theta + r)$, $\tan(\theta - r)$ and $-\cot r$, should be $1, 1$ and 3, respectively. Then we have an orthonormal frame field $\{X_i\}_{i=1}^5$ such that

$$\begin{cases} AX_1 = \tan(\theta + r)X_1,\ AX_2 = \tan(\theta - r)X_2,\ AX_3 = -\cot rX_3, \\ AX_4 = -\cot rX_4,\ AX_5 = -\cot rX_5,\ X_5 = U. \end{cases}$$

Applying the condition $A\phi + \phi A = 0$, we have

$$A\phi X_1 = -\tan(\theta + r)\phi X_1,\ \ A\phi X_2 = -\tan(\theta - r)\phi X_2,\ \ A\phi X_3 = \cot r\phi X_3.$$

Taking $X = X_1$ and $Y = \phi X_1$ in (3.2), and using $A\phi + \phi A = 0$, we get $\tan(\theta + r) = 0$. Analogously, taking $X = X_2$ and $Y = \phi X_2$ in (3.2), we get $\tan(\theta - r) = 0$, which is a contradiction with $\tan(\theta + r) \neq \tan(\theta - r)$. Thus, **Case I** does not occur.

Case II. $\nu = 2$.

In this case, M has exactly two distinct principal curvatures, that is, two of the three principal curvatures $\tan(\theta + r)$, $\tan(\theta - r)$ and $-\cot r$ are equal. Without loss of generality, we assume that $\tan(\theta + r) = -\cot r$, so that the multiplicities of the distinct principal curvatures, namely $\tan(\theta - r)$ and $-\cot r$, are 1 and 4, respectively. Then, we have an orthonormal frame field $\{X_i\}_{i=1}^5$ such that

$$\begin{cases} AX_1 = \tan(\theta - r)X_1,\ AX_2 = -\cot rX_2,\ AX_3 = -\cot rX_3, \\ AX_4 = -\cot rX_4,\ AX_5 = -\cot rX_5,\ X_5 = U. \end{cases}$$

Applying $A\phi + \phi A = 0$, we get $A\phi X_1 = -\tan(\theta - r)\phi X_1$ and $A\phi X_2 = \cot r\phi X_2$. Then taking in (3.2) that $(X, Y) = (X_1, \phi X_1)$ and $(X, Y) = (X_2, \phi X_2)$, respectively, we immediately get $\tan(\theta - r) = -\cot r = 0$. This is again a contradiction.

This completes the proof of Theorem 1.2. ∎

4 Proof of Theorem 1.3

To give the proof, we assume that M is a hypersurface of the NK $\mathbb{S}^3 \times \mathbb{S}^3$ which satisfies the condition $A\phi + \phi A = 0$. Then, by the fact $\phi U = 0$, we see that M is a Hopf hypersurface with $AU = \mu U$. Moreover, if $X \in \{U\}^\perp$ is a principal vector field with principal curvature function λ, i.e., $AX = \lambda X$, then $A\phi X = -\phi AX = -\lambda\phi X$ implies that ϕX is also a principal vector field with principal curvature function $-\lambda$. We denote λ, $-\lambda$, β, $-\beta$ with $\lambda \geq 0$ and $\beta \geq 0$ the four principal curvatures on distribution $\{U\}^\perp$. Since the only possible dimension of $\mathfrak{D}$ is 2 or 4, we will divide the proof of Theorem 1.3 into the proofs of two Lemmas. First, we have the following Lemma.

Lemma 4.1. *The case* $\dim\mathfrak{D} = 4$ *does not occur.*

Proof. Suppose that $\dim\mathfrak{D} = 4$ does occur on some point of M. We denote by $\Omega = \{x \in M|$ the dimension of $\mathfrak{D}$ is 4 at $x\}$, then Ω is an open subset of M. Since $A\phi + \phi A = 0$, we can write (2.16) on Ω as

$$\begin{aligned} \tfrac{1}{6}g(\phi X, Y) - \tfrac{2}{3}\big[g(PX,\xi)g(PY,U) - g(PX,U)g(PY,\xi)\big] = -2g(\phi A^2X, Y) \\ + g\big((\mu I - A)G(X,\xi), Y\big) + g\big(G((\mu I - A)X,\xi), Y\big), \quad X, Y \in \{U\}^\perp. \end{aligned} \tag{4.1}$$

