Order theoretic and topological Characterizations of the Divided Spectrum of a Ring

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Abstract

Let R be a commutative ring with identity. We denote by $\mathcal{D}\text{iv}(R)$ the divided spectrum of R (the set of all divided prime ideals of R). By a divspectral space, we mean a topological space homeomorphic with the subspace $\mathcal{D}\text{iv}(R)$ of $\text{Spec}(R)$ endowed with the Zariski topology, for some ring R. A divspectral set is a poset which is order isomorphic to $(\mathcal{D}\text{iv}(R), \subseteq)$, for some ring R. The main purpose of this paper is to provide some topological (resp., algebraic) characterizations of of divspectral spaces (resp., sets).

Introduction

The algebraic concepts of G-domains and G-ideals have been introduced by Kaplansky. Later on, some topological characterizations for G-ideals have been investigated. Let us recall that $\mathfrak{p} \in \text{Spec}(R)$ is a G-ideal if and only if $\{\mathfrak{p}\}$ is locally closed in $\text{Spec}(R)$ endowed with its Zariski topology. Moreover, if $\text{Spec}(R)$ is *linearly ordered*, then $\mathfrak{p}$ is a G-ideal if and only if $(\downarrow \mathfrak{p})$ is open, where $(\downarrow \mathfrak{p})$ denotes the set of primes ideals contained in $\mathfrak{p}$.

The aim of the paper is threefold. Firstly, to extend these characterizations to the more general setting of a linearly ordered poset $(X, \leq)$ equipped with $\mathcal{T}$, a topology compatible with the order. We prove that if $x \in X$ has an immediate

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successor, then $\{x\}$ is locally closed, and the converse remains true if the topology $\mathcal{T}$ is sober. We also provide an example showing that the condition being sober is essential.

Next, we give a complete description of Gold(R), the set of G-ideals of R, and Spec(R), when the latter is linearly ordered with maximal ideal $\mathfrak{m}$. Our main tool is the prime ideal

$$\mathcal{J}_\rho = \bigcup[\mathfrak{q} \in \text{Spec}(R) \,:\, \rho \notin \mathfrak{q}], \text{ for } \rho \in \mathfrak{m} \setminus \{0\}.$$

We prove that $(\mathcal{J}_\rho)_{\rho \in \mathfrak{m}\setminus\{0\}}$ constitute the set of all non-maximal G-ideals of R. Thus any non-maximal prime ideal is the intersection of $\mathcal{J}_\rho$ for ρ varying in an arbitrarily subset of $\mathfrak{m} \setminus \{0\}$.

As a second goal of the present paper, we provide new characterizations of divided domains. Let us recall that a commutative integral domain R is said to be divided in case each prime ideal $\mathfrak{p}$ of R is divided; that is $\mathfrak{p} = \mathfrak{p}R_{\mathfrak{p}}$. An important class of divided integral domains is provided by pseudo-valuation domains. It is worth noting that these domains were studied by Akiba [1] as AV-domains (almost Valuation domains) and have been also studied by Dobbs [7], and Fontana [11].

For a nonzero and nonunit element ρ of a domain R, we denote by

$$\mathcal{I}_\rho := \bigcap_{n \geq 1} \rho^n R.$$

These ideals are related to the notion of "power-Ahmes domains" (or "pointwise non-Archimedean domains"). Recall that a domain R is said to be pointwise non-Archimedean if $\mathcal{I}_\rho \neq 0$, for all $\rho \in R \setminus \{0\}$. For a divided domain R, R is power-Ahmes if and only if the zero ideal has no immediate successor, see [6].

The ideals $\mathcal{I}_\rho$ are also related to fragmented domains. Recall that a domain R is said to be fragmented, if each nonunit and nonzero element of R is divisible by all positive integral powers of some corresponding nonunit and nonzero element of R (see [8]). In other words for all $r \in R \setminus (\mathcal{U}(R) \cup \{0\})$, there exists $s \in R \setminus (\mathcal{U}(R) \cup \{0\})$ such that $r \in \mathcal{I}_s$, where $\mathcal{U}(R)$ is the set of all unit elements of R.

We show that a domain R is divided if and only if Spec(R) is linearly ordered and each $\mathcal{J}_\rho$ is divided, equivalently, for every nonunit and nonzero element ρ of R, R_ρ is quasi-local with maximal ideal $\mathcal{I}_\rho$, where R_ρ is the quotient ring of R with respect to the multiplicative set $S = \{1, \rho, \rho^2, \rho^3, \ldots\}$, see Theorem 4.4. We also prove that $\mathcal{J}_\rho = \mathcal{I}_\rho$ if and only if $\mathcal{J}_\rho$ is divided. As a corollary, we obtain a complete description of the prime spectrum of divided domains.

It is worth noting that some subspaces of spectral spaces have been characterized in [13], [14], [9]. More precisely, in 1969, Hochster [13] showed that a topological space is homeomorphic to the the subspace of maximal ideals of Spec(R) if and only if it is a T_1 compact space. Two years later, Hochster gave a topological characterisation of minspectral spaces (spaces which are homeomorphic to the subspace of minimal prime ideals of a ring). In 2000, Echi [9] provided a topological characterization of the Goldman prime spectrum of a commutative ring.

By a divspectral set, we mean a poset which is isomorphic to $(\mathcal{D}\text{iv}(R), \subseteq)$, the set of divided ideals of R. And by a divspectral space, we mean a topological space which is homeomorphic to some $\mathcal{D}\text{iv}(R)$ endowed with the topology inherited by the Zariski topology on $\text{Spec}(R)$. Finally as a third goal of the present paper, we provide some topological characterizations of *divspectral spaces* and a complete algebraic characterization of *divspectral sets*.

It is worth noting that most of the results of this paper have been used in our recent paper [10].

Throughout this paper "$\subset$" stands for proper containment and "$\subseteq$" for large containment, all rings considered are commutative with identity.

1 Preliminaries

Let $\text{Spec}(R)$ denote the set of all prime ideals of a commutative ring R. The *Zariski topology* for $\text{Spec}(R)$ is defined by letting $C \subseteq \text{Spec}(R)$ be closed if and only if there exists an ideal $\mathfrak{a}$ of R such that

$$C = \{\mathfrak{p} \in \text{Spec}(R) \,:\, \mathfrak{a} \subseteq \mathfrak{p}\} =: V(\mathfrak{a}).$$

This topology has a basis of compact special open sets formed by

$$\mathcal{D}(x) := \{\mathfrak{p} \in \text{Spec}(R) : x \notin \mathfrak{p}\} = \text{Spec}(R) \setminus V(xR),$$

and satisfies the property of compatibility with the inclusion order; that is, for each prime ideal $\mathfrak{p}$, we have

$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{\mathfrak{q} \in \text{Spec}(R) : \mathfrak{p} \subseteq \mathfrak{q}\}.$$

According to Hochster [13], a topological space $(X, \mathcal{T})$ is homeomorphic to the prime spectrum of a ring equipped with the Zariski topology if and only if the following properties hold:

- $(X, \mathcal{T})$ is sober (that is, every nonempty irreducible closed set is the closure of a unique point).
- $(X, \mathcal{T})$ is compact.
- The compact open sets form a basis of $\mathcal{T}$.
- The family of compact open sets of X is closed under finite intersections.

Such topological spaces are called *spectral spaces*.

Now, let us recall the concept of G-ideals and give some topological property of them.

According to Kaplansky [15], a *G-domain* is a domain R such that the quotient field K of R is of the form $R[1/t]$, for some $t \in R \setminus \{0\}$. A prime ideal $\mathfrak{p}$ of a ring

R is said to be a *G-ideal* if $R/\mathfrak{p}$ is a G-domain. It is well known that $\mathfrak{p}$ is a G-ideal if and only if $\{\mathfrak{p}\}$ is locally closed (i.e., an intersection of an open set and a closed set) in $\mathrm{Spec}(R)$ endowed with the Zariski topology.

A subset S of a topological space X is said to be *strongly dense* [5] if it meets every nonempty locally closed set of X. Recall, from the folklore of commutative algebra, that the set $\mathrm{Gold}(R)$ of all G-ideals of a ring R is strongly dense in $\mathrm{Spec}(R)$ [5, **0**.2.6.2].

Let $(X, \leq)$ be a poset. Following Lewis-Ohm [18], a topology $\mathcal{T}$ is said to be compatible with the order if $\overline{\{x\}} = (x\uparrow)$, where $(x\uparrow) := \{y \in X : x \leq y\}$ and $(\downarrow x) := \{y \in X : y \leq x\}$. The finest topology compatible with the order is called the *Alexandroff topology*; it has $\mathcal{B} := \{(\downarrow x) : x \in X\}$ as a basis of open sets. Thus every open set O of a compatible topology $\mathcal{T}$ of a poset is Alexandroff-open.

2 Topological Properties of G-ideals

Let us recall that if $\mathrm{Spec}(R)$ is linearly ordered, then $\mathfrak{p}$ is a G-ideal if and only if $(\downarrow \mathfrak{p})$ is Zariski-open. We extend this characterization to a linearly ordered poset. We prove the following:

Lemma 2.1. *Let $(X, \leq)$ be a linearly ordered poset, $\mathcal{T}$ be a topology on X which is compatible with the order and $x \in X$.*

Then the following statements are equivalent.

1. *$\{x\}$ is locally closed.*
2. *$(\downarrow x)$ is an open set.*

Proof. Assume $\{x\}$ is locally closed, then there exists an open set U of X such that $\{x\} = U \cap \overline{\{x\}}$. We will show that $U = (\downarrow x)$.

Indeed, as $\mathcal{T}$ is compatible with the order and $x \in U$, we have $(\downarrow x) \subseteq U$. Now let $y \in U$. If we suppose that $y \notin (\downarrow x)$, then $x < y$, and consequently, $y \in U \cap \overline{\{x\}}$, a contradiction. Therefore $U = (\downarrow x)$ is open.

Conversely, assume $(\downarrow x)$ is open, then $\{x\} = \overline{\{x\}} \cap (\downarrow x)$ is locally closed. ∎

Remark 2.2. The assumption $(X, \leq)$ being linearly ordered is essential in Lemma 2.1. Indeed, in an infinite set X equipped with the co-finite topology, every one-point set is closed (so locally closed), but no one-point set is open.

Recall that, in a linearly ordered set $(X, \leq)$, $x < y$ are said to be *adjacent* (or *consecutive*, or y is an *immediate successor* of x, or x is an *immediate predecessor* of y) if there is no $z \in X$ such that $x < z < y$.

Using purely topological arguments, the next result provides a characterization of G-ideals in term of order (in a ring with linearly ordered prime spectrum).

Theorem 2.3. *Let $(X, \leq)$ be a linearly ordered poset, $\mathcal{T}$ be a topology on X which is compatible with the order and $x \in X$.*

1. *If x has an immediate successor, then $\{x\}$ is locally closed.*
2. *If in addition $\mathcal{T}$ is a sober topology, then the following statements are equivalent.*

   (i) *$\{x\}$ is locally closed.*

   (ii) *x is maximal or has an immediate successor.*

Proof.
(1) Assume x has an immediate successor y; then $(\downarrow x) = X \setminus \overline{\{y\}}$ is open. Thus, according to Lemma 2.1, $\{x\}$ is locally closed.

(2) If x is maximal, then $\{x\}$ is closed, a fortiori $\{x\}$ is locally closed. Now, taking into consideration (1), we obtain the implication $(i) \Longrightarrow (ii)$.

Conversely, assume $\{x\}$ is locally closed and x non maximal; then $X \setminus (\downarrow x)$ is a nonempty closed set (by Lemma 2.1). But as the set is linearly ordered $X \setminus (\downarrow x)$ is irreducible; so thanks to the sobriety property of $\mathcal{T}$, $X \setminus (\downarrow x)$ has a unique generic point y. Clearly, y is an immediate successor of x. ∎

Remark 2.4. Let $(X, \leq)$ be a linearly ordered poset equipped with a topology $\mathcal{T}$ which is compatible with the order. If $\mathcal{T}$ is not sober and $x \in X$ is locally closed, then x need not have an immediate successor.

For example, let $X := \{0\} \cup \{\frac{1}{n} : n \text{ is a positive integer}\}$ equipped with the usual order and let $\mathcal{T}$ be the topology on X whose open sets are $\emptyset$ and the $(\downarrow x)$, with $x \in X$. Then, $\mathcal{T}$ is an order compatible topology which is not sober, as $X \setminus \{0\}$ is an irreducible closed set with no generic point. Then every point of X is locally closed, and clearly 0 has no immediate successor.

It is well known that if R is a ring, then $\mathrm{Gold}(R)$ is strongly dense in $\mathrm{Spec}(R)$ and that every prime ideal $\mathfrak{p}$ of R is the intersection of all G-ideals containing $\mathfrak{p}$ (see [15, Theorem 26, page 17]). The following theorem provides a topological result close to this fact.

Theorem 2.5. *Let $(X, \leq)$ be a poset and $\mathcal{T}$ be a topology on X which is compatible with the order. We denote by $\mathrm{Lc}(X)$ the set of all locally closed points of X.*

Consider the following statements.

1. $\mathrm{Lc}(X)$ *is strongly dense in X.*
2. *For all $x \in X$, if we denote $\mathrm{Lc}_x := \{y \in \mathrm{Lc}(X) : x \leq y\}$, then $x = \inf(\mathrm{Lc}_x)$.*

Then (1) *implies* (2). *If, in addition, $(X, \leq)$ is linearly ordered, then* (1) *and* (2) *are equivalent.*

Proof. (1) $\Longrightarrow$ (2). Let a be a lower bound of $\mathrm{Lc}(X)$. Assume $a \nleq x$; then $x \in X \setminus \overline{\{a\}}$. So $\overline{\{x\}} \cap (X \setminus \overline{\{a\}})$ is a nonempty locally closed set of X. Thus

$$\overline{\{x\}} \cap (X \setminus \overline{\{a\}}) \cap \mathrm{Lc}(X) \neq \emptyset.$$

Consequently, there exists $b \in \mathrm{Lc}(X)$, such that $b \in \overline{\{x\}}$ and $b \in X \setminus \overline{\{a\}}$; this implies that $a \nleq b$, contradicting the fact that a is a lower bound of $\mathrm{Lc}(X)$.

Therefore, $a \leq x$, and so $x = \inf(\mathrm{Lc}_x)$.

Conversely, assume $(X, \leq)$ is linearly ordered. Assume (2) holds. Let us show that $\mathrm{Lc}(X)$ is strongly dense in X.

Indeed, let $L = O \cap C$ be a nonempty locally closed set of X, where O is an open set and C is a closed set. Let $x \in L$, we will show that $\overline{\{x\}} \cap O$ meets $\mathrm{Lc}(X)$.

– If $\overline{\{x\}} \cap O = \{x\}$, then $x \in \mathrm{Lc}(X)$.

– Now, assume $\overline{\{x\}} \cap O \neq \{x\}$. Then there exists $y \neq x$ such that $y \in \overline{\{x\}} \cap O \neq \{x\}$. As $x = \inf(\mathrm{Lc}_x)$ and $x < y$, there exists $z \in \mathrm{Lc}_x)$ such that $y \nleq z$. So $z < y$; hence $z \in \overline{\{x\}} \cap (\downarrow y)$. But as O is open and the topology is compatible with the order, O is Alexandroff-open, and consequently $(\downarrow y) \subseteq O$. It follows that $z \in (\overline{\{x\}} \cap O) \cap \mathrm{Lc}(X)$. We conclude that $\mathrm{Lc}(X)$ is strongly dense in X. ∎

3 G-ideals in a ring with linearly ordered spectrum

Let R be a domain, ρ be a nonzero and non invertible element of R and

$$S = \{1, \rho, \rho^2, \rho^3, \ldots, \rho^n, \ldots\},$$

we denote by R_ρ the quotient ring of R with respect to the multiplicative set S.

For a ring R with linearly ordered prime spectrum, and maximal ideal $\mathfrak{m}$, and $\rho \in \mathfrak{m} \setminus \{0\}$, we denote by $\mathcal{J}_\rho$ the union of all prime ideals of R not containing ρ, that is,

$$\mathcal{J}_\rho = \bigcup [\mathfrak{q} \in \mathrm{Spec}(R) \ : \ \rho \notin \mathfrak{q}].$$

Proposition 3.1. *Let $(R, \mathfrak{m})$ be a domain with linearly ordered prime spectrum, $\mathfrak{p}$ be a non maximal prime ideal of R. Then the following statements are equivalent.*

1. *$R_\mathfrak{p} = R_\rho$, for some $\rho \in \mathfrak{m} \setminus \{0\}$.*
2. *There exists $\rho \in \mathfrak{m} \setminus \mathfrak{p}$ such that $\mathfrak{p} = \mathcal{J}_\rho$.*
3. *$\mathcal{D}(\rho) = (\downarrow \mathfrak{p})$, for some $\rho \in \mathfrak{m} \setminus \mathfrak{p}$.*
4. *$(\downarrow \mathfrak{p})$ is open in $\mathrm{Spec}(R)$.*

We need the following lemma.

Lemma 3.2 ([19, Proposition 2.1]). *Let R be a domain, $\mathfrak{p}$ be a prime ideal of R and $\rho \in R \setminus \mathfrak{p}$. Then the following statements are equivalent.*

1. *$R_\mathfrak{p} = R_\rho$.*
2. *For each $b \in R \setminus \mathfrak{p}$, $\rho \in \sqrt{bR}$.*
3. *If $\mathfrak{q} \in \mathrm{Spec}(R)$ such that $\mathfrak{q} \nsubseteq \mathfrak{p}$, then $\rho \in \mathfrak{q}$.*
4. *$\mathcal{D}(\rho) = (\downarrow \mathfrak{p})$.*

Proof of Proposition 3.1. As Spec(R) is linearly ordered, Lemma 3.2 guarantees the equivalences

$$(1) \iff (2) \iff (3).$$

(3) $\Longrightarrow$ (4). Straightforward.

(4) $\Longrightarrow$ (3). Assume $(\downarrow \mathfrak{p})$ is open in Spec(R); then it is compact. Hence there exist $x_1, x_2, \ldots, x_n \in R$ such that

$$(\downarrow \mathfrak{p}) = \mathcal{D}(x_1) \cup \mathcal{D}(x_n) \ldots \cup \mathcal{D}(x_n).$$

But, as the ideals $\sqrt{x_i R}$ are comparable, the $\mathcal{D}(x_i)$s are also comparable. It follows that $(\downarrow \mathfrak{p}) = \mathcal{D}(\rho)$, for some $\rho \in \mathfrak{m} \setminus \{0\}$; and consequently $\mathfrak{p} = \mathcal{J}_\rho$, for some ρ. ■

Combining Lemma 2.1 and Proposition 3.1, one may check easily the following corollary.

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a domain with linearly ordered prime spectrum and $\mathfrak{p} \subset \mathfrak{q}$ are prime ideals of R, then the the following statements are equivalent.*

1. *$\mathfrak{p} \subset \mathfrak{q}$ are consecutive.*
2. *There exists $\rho \in \mathfrak{q} \setminus \mathfrak{p}$, such that $\mathfrak{p} = \mathcal{J}_\rho$ and $\mathfrak{q} = \sqrt{\rho R}$.*

The following result gives a complete description of Gold(R) and Spec(R) when the latter space is linearly ordered.

Theorem 3.4. *Let R with linearly ordered prime spectrum, and $\mathfrak{m}$ its maximal ideal. Then the following properties hold.*

1. Gold$(R) = \{\mathcal{J}_\rho : \rho \in \mathfrak{m} \setminus \{0\}\} \cup \{\mathfrak{m}\}$.
2. $\mathrm{Spec}(R) = \{\mathfrak{m}\} \bigcup \left\{ \bigcap_{\rho \in T} \mathcal{J}_\rho \ : \ \emptyset \neq T \subset \mathfrak{m} \setminus \{0\} \right\}$.

Proof.

(1) Let $\mathfrak{p}$ be a nonmaximal G-ideal, then by Lemma 2.1, $(\downarrow \mathfrak{p})$ is open in Spec(R). Hence, according to Proposition 3.1, there exists $\rho \in \mathfrak{m} \setminus \mathfrak{p}$ such that $\mathfrak{p} = \mathcal{J}_\rho$.

Conversely, if $\rho \in \mathfrak{m} \setminus \{0\}$, then $(\downarrow \mathcal{J}_\rho) = \mathcal{D}(\rho)$; therefore according to Lemma 2.1 $\mathcal{J}_\rho \in$ Gold(R).

(2) Let $\mathfrak{p}$ be a nonmaximal prime ideal of R. We claim that $\mathfrak{p} = \bigcap_{\rho \in \mathfrak{m} \setminus \mathfrak{p}} \mathcal{J}_\rho$.

Indeed, we know that $\mathfrak{p}$ is the intersection of all G-ideals containing $\mathfrak{p}$; so by Fact (1), $\mathfrak{p} = \cap\{\mathcal{J}_\rho : \mathfrak{p} \subseteq \mathcal{J}_\rho\}$. This yields $\mathfrak{p} = \bigcap_{\rho \in \mathfrak{m} \setminus \mathfrak{p}} \mathcal{J}_\rho$.

Conversely, if $\emptyset \neq T \subset \mathfrak{m}$, then $\bigcap_{\rho \in T} \mathcal{J}_\rho$ is in Spec(R), by [15, Theorem 9, page 6]. ■

4 Divided Ideals

The following result provides the structure of some divided prime ideals.

Proposition 4.1. *Let $\mathfrak{p}$ be a prime ideal of a domain R. Assume there exists a nonunit element of R not contained in $\mathfrak{p}$. If we let $\mathcal{NU}(R)$ be the set of all nonunit elements of R, then $\mathfrak{p}$ is divided if and only if $\mathfrak{p} = \bigcap_{\rho \in \mathcal{NU}(R) \setminus \mathfrak{p}} \mathcal{I}_\rho$.*

Proof.
• Assume $\mathfrak{p}$ is a divided prime ideal; then it is comparable to any principal ideal of R. So, if $\rho \in \mathcal{NU}(R) \setminus \mathfrak{p}$, we have $\mathfrak{p} \subseteq \rho^n R$, for all $n \geq 1$. This leads to $\mathfrak{p} \subseteq \mathcal{I}_\rho$; and consequently $\mathfrak{p} \subseteq \bigcap_{\rho \in \mathcal{NU}(R) \setminus \mathfrak{p}} \mathcal{I}_\rho$.

Conversely, if $x \in \bigcap_{\rho \in \mathcal{NU}(R) \setminus \mathfrak{p}} \mathcal{I}_\rho$ and $x \notin \mathfrak{p}$, then $x \in \mathcal{I}_x$, a contradiction.

• Now, suppose that $\mathfrak{p} = \bigcap_{\rho \in \mathcal{NU}(R) \setminus \mathfrak{p}} \mathcal{I}_\rho$. It is enough to establish the containment $\mathfrak{p}R_\mathfrak{p} \subseteq \mathfrak{p}$.

For, let $x \in \mathfrak{p}$ and $s \in R \setminus \mathfrak{p}$. Hence $\mathfrak{p} \subseteq \mathcal{I}_s \subseteq sR$. Thus there exists $r \in R$ such that $x = sr$. But as $x \in \mathfrak{p}$ and $s \notin \mathfrak{p}$, we get $r \in \mathfrak{p}$. It follows that $\frac{x}{s} = r \in \mathfrak{p}$. This yields $\mathfrak{p}R_\mathfrak{p} \subseteq \mathfrak{p}$. Therefore $\mathfrak{p}$ is divided. ■

The following result compares the ideals $\mathcal{I}_\rho$ and $\mathcal{J}_\rho$ and provides an answer to the problem when they are equal.

Proposition 4.2. *Let R be a domain with linearly prime spectrum and $\mathfrak{m}$ be its maximal ideal. