Topological groups have representable actions*

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Abstract

This paper shows that the group of auto-homeomorphisms of a topological group can be endowed with a topology so that the resulting topological group plays, for topological groups, the role of the group of automorphisms of a group: it represents the internal actions on the given topological group.

1 Introduction

It is well known that the category **Grp** of groups is *action representative*, with internal actions represented by the group $\mathrm{Aut}(X)$ of automorphisms of X. This means that the functor

$$\mathrm{Act}(-, X) : \mathbf{Grp} \to \mathbf{Set},$$

assigning to each group Y the set of internal actions of Y on X, is represented by the group $\mathrm{Aut}(X)$. As observed in [2], representability of this functor is equivalent to the existence of *split extension classifier* for the group X, meaning that, for each split extension with kernel X

$$0 \longrightarrow X \longrightarrow A \underset{p}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

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there exists a unique homomorphism $\varphi : B \to \mathrm{Aut}(X)$ making the following diagram commute

$$\begin{array}{ccccc}
X & \xrightarrow{\;k\;} & A & \overset{s}{\underset{p}{\rightleftarrows}} & B \\
\| & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi} \\
X & \longrightarrow & \mathrm{Hol}(X) & \rightleftarrows & \mathrm{Aut}(X),
\end{array}$$

where $\mathrm{Hol}(X)$ is the semidirect product of X and $\mathrm{Aut}(X)$, with respect to the evaluation action (i.e. the classical holomorph of the group X).

In [1] it was investigated whether the category **TopGrp** of topological groups has the same property. As shown there, this ends up on investigating whether, for a given topological group X, the set $\underline{\mathrm{Aut}}(X)$ of auto-homeomorphisms of X, as a subspace of the pseudotopological space $X^X \times X^X$, is topological. When X is quasi-locally compact, so that X is exponentiable in **Top**, or, equivalently, the pseudotopological space Y^X is topological for every topological space Y, $\underline{\mathrm{Aut}}(X)$ is surely topological, since $X^X \times X^X$ is.

In this paper we show that, in fact, $\underline{\mathrm{Aut}}(X)$ is topological for every topological group X, concluding that the category **TopGrp** has representable actions. In order to do that, in Section 2 we revisit the results used in [1], in Section 3 we reduce the problem to the study of the pseudotopology on the subspace $\mathrm{Iso}(X)$ of X^X consisting of homeomorphisms of a Tychonoff space X, and finally in Section 4 we prove our key result:

Theorem. *If X is a Tychonoff space, then* $\mathrm{Iso}(X)$ *is a subspace of* $\mathrm{Iso}(\beta X^{\beta X})$. *In particular, it is a topological space.*

2 $\underline{\mathrm{Aut}}(X)$ as a subspace of $X^X \times X^X$

We start by recalling a key result published (without proof) in [3]:

Theorem. *If* **C** *is a finitely complete Cartesian closed category, then the category of internal groups in* **C** *is action representative.*

The proof of this Theorem presented in [1] shows how to build the internal group that represents the functor $\mathrm{Act}(-, X)$ out of the exponential X^X. Here we just detail the construction described in [1] for internal groups in the category **Top** of topological spaces. Since **Top** is not Cartesian closed, one embeds **Top** in the (complete and) Cartesian closed category **PsTop** of pseudotopological spaces, so that for every pseudotopological space X there is an adjunction:

$$\mathbf{PsTop} \underset{(\;)^X}{\overset{(\;)\times X}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{PsTop}$$

The category **Top** is in fact a bireflective subcategory of **PsTop**, meaning that the unit of the adjunction is pointwise both a mono and an epimorphism. (For more information on **PsTop** see [4].)

Given a topological group X, one first considers the pseudotopological space

$$X^X = \{f : X \to X \mid f \text{ is continuous}\},$$

which is in fact an internal monoid with respect to the composition, then its subspace – and submonoid –

$$\text{Hom}(X, X) = \{f : X \to X \mid f \text{ is a continuous homomorphism}\},$$

and finally one obtains $\underline{\text{Aut}}(X)$ as the pullback

$$\begin{array}{ccc} \underline{\text{Aut}}(X) & \longrightarrow & 1 \\ \downarrow & & \downarrow {\scriptstyle <1_X, 1_X>} \\ \text{Hom}(X,X) \times \text{Hom}(X,X) & \xrightarrow[\mu \times \mu^{\text{op}}]{} & \text{Hom}(X,X) \times \text{Hom}(X,X) \end{array}$$

(where μ is the composition and μ^{op} the reverse composition); that is, we identify the internal group

$$\underline{\text{Aut}}(X) = \{f : X \to X \mid f \text{ is an auto-homeomorphism}\}$$

with the subspace $\{(f, f^{-1}) \mid f : X \to X \text{ is an auto-homeomorphism}\}$ of $X^X \times X^X$. We point out that, when X is compact and Hausdorff, the topology in $\underline{\text{Aut}}(X)$ is not the compact-open topology but the product of the compact-open topology in each factor.

In order to show that the pseudotopology of $\underline{\text{Aut}}(X)$ is in fact a topology for every topological group X, in the next sections we will show that the subspace

$$\text{Iso}(X) = \{f \in X^X \mid f \text{ is a homeomorphism}\}$$

of X^X is a topological space. Then we may conclude that $\underline{\text{Aut}}(X)$, as a subspace of $\text{Iso}(X) \times \text{Iso}(X)$, is also a topological space.

3 $\underline{\text{Aut}}(X)$ in **PsTop**

Throughout this section X is a topological space.

Given a directed set Λ, we denote by Λ_∞ the set $\Lambda \dot{\cup} \{\infty\}$ equipped with the topology

$$\{A \subseteq \Lambda_\infty \mid \infty \notin A \text{ or there exists } \lambda \in \Lambda \text{ such that } \uparrow \lambda \subseteq A\}.$$

Lemma. *Given a net $(f_\lambda)_{\lambda \in \Lambda}$ and f in X^X, (f_λ) converges to f if, and only if, the map $F : \Lambda_\infty \times X \to X$, defined by $F(\lambda, x) = f_\lambda(x)$ and $F(\infty, x) = f(x)$, is continuous.*

Proof. Continuity of F is equivalent to continuity of the map $\tilde{F} : \Lambda_\infty \to X^X$, with $\tilde{F}(\lambda) = f_\lambda$ and $\tilde{F}(\infty) = f$, and this is clearly equivalent to the convergence of (f_λ) to f. (Note that in general $\tilde{F}$ is a morphism in **PsTop**, not in **Top**.) ∎

From now on we denote by $\eta = (\eta_X : X \to RX)_{X \in \mathbf{Top}}$ the unit of the adjunction

$$\mathbf{Top}_0 \underset{\longrightarrow}{\overset{R}{\longleftarrow}}_{\perp} \mathbf{Top}$$

that is η_X is the $T0$-reflection of X. Note that $T0$-reflections are both initial and final maps (see [5]).

Theorem. *If* $\mathrm{Iso}(RX)$ *is a topological space, then so is* $\mathrm{Iso}(X)$*.*

Proof. Since **Top** is bireflective in **PsTop**, it is closed under initial continuous maps. We will show now that the map $\varphi : \mathrm{Iso}(X) \to \mathrm{Iso}(RX)$, assigning to each element f of $\mathrm{Iso}(X)$ its $T0$-reflection Rf, is both continuous and initial. Let $(f_\lambda)_{\lambda \in \Lambda}$ and f belong to $\mathrm{Iso}(X)$. Observing that the $T0$-reflection of $\Lambda_\infty \times X$ is $\Lambda_\infty \times RX$, we have the following commutative diagram

$$\begin{array}{ccc} \Lambda_\infty \times X & \xrightarrow{\eta_{\Lambda_\infty \times X}} & \Lambda_\infty \times RX \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \overline{F}} \\ X & \xrightarrow[\eta_X]{} & RX, \end{array}$$

where $\overline{F}(\lambda, y) = Rf_\lambda(y)$ and $\overline{F}(\infty, y) = Rf(y)$. Then φ is continuous and initial if, and only if, continuity of F is equivalent to continuity of $\overline{F}$. If (f_λ) converges to f in $\mathrm{Iso}(X)$, so that F is continuous, then $\overline{F} \cdot \eta_{\Lambda_\infty \times X}$ is continuous, from which it follows that $\overline{F}$ is continuous, due to finality of η. Conversely, if (Rf_λ) converges to Rf in $\mathrm{Iso}(RX)$, that is, $\overline{F}$ is continuous, then $\eta_X \cdot F$ is continuous, and initiality of η_X gives the continuity of F. ∎

4 $\mathrm{Iso}(X)$ is topological

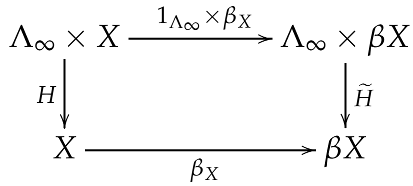
Throughout this section X is a Tychonoff space, so that we consider its embedding $\beta_X : X \to \beta X$ into its Stone-Čech compactification.

Theorem. *If X is a Tychonoff space, then* $\mathrm{Iso}(X)$ *is a subspace of* $\mathrm{Iso}(\beta X)$*. In particular, it is a topological space.*

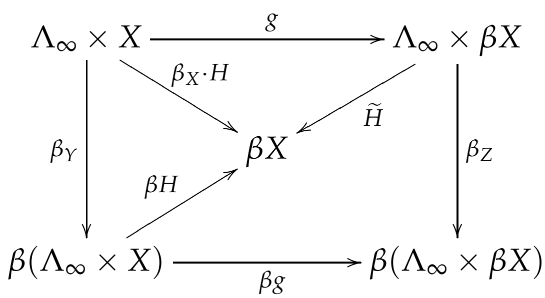
Proof. First we point out that, since βX is compact and Hausdorff, $\beta X^{\beta X}$ is a topological space, endowed with the compact-open topology. Therefore its subspace $\mathrm{Iso}(\beta X)$ is also topological.

Consider the map $\widetilde{\beta}_X : \mathrm{Iso}(X) \to \mathrm{Iso}(\beta X)$ which assigns to each homeomorphism $h : X \to X$ the homeomorphism $\beta(h)$. Then $\widetilde{\beta}_X$ is injective since β_X is. To conclude that it is an embedding we need to show that, for $(h_\lambda)_{\lambda \in \Lambda}$ and h in $\mathrm{Iso}(X)$, (h_λ) converges to h if, and only if, $(\widetilde{\beta}_X(h_\lambda))$ converges to $\widetilde{\beta}_X(h)$. As we have already observed, (h_λ) converges to h if, and only if, the corresponding map $H : \Lambda_\infty \times X \to X$ is continuous, while $(\widetilde{\beta}_X(h_\lambda))$ converges to $\widetilde{\beta}_X(h)$ exactly when the corresponding map $\widetilde{H} : \Lambda_\infty \times \beta X \to \beta X$ is continuous.

Consider the following commutative diagram



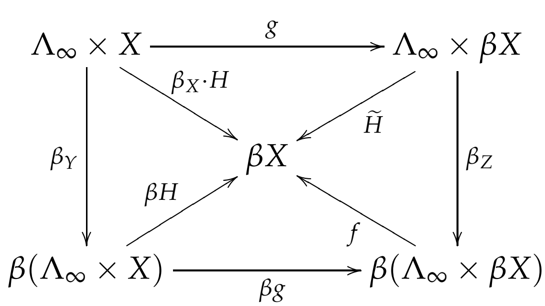
If $\widetilde{H}$ is continuous, then $\beta_X \cdot H$ is continuous, and so is H since β_X is an embedding. The converse implication is the non-trivial one. In order to prove it, assume that H is continuous and consider the following commutative diagram, where $g := 1_X \times \beta_X$, $Y = \Lambda_\infty \times X$ and $Z = \Lambda_\infty \times \beta X$:



The map βg is dense, since both β_Z and g are. As a continuous map between compact Hausdorff spaces, it is thus surjective, and, moreover, a quotient. Below we will show that:

$$\forall \mathfrak{y}, \mathfrak{y}' \in \beta(\Lambda_\infty \times X) \;\; \beta g(\mathfrak{y}) = \beta g(\mathfrak{y}') \implies \beta H(\mathfrak{y}) = \beta H(\mathfrak{y}'). \qquad (\Diamond)$$

From ($\Diamond$) it follows that βH factors through βg, via a map $f : \beta(\Lambda_\infty \times \beta X) \to \beta X$. Since βg is a quotient and $f \cdot \beta g = \beta H$ is continuous, f is in fact continuous.



Now, from $(f \cdot \beta_Z) \cdot g = \beta H \cdot \beta_Y = \beta_X \cdot H = \widetilde{H} \cdot g$ and since, by construction, $\widetilde{H}$ is the unique map such that $\widetilde{H} \cdot g = \beta_X \cdot H$, it follows that $\widetilde{H} = f \cdot \beta_Z$, and then it is continuous as claimed.

Therefore to finish our proof we only need to show ($\Diamond$).

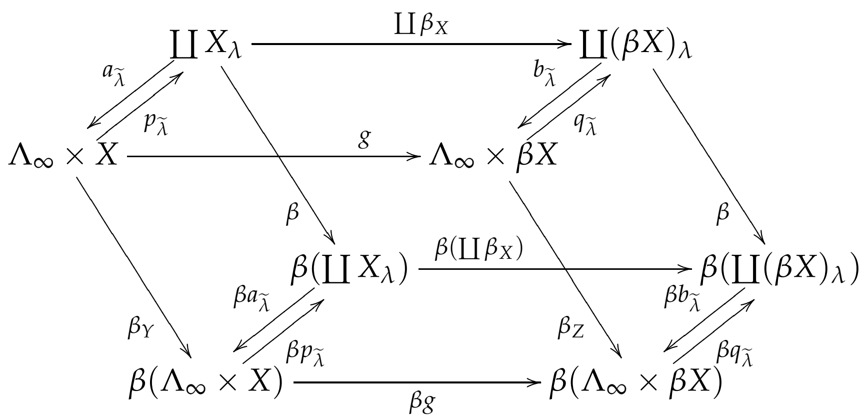
First we prove it for $\mathfrak{y}, \mathfrak{y}' \in \beta(\coprod_{\widetilde{\lambda}} X_\lambda) \subseteq \beta(\Lambda_\infty \times X)$, where $\coprod_{\widetilde{\lambda}} X_\lambda$ is the coproduct in **Top** of $(X_\lambda)_{\lambda \leq \widetilde{\lambda}}$, with $\widetilde{\lambda} \in \Lambda$ and $X_\lambda = X$ for every $\lambda \leq \widetilde{\lambda}$. Note that the maps

$$\coprod_{\widetilde{\lambda}} X_\lambda \underset{p_{\widetilde{\lambda}}}{\overset{a_{\widetilde{\lambda}}}{\rightleftarrows}} \Lambda_\infty \times X,$$

with $a_{\widetilde{\lambda}}(x) = (\lambda, x)$ for $x \in X_\lambda$, and $p_{\widetilde{\lambda}}(\mu, x) = x \in X_\mu$ if $\mu \leq \widetilde{\lambda}$ and $p_{\widetilde{\lambda}}(\mu, x) = x \in X_{\widetilde{\lambda}}$ otherwise, are continuous, and $p_{\widetilde{\lambda}} \cdot a_{\widetilde{\lambda}} = 1$. Analogously we define

$$\coprod_{\widetilde{\lambda}} (\beta X)_\lambda \underset{q_{\widetilde{\lambda}}}{\overset{b_{\widetilde{\lambda}}}{\rightleftarrows}} \Lambda_\infty \times \beta X.$$

Observing that the following diagram commutes and $\beta(\coprod \beta_X)$ is an isomorphism because the functor β is a left adjoint

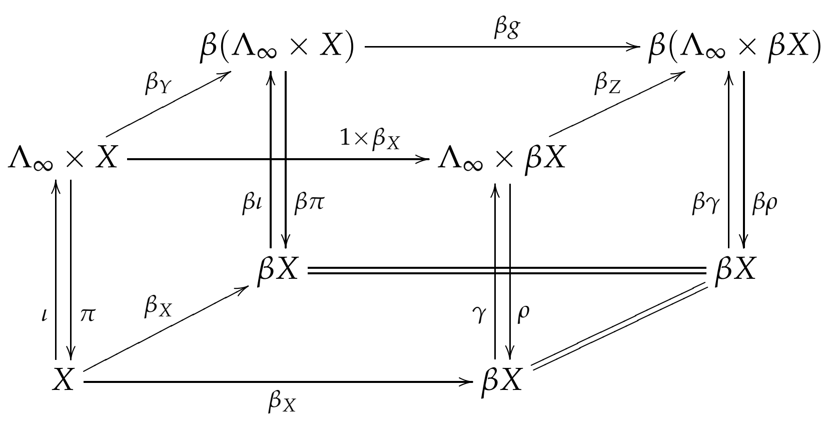


we conclude that $\beta g \cdot \beta a_{\widetilde{\lambda}} = \beta b_{\widetilde{\lambda}} \cdot \beta(\coprod \beta_X)$ is an embedding. This leads to the conclusion that βg is injective when restricted to the image of the closure of $\coprod_{\widetilde{\lambda}} X_\lambda$: $\beta(\overline{\coprod_{\tilde{\lambda}} X_\lambda}) \subseteq \overline{\beta(\coprod_{\tilde{\lambda}} X_\lambda)} = \beta(\coprod_{\tilde{\lambda}} X_\lambda)$. Now, for each $\tilde{\mu} \geq \tilde{\lambda}$, we have a section $\iota_{\tilde{\lambda},\tilde{\mu}}$, with retraction $\pi_{\tilde{\lambda},\tilde{\mu}}$,

$$\coprod_{\tilde{\lambda}} X_\lambda \underset{\pi_{\tilde{\lambda},\tilde{\mu}}}{\overset{\iota_{\tilde{\lambda},\tilde{\mu}}}{\rightleftarrows}} \coprod_{\tilde{\mu}} X_\lambda$$

where $\iota_{\tilde{\lambda},\tilde{\mu}}(\lambda, x) = (\lambda, x)$, and $\pi_{\tilde{\lambda},\tilde{\mu}}(\lambda, x) = (\lambda, x)$ if $\lambda \leq \tilde{\lambda}$ and $\pi_{\tilde{\lambda},\tilde{\mu}}(\lambda, x) = (\tilde{\lambda}, x)$ elsewhere. Therefore, since Λ is directed, this allows us to conclude that βg *is injective when restricted to* $\bigcup_{\widetilde{\lambda} \in \Lambda} \beta(\coprod_{\widetilde{\lambda}} X_\lambda) \hookrightarrow \beta(\Lambda_\infty \times X)$.

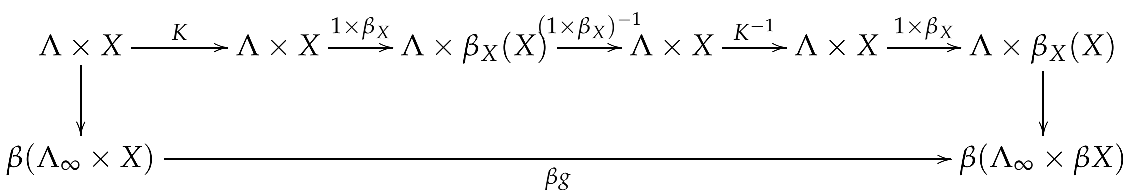
For the remaining proof it is useful to consider the diagram



where π and ρ are projections, $\iota(x) = (\infty, x)$, and $\gamma(\mathfrak{x}) = (\infty, \mathfrak{x})$. (For simplicity we will also denote by $(\infty, \mathfrak{x})$ the image of $\mathfrak{x} \in \beta X$ under $\beta\gamma$, i.e. we will identify $(\infty, \mathfrak{x})$ with $\beta_Z(\infty, \mathfrak{x})$.)

Let $\mathfrak{y} \in \widetilde{Y} := \beta(\Lambda_\infty \times X) \setminus \bigcup_{\widetilde{\lambda}\in\Lambda} \beta(\coprod_{\widetilde{\lambda}} X_\lambda)$. Then $\mathfrak{y}$ must be the limit point of a net $(\mathfrak{y}_\mu = \beta_Y(\lambda_\mu, x_\mu))_{\mu\in M}$, in the image of $\Lambda \times X$ via β_Y, cofinal with Λ, since $\Lambda \times X \to \Lambda_\infty \times X \to \beta(\Lambda_\infty \times X)$ is dense. The net $(\beta g(\mathfrak{y}_\mu))_\mu$ converges to $\beta g(\mathfrak{y})$, and its image under $\beta\rho$ converges to $\mathfrak{x} := \beta\rho(\beta g(\mathfrak{y})) = \beta\pi(\mathfrak{y})$. Any neighbourhood of $\beta\gamma(\mathfrak{x}) = (\infty, \mathfrak{x})$ contains a queue of $(\beta g(\mathfrak{y}_\mu))_\mu$, since this net is cofinal with Λ. We may choose a neighbourhood U of $\beta\gamma(\mathfrak{x})$ so that $U \cap (\Lambda_\infty \times \beta X) = V \times A$, with $V = \{\lambda\,;\, \lambda \geq \overline{\lambda}\} \cup \{\infty\}$, and A an open subset of βX containing $\mathfrak{x}$. Hence in A there is a queue of $(\beta\rho(\beta g(\mathfrak{y}_\mu)))_\mu$ and then in U there is a queue of $(\beta g(\mathfrak{y}_\mu))_\mu$, which implies that $\beta g(\mathfrak{y}) = (\infty, \mathfrak{x})$. We may then conclude that any $\mathfrak{y}' \in Y$ with $\beta g(\mathfrak{y}) = \beta g(\mathfrak{y}')$ belongs necessarily to $\widetilde{Y}$.

To complete the proof we consider the continuous map $K = \langle 1, H\rangle : \Lambda_\infty \times X \to \Lambda_\infty \times X$ defined by $K(\lambda, x) = (\lambda, h_\lambda(x))$ and $K(\infty, x) = (\infty, h(x))$, so that $H = \pi \cdot K$. Note that in the diagram below βg is the (unique) continuous extension of the top composite, since the vertical maps are dense

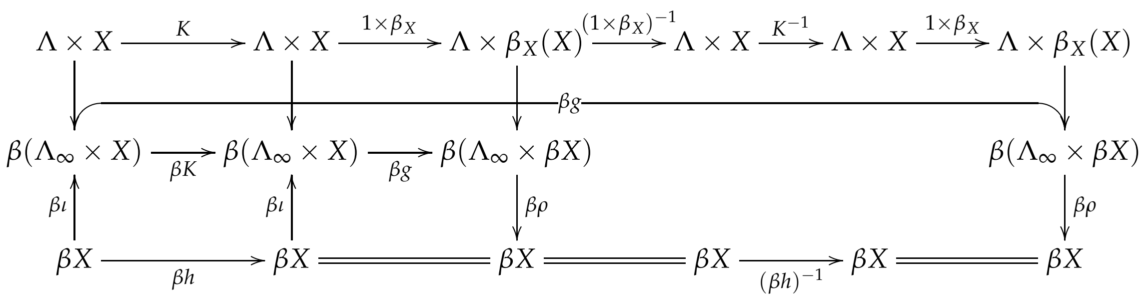


(here we consider (co)restrictions of K and β_X although we use the same notations).

Our next goal is to show that

$$\beta g(\beta K(\mathfrak{y})) = \beta\gamma(\beta h(\beta\pi(\mathfrak{y}))). \qquad (\triangledown)$$

By definition of K it is clear that $\beta K(\mathfrak{y}) \in \widetilde{Y}$. Therefore $\beta g(\beta K(\mathfrak{y})) = (\infty, \beta\pi(\beta K(\mathfrak{y})))$. To show that $(\infty, \beta\pi(\beta K(\mathfrak{y}))) = (\infty, \beta h(\beta\pi(\mathfrak{y})))$ – that is, $(\triangledown)$ – we use the following diagram



Indeed, applying $(\beta h)^{-1} \cdot \beta\rho$ to the latter, one gets $\beta\pi(\mathfrak{y})$; but $(\beta h)^{-1}(\beta\rho(\beta\pi(\beta K(\mathfrak{y}))))$ must be also $\beta\pi(\mathfrak{y})$, since $\beta\pi(\mathfrak{y}) = \beta\rho(\beta g(\mathfrak{y})) = (\beta h)^{-1}(\beta\rho(\beta g(\beta K(\mathfrak{y}))))$, and so $(\triangledown)$ holds.

Now we are able to conclude $(\Diamond)$ for $\mathfrak{y}, \mathfrak{y}' \in \widetilde{Y}$. From $\beta g(\mathfrak{y}) = \beta g(\mathfrak{y}')$ it follows that $\beta\pi(\mathfrak{y}) = \beta\pi(\mathfrak{y}')$, and then, by equality $(\triangledown)$, $\beta g(\beta K(\mathfrak{y})) = \beta g(\beta K(\mathfrak{y}'))$, with $\beta K(\mathfrak{y})$ and $\beta K(\mathfrak{y}')$ in $\widetilde{Y}$. Therefore also $\beta\pi(\beta K(\mathfrak{y})) = \beta\pi(\beta K(\mathfrak{y}'))$, which means exactly that $\beta H(\mathfrak{y}) = \beta H(\mathfrak{y}')$, and this ends the proof. ∎

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