We denote by ν ($\nu \leq 5$) the maximum number on Ω of distinct principal curvatures, then the set $\Omega_\nu := \{x \in \Omega \,|\, M$ has exactly ν distinct principal curvatures at $x\}$ is a non-empty open subset of M. By the continuity of the principal curvature function, each connected component of Ω_ν is an open subset, the multiplicities of distinct principal curvatures remain unchanged on each connected component of Ω_ν, so we can find a local smooth frame field with respect to the principal curvatures. From Theorem 1.1 of [11], we know that M can not be totally umbilical, even locally. So the following discussion will be divided into four cases, depending on the value of ν, $2 \leq \nu \leq 5$.

Case I. $\nu = 5$.

In this case, on each connected component of Ω_5, we can have an orthonormal frame field $\{X_i\}_{i=1}^5$ such that

$$AX_1 = \lambda X_1,\ \ AX_2 = \beta X_2,\ \ AX_3 = -\lambda X_3,\ \ AX_4 = -\beta X_4,\ \ AX_5 = \mu X_5, \tag{4.2}$$

where $X_3 = JX_1$, $X_4 = JX_2$, $X_5 = U$. As $\nu = 5$, we have $\lambda > 0$, $\beta > 0$, $\lambda \neq \beta$ and $\mu \notin \{\lambda, -\lambda, \beta, -\beta\}$. Let $\{e_i\}_{i=1}^5$ be the frame field as described in (2.22). Then, by assuming that $X_i = \sum_{j=1}^4 a_{ij} e_j$ for $1 \le i \le 4$, we have $(a_{ij}) \in SO(4)$, and by the choice of $\{e_i\}_{i=1}^5$ it holds that

$$a_{i+2,j} = (-1)^j a_{i,3-j}, \ \ a_{i+2,j+2} = (-1)^j a_{i,5-j}, \ \ i,j = 1,2. \tag{4.3}$$

First, taking $X = X_i$ and $Y = X_j$ in (4.1) for $1 \le i < j \le 4$, using (2.3)–(2.5) and (2.22), we can derive the following equations:

$$-\tfrac{1}{6} + \tfrac{2}{3}c^2 a_{11}^2 + \tfrac{2}{3}c^2 a_{12}^2 = 2\lambda^2, \tag{4.4}$$

$$-\tfrac{1}{6} + \tfrac{2}{3}c^2 a_{21}^2 + \tfrac{2}{3}c^2 a_{22}^2 = 2\beta^2, \tag{4.5}$$

$$\tfrac{2}{3}c^2 a_{11}a_{21} + \tfrac{2}{3}c^2 a_{12}a_{22} = (2\mu + \lambda - \beta) g(G(X_1, X_2), U), \tag{4.6}$$

$$\tfrac{2}{3}c^2 a_{11}a_{21} + \tfrac{2}{3}c^2 a_{12}a_{22} = -(2\mu - \lambda + \beta) g(G(X_1, X_2), U), \tag{4.7}$$

$$\tfrac{2}{3}c^2 a_{11}a_{22} - \tfrac{2}{3}c^2 a_{12}a_{21} = (2\mu - \lambda - \beta) g(G(X_1, X_2), \xi), \tag{4.8}$$

$$\tfrac{2}{3}c^2 a_{11}a_{22} - \tfrac{2}{3}c^2 a_{12}a_{21} = -(2\mu + \lambda + \beta) g(G(X_1, X_2), \xi). \tag{4.9}$$

The equations (4.6) and (4.7), (4.8) and (4.9) imply that

$$4\mu g(G(X_1, X_2), U) = 0, \ \ 4\mu g(G(X_1, X_2), \xi) = 0. \tag{4.10}$$

From (2.3), (2.4) and (2.5) we see that, for $1 \le i \le 4$, it holds $g(G(X_1, X_2), X_i) = 0$. Thus, $G(X_1, X_2) \in \operatorname{Span}\{\xi, U\}$. On the other hand, from (2.6), we have

$$g(G(X_1, X_2), G(X_1, X_2)) = \tfrac{1}{3}. \tag{4.11}$$

It follows from (4.10) that $\mu = 0$.

Second, from the fact $AU = 0$, we have

$$(\nabla_X A)U - (\nabla_U A)X = -A\nabla_X U - \nabla_U AX + A\nabla_U X. \tag{4.12}$$

On the other hand, applying (2.22) to the Codazzi equation (2.15), we can get

$$(\nabla_{e_1} A)U - (\nabla_U A)e_1 = -\tfrac{1}{12}e_2 - \tfrac{1}{3}\big[2acU - 2abe_1 + (2a^2 - 1)e_2\big], \tag{4.13}$$

$$(\nabla_{e_2} A)U - (\nabla_U A)e_2 = \tfrac{1}{12}e_1 - \tfrac{1}{3}\big[2bcU + (1 - 2b^2)e_1 + 2abe_2\big]. \tag{4.14}$$

Then, from (4.12) and (4.13), calculating the U-component of both the right hand sides, we can get $ac = 0$. Analogously, from (4.12) and (4.14), we can get $bc = 0$. Therefore, according to (2.21), we have $a = b = 0$ and $c = 1$.

Third, in order to apply the Codazzi equations, we need to calculate the connections $\{\nabla_{X_i} X_j\}$. Put $\nabla_{X_i} X_j = \sum \Gamma_{ij}^k X_k$ with $\Gamma_{ij}^k = -\Gamma_{ik}^j$, $1 \le i,j,k \le 5$. Assume that

$$g(G(X_1, X_2), \xi) = k, \ \ g(G(X_1, X_2), U) = l. \tag{4.15}$$

Then (4.11) and the fact $G(X_1, X_2) \in \operatorname{Span}\{\xi, U\}$ show that $k^2 + l^2 = \tfrac{1}{3}$.

By definition and the Gauss and Weingarten formulas, we have the calculation

$$G(X_1,\xi) = -\sum_{i=1}^{5} \Gamma_{15}^{i} X_i + \lambda X_3.$$

However, according to (2.3)–(2.5) and (4.15), we also have $G(X_1,\xi) = -kX_2 + lX_4$. Hence, we obtain

$$\Gamma_{15}^1 = 0,\ \ \Gamma_{15}^2 = k,\ \ \Gamma_{15}^3 = \lambda,\ \ \Gamma_{15}^4 = -l. \tag{4.16}$$

Similarly, taking $(X,Y) = (X_i,\xi)$ in $G(X,Y) = (\tilde{\nabla}_X J)Y$ for $2 \le i \le 4$, and by use of (2.3)–(2.5) and (4.15), we further obtain

$$\begin{cases} \Gamma_{25}^1 = -k,\ \ \Gamma_{25}^2 = 0,\ \ \Gamma_{25}^3 = l,\ \ \Gamma_{25}^4 = \beta, \\ \Gamma_{35}^1 = \lambda,\ \ \Gamma_{35}^2 = -l,\ \ \Gamma_{35}^3 = 0,\ \ \Gamma_{35}^4 = -k, \\ \Gamma_{45}^1 = l,\ \ \Gamma_{45}^2 = \beta,\ \ \Gamma_{45}^3 = k,\ \ \Gamma_{45}^4 = 0. \end{cases} \tag{4.17}$$

Moreover, by using (4.15) and the Gauss and Weingarten formulas, we get

$$lX_2 + kX_4 = G(U,X_1) = \sum_{i=1}^{5} \Gamma_{53}^{i} X_i - \sum_{i=1}^{5} \Gamma_{51}^{i} JX_i. \tag{4.18}$$

It follows that

$$\Gamma_{53}^2 + \Gamma_{51}^4 = l,\ \ \Gamma_{53}^4 - \Gamma_{51}^2 = k. \tag{4.19}$$

Finally, we will calculate the expressions $(\nabla_U A)e_i - (\nabla_{e_i} A)U$ for $1 \le i \le 4$.

On one hand, for each $1 \le i \le 4$, we directly calculate $(\nabla_U A)e_i - (\nabla_{e_i} A)U$, with the use of $e_i = \sum_{j=1}^4 a_{ji} X_j$ and the preceding results (4.16) and (4.17). Then we get an expression for $(\nabla_U A)e_i - (\nabla_{e_i} A)U$ in terms of the frame field $\{X_i\}_{i=1}^4$.

On the other hand, for each $1 \le i \le 4$, we calculate $(\nabla_U A)e_i - (\nabla_{e_i} A)U$ by the Codazzi equation (2.15). Then, by using (2.22) and $e_i = \sum_{j=1}^4 a_{ji} X_j$, we get another expression of $(\nabla_U A)e_i - (\nabla_{e_i} A)U$ in terms of the frame field $\{X_i\}_{i=1}^4$.

In this way, comparing both calculations of $(\nabla_U A)e_i - (\nabla_{e_i} A)U$ for each $1 \le i \le 4$, we get a matrices equation $C = (a_{ij})^T B$, where

$$C = \begin{pmatrix} -\frac{1}{4}a_{12} & -\frac{1}{4}a_{22} & -\frac{1}{4}a_{11} & -\frac{1}{4}a_{21} \\ \frac{1}{4}a_{11} & \frac{1}{4}a_{21} & -\frac{1}{4}a_{12} & -\frac{1}{4}a_{22} \\ \frac{1}{12}a_{14} & \frac{1}{12}a_{24} & \frac{1}{12}a_{13} & \frac{1}{12}a_{23} \\ -\frac{1}{12}a_{13} & -\frac{1}{12}a_{23} & \frac{1}{12}a_{14} & \frac{1}{12}a_{24} \end{pmatrix},$$

$$B = \begin{pmatrix} U(\lambda) & (\lambda-\beta)\Gamma_{51}^2 + \beta k & 2\lambda\Gamma_{51}^3 - \lambda^2 & (\lambda+\beta)\Gamma_{51}^4 + \beta l \\ (\beta-\lambda)\Gamma_{52}^1 - \lambda k & U(\beta) & (\lambda+\beta)\Gamma_{52}^3 - \lambda l & 2\beta\Gamma_{52}^4 - \beta^2 \\ -2\lambda\Gamma_{53}^1 + \lambda^2 & (-\lambda-\beta)\Gamma_{53}^2 - \beta l & -U(\lambda) & (\beta-\lambda)\Gamma_{53}^4 + \beta k \\ (-\lambda-\beta)\Gamma_{54}^1 + \lambda l & -2\beta\Gamma_{54}^2 + \beta^2 & (\lambda-\beta)\Gamma_{54}^3 - \lambda k & -U(\beta) \end{pmatrix}.$$

Thus, $B = (a_{ij})C := (B_{ij})$. Using (4.3), it is straightforward to verify that $B = (a_{ij})C$ is skew-symmetric. From the facts $B_{12} + B_{21} = 0$ and $\lambda \neq \beta$, we have $\Gamma_{51}^2 = \frac{k}{2}$. Moreover, from the facts $B_{34} + B_{43} = 0$ and $\lambda \neq \beta$, we have

$\Gamma_{53}^4 = -\frac{k}{2}$. Combining these with (4.19) we get $k = 0$. Analogously, from the facts $B_{23}+B_{32} = 0$, $B_{14}+B_{41} = 0$, $\lambda+\beta \neq 0$ and (4.19), we can further get $l = 0$. Thus, we get a contradiction to $k^2+l^2 = \frac{1}{3}$. This implies that **Case I** does not occur.

Case II. $\nu = 4$.

In this case, on a connected component of Ω_4, without loss of generality, we are sufficient to consider the following two subcases:
II-(i): $\lambda \neq \beta,\ \lambda > 0,\ \beta > 0$ and $\mu \in \{\lambda, \beta, -\lambda, -\beta\}$.
II-(ii): $\lambda = 0,\ \beta > 0$ and $\mu \notin \{0, \beta, -\beta\}$.

For both of the above two subcases, following similar arguments as the discussion of Case I from (4.2) up to (4.11), we can also get $\mu = 0$. This is a contradiction, showing that **Case II** does not occur.

Case III. $\nu = 3$.

In this case, on a connected component of Ω_3, without loss of generality, we are sufficient to consider the following three subcases:
III-(i): $\lambda = 0,\ \beta > 0$ and $\mu \in \{\beta, -\beta\}$.
III-(ii): $\lambda = \mu = 0$ and $\beta > 0$.
III-(iii): $\lambda = \beta > 0$ and $\mu \notin \{\lambda, -\lambda\}$.

In case **III-(i)**, similar arguments as the discussion of Case I from (4.2) up to (4.11), we can get $\mu = 0$. Thus, we get a contradiction.

In case **III-(ii)**, taking an orthonormal frame field $\{X_i\}_{i=1}^5$ satisfying (4.2), we still have the equations from (4.4) up to (4.14). Then we can get $c = 1$. By calculating (4.4)+(4.5) and that $(a_{ij}) \in SO(4)$, we further have the conclusion

$$\lambda^2+\beta^2 = \tfrac{1}{6}. \tag{4.20}$$

By $\lambda = 0$, we have $\beta = \frac{\sqrt{6}}{6}$. Then (4.4) and (4.5) give that

$$a_{11}^2+a_{12}^2 = \tfrac{1}{4},\ \ a_{21}^2+a_{22}^2 = \tfrac{3}{4}. \tag{4.21}$$

On the other hand, making the summation $(4.6)^2+(4.8)^2$, we easily see that

$$(a_{11}^2+a_{12}^2)(a_{21}^2+a_{22}^2) = \tfrac{1}{8},$$

which is a contradiction to (4.21).

In case **III-(iii)**, taking an orthonormal frame field $\{X_i\}_{i=1}^5$ satisfying (4.2), we can also derive the equations from (4.4) up to (4.11), thus we have $\mu = 0$. Then, similarly, we have the equations from (4.12) up to (4.14), so we get in (2.22) that $a = b = 0$ and $c = 1$, and by calculating (4.4)+(4.5), we get $\lambda = \beta = \frac{\sqrt{3}}{6}$. It follows from (4.4), (4.5) and (4.6) that

$$a_{11}^2+a_{12}^2 = \tfrac{1}{2},\ \ a_{21}^2+a_{22}^2 = \tfrac{1}{2},\ \ a_{11}a_{21}+a_{12}a_{22} = 0. \tag{4.22}$$

Let us put $a_{11} = \frac{1}{\sqrt{2}}\cos\theta_1$, $a_{12} = \frac{1}{\sqrt{2}}\sin\theta_1$, $a_{21} = \frac{1}{\sqrt{2}}\cos\theta_2$ and $a_{22} = \frac{1}{\sqrt{2}}\sin\theta_2$. Then $0 = a_{11}a_{21}+a_{12}a_{22} = \frac{1}{2}\cos(\theta_1-\theta_2)$ implies that $\theta_1-\theta_2 = \frac{\pi}{2}(2k+1)$, $k \in \mathbb{Z}$. Therefore, we have either $(a_{21}, a_{22}) = (a_{12}, -a_{11})$ or $(a_{21}, a_{22}) = (-a_{12}, a_{11})$. If necessary by taking $-X_2$ instead of X_2, we are sufficient to consider the case that $a_{21} = a_{12}$ and $a_{22} = -a_{11}$.

From (4.22) and that $(a_{ij}) \in SO(4)$, we further have

$$a_{13}^2 + a_{14}^2 = \tfrac{1}{2},\ \ a_{23}^2 + a_{24}^2 = \tfrac{1}{2},\ \ a_{13}a_{23} + a_{14}a_{24} = 0.$$

This implies that, similar to the preceding paragraph, $(a_{23}, a_{24}) = (a_{14}, -a_{13})$ or $(a_{23}, a_{24}) = (-a_{14}, a_{13})$. If $a_{23} = a_{14}$ and $a_{24} = -a_{13}$, then $X_2 = -X_3$, which is impossible. Thus, $a_{23} = -a_{14}$ and $a_{24} = a_{13}$ hold.

For simplicity, we put $m = -\frac{2\sqrt{6}}{3}a_{13}a_{14}$ and $n = \frac{\sqrt{6}}{3}(a_{14}^2 - a_{13}^2)$. Then $m^2 + n^2 = \frac{1}{6}$.

Now, from (2.22) we can express $\{PX_i\}_{i=1}^4$ as follows:

$$\begin{cases} PX_1 = a_{11}\xi + a_{12}U - \frac{\sqrt{6}}{2}nX_1 + \frac{\sqrt{6}}{2}mX_2 + \frac{\sqrt{6}}{2}mX_3 + \frac{\sqrt{6}}{2}nX_4, \\ PX_2 = a_{12}\xi - a_{11}U + \frac{\sqrt{6}}{2}mX_1 + \frac{\sqrt{6}}{2}nX_2 + \frac{\sqrt{6}}{2}nX_3 - \frac{\sqrt{6}}{2}mX_4, \\ PX_3 = -a_{12}\xi + a_{11}U + \frac{\sqrt{6}}{2}mX_1 + \frac{\sqrt{6}}{2}nX_2 + \frac{\sqrt{6}}{2}nX_3 - \frac{\sqrt{6}}{2}mX_4, \\ PX_4 = a_{11}\xi + a_{12}U + \frac{\sqrt{6}}{2}nX_1 - \frac{\sqrt{6}}{2}mX_2 - \frac{\sqrt{6}}{2}mX_3 - \frac{\sqrt{6}}{2}nX_4. \end{cases} \tag{4.23}$$

Then, applying the Codazzi equation (2.15), we get

$$\begin{aligned}(\nabla_{X_1}A)X_3 - (\nabla_{X_3}A)X_1 = \tfrac{1}{6}U &+ \tfrac{\sqrt{6}}{3}(a_{11}m - a_{12}n)X_1 + \tfrac{\sqrt{6}}{3}(a_{11}n + a_{12}m)X_2 \\ &+ \tfrac{\sqrt{6}}{3}(a_{11}n + a_{12}m)X_3 + \tfrac{\sqrt{6}}{3}(-a_{11}m + a_{12}n)X_4,\end{aligned} \tag{4.24}$$

$$\begin{aligned}(\nabla_{X_1}A)X_4 - (\nabla_{X_4}A)X_1 =& \tfrac{\sqrt{6}}{3}(a_{11}n + a_{12}m)X_1 + \tfrac{\sqrt{6}}{3}(-a_{11}m + a_{12}n)X_2 \\ &+ \tfrac{\sqrt{6}}{3}(-a_{11}m + a_{12}n)X_3 + \tfrac{\sqrt{6}}{3}(-a_{11}n - a_{12}m)X_4.\end{aligned} \tag{4.25}$$

Let $\nabla_{X_i}X_j = \sum \Gamma_{ij}^k X_k$ with $\Gamma_{ij}^k = -\Gamma_{ik}^j$, $1 \le i, j, k \le 5$. Then, from (4.24) and (4.25), after calculating the left hand sides of (4.24) and (4.25) respectively, we get

$$\begin{cases} \Gamma_{13}^1 = -\sqrt{2}(a_{11}m - a_{12}n),\ \ \Gamma_{13}^2 = -\sqrt{2}(a_{11}n + a_{12}m), \\ \Gamma_{14}^1 = -\sqrt{2}(a_{11}n + a_{12}m),\ \ \Gamma_{14}^2 = -\sqrt{2}(-a_{11}m + a_{12}n). \end{cases} \tag{4.26}$$

Next, (4.8) gives that $g(G(X_1, X_2), \xi) = \frac{\sqrt{3}}{3}$, and so that $g(G(X_1, X_2), U) = 0$ from (4.11). Then by the relations (2.3)–(2.5) we can easily solve $G(X_1, \xi) = -\frac{\sqrt{3}}{3}X_2$. Thus, by the Gauss and Weingarten formulas, a direct calculation gives that

$$G(X_1, \xi) = (\tilde{\nabla}_{X_1}J)\xi = -\sum_{i=1}^{5} \Gamma_{15}^i X_i + \tfrac{\sqrt{3}}{6}X_3. \tag{4.27}$$

Hence, we have

$$\Gamma_{15}^2 = \tfrac{\sqrt{3}}{3},\ \ \Gamma_{15}^3 = \tfrac{\sqrt{3}}{6},\ \ \Gamma_{15}^1 = \Gamma_{15}^4 = 0. \tag{4.28}$$

By (4.26) and (4.28), we obtain

$$\begin{cases}\nabla_{X_1}U=\frac{\sqrt{3}}{3}X_2+\frac{\sqrt{3}}{6}X_3,\\ \nabla_{X_1}X_1=\Gamma_{11}^2X_2+\sqrt{2}(a_{11}m-a_{12}n)X_3+\sqrt{2}(a_{11}n+a_{12}m)X_4,\\ \nabla_{X_1}X_2=\Gamma_{12}^1X_1+\sqrt{2}(a_{11}n+a_{12}m)X_3+\sqrt{2}(-a_{11}m+a_{12}n)X_4-\frac{\sqrt{3}}{3}U,\\ \nabla_{X_1}X_3=-\sqrt{2}(a_{11}m-a_{12}n)X_1-\sqrt{2}(a_{11}n+a_{12}m)X_2+\Gamma_{13}^4X_4-\frac{\sqrt{3}}{6}U,\\ \nabla_{X_1}X_4=-\sqrt{2}(a_{11}n+a_{12}m)X_1-\sqrt{2}(-a_{11}m+a_{12}n)X_2+\Gamma_{14}^3X_3.\end{cases} \tag{4.29}$$

Now, using that $G(X_1,X_2)=\frac{\sqrt{3}}{3}\xi$ and $G(X_1,\xi)=-\frac{\sqrt{3}}{3}X_2$, $a_{11}^2+a_{12}^2=\frac{1}{2}$ and $m^2+n^2=\frac{1}{6}$, (4.23) and (4.29), by direct calculations of both sides of

$$2(\tilde{\nabla}_{X_1}P)X_2=JG(X_1,PX_2)+JPG(X_1,X_2),$$

we obtain the following equations:

$$2X_1(a_{12})+2\sqrt{2}m-2a_{11}\Gamma_{12}^1=0, \tag{4.30}$$

$$-2X_1(a_{11})-2\sqrt{2}n-2a_{12}\Gamma_{12}^1=0, \tag{4.31}$$

$$\sqrt{6}X_1(m)+2\sqrt{6}n\Gamma_{12}^1=0, \tag{4.32}$$

$$-\frac{4\sqrt{3}}{3}a_{11}+\sqrt{6}X_1(n)-2\sqrt{6}m\Gamma_{12}^1=0. \tag{4.33}$$

Then, carrying the computations (4.30) $\times\, a_{12}$ − (4.31) $\times\, a_{11}$ and (4.32) $\times\, m$ + (4.33) $\times\, n$, respectively, we get

$$a_{11}n=0,\ \ a_{12}m=0.$$

If $a_{11}=0$, we get $a_{12}^2=\frac{1}{2}$, $m=0$ and $n^2=\frac{1}{6}$. Inserting these into (4.32), we obtain $\Gamma_{12}^1=0$. Then by (4.31), we have $n=0$. This yields a contradiction.

If $a_{11}\neq 0$, it holds that $a_{11}^2=\frac{1}{2}$, $a_{12}=0$, $m^2=\frac{1}{6}$ and $n=0$. Then by (4.30) and (4.33), we have $\frac{\sqrt{2}m}{a_{11}}=\Gamma_{12}^1=-\frac{\sqrt{2}a_{11}}{3m}$. This contradicts to the facts $a_{11}^2=\frac{1}{2}$ and $m^2=\frac{1}{6}$.

Thus, **Case III** does not occur.

Case IV. $\nu=2$.

In this case, we restrict the discussion on a connected component of Ω_2. It is easily seen that we are sufficient, without loss of generality, to consider the following two subcases:

IV-(i): $\lambda=\beta>0,\ \mu\in\{\lambda,-\lambda\}$.

IV-(ii): $\lambda=\beta=0,\ \mu\neq 0$.

Actually, for both of the above two subcases, following similar arguments as in the discussion of Case I from (4.2) up to (4.11), we can also get $\mu=0$. This is a contradiction, showing that **Case IV** does not occur.

We have completed the proof of Lemma 4.1. ■

Next, we have the following Lemma.

Lemma 4.2. *The case* $\dim \mathfrak{D} = 2$ *does not occur either.*

Proof. In this case, we denote still by ν, $\nu \leq 5$, the maximum number of distinct principal curvatures of M. Then the set $M_\nu = \{x \in M \,|\, M$ has exactly ν distinct principal curvatures at $x\}$ is a non-empty open subset of M. By the continuity of the principal curvature function, each connected component of M_ν is an open subset, the multiplicities of distinct principal curvatures remain unchanged on each connected component of M_ν. So we can choose a local smooth frame field with respect to the principal curvatures.

Now, by assumption $A\phi + \phi A = 0$ and Lemma 2.1, we can write (2.16) as:

$$\begin{aligned}\tfrac{1}{6}g(\phi X, Y) = g((\mu I - A)G(X,\xi), Y) + g(G((\mu I - A)X, \xi), Y) \\ - 2g(\phi A^2 X, Y), \quad X, Y \in \{U\}^\perp.\end{aligned} \tag{4.34}$$

In a connected component of M_ν, we take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \lambda X_1, \;\; AX_2 = \beta X_2, \;\; AX_3 = -\lambda X_3, \;\; AX_4 = -\beta X_4, \;\; AX_5 = \mu X_5,$$

where $X_3 = JX_1$, $X_4 = JX_2$, $X_5 = U$. Then, taking $(X,Y) = (X_1, \phi X_1)$ in (4.34), with using $AX_1 = \lambda X_1$ and $A\phi X_1 = -\lambda \phi X_1$, we get $-\frac{1}{6} = 2\lambda^2$, this is impossible and hence, we have proved Lemma 4.2. ∎

Finally, from Lemmas 4.1, 4.2 and the fact that dim $\mathfrak{D}$ can only be 2 or 4 at each point of M, we get immediately the assertion of Theorem 1.3. ∎

References

[1] B. Bektas, M. Moruz, J. Van der Veken and L. Vrancken, *Lagrangian submanifolds with constant angle functions of the nearly Kähler* $\mathbb{S}^3 \times \mathbb{S}^3$, J. Geom. Phys. **127** (2018), 1–13.

[2] B. Bektas, M. Moruz, J. Van der Veken and L. Vrancken, *Lagrangian submanifolds of the nearly Kähler* $\mathbb{S}^3 \times \mathbb{S}^3$ *from minimal surfaces in* $\mathbb{S}^3$, Proc. Roy. Soc. Edinburgh Sect. A, Proc. Roy. Soc. Edinburgh Sect. A, **149** (2019), 655–689.

[3] J. Berndt, J. Bolton and L.M. Woodward, *Almost complex curves and Hopf hypersurfaces in the nearly Kähler 6-sphere*, Geom. Dedicata **56** (1995), 237–247.

[4] J. Bolton, F. Dillen, B. Dioos and L. Vrancken, *Almost complex surfaces in the nearly Kähler* $\mathbb{S}^3 \times \mathbb{S}^3$, Tohoku Math. J. **67** (2015), 1–17.

[5] J. B. Butruille, *Classification des variétés approximativement kähleriennes homogènes*, Ann. Glob. Anal. Geom. **27** (2005), 201–225.

[6] J. B. Butruille, *Homogeneous nearly Kähler manifolds*, in: Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. Theor. Phys., **16**, Eur. Math. Soc., Zürich, 2010. pp. 399–423.

[7] F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken, *On totally real 3-dimensional submanifolds of the nearly Kaehler 6-sphere*, Proc. Amer. Math. Soc. **99** (1987), 741–749.

[8] B. Dioos, H. Li, H. Ma and L. Vrancken, *Flat almost complex surfaces in the homogeneous nearly Kähler* $\mathbb{S}^3 \times \mathbb{S}^3$. Results Math. **73** (2018), Art. 38, 24 pp.

[9] B. Dioos, L. Vrancken and X. Wang, *Lagrangian submanifolds in the homogeneous nearly Kähler* $\mathbb{S}^3 \times \mathbb{S}^3$, Ann. Global Anal. Geom. **53** (2018), 39–66.

[10] L. Foscolo and M. Haskins, *New* G_2*-holonomy cones and exotic nearly Kähler structures on* $\mathbb{S}^6$ *and* $\mathbb{S}^3 \times \mathbb{S}^3$, Ann. Math. **185** (2017), 59–130.

[11] Z. Hu, Z. Yao and Y. Zhang, *On some hypersurfaces of the homogeneous nearly Kähler* $\mathbb{S}^3 \times \mathbb{S}^3$, Math. Nachr. **291** (2018), 343–373.

[12] Z. Hu and Y. Zhang, *Rigidity of the almost complex surfaces in the nearly Kähler* $\mathbb{S}^3 \times \mathbb{S}^3$, J. Geom. Phys. **100** (2016), 80–91.

[13] Z. Hu and Y. Zhang, *Isotropic Lagrangian submanifolds in the homogeneous nearly Kähler* $\mathbb{S}^3 \times \mathbb{S}^3$, Sci. China Math. **60** (2017), 671–684.

[14] U. Ki and Y. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama Univ. **32** (1990), 207–221.

[15] J. K. Martins, *Congruence of hypersurfaces in* S^6 *and in* CP^n, Bol. Soc. Brasil. Mat. **32** (2001), 83–105.

[16] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geom. Dedicata **20** (1986), 245–261.

[17] M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.

[18] Y. Zhang, B. Dioos, Z. Hu, L. Vrancken and X. Wang, *Lagrangian submanifolds in the 6-dimensional nearly Kähler manifolds with parallel second fundamental form*, J. Geom. Phys. **108** (2016), 21–37.

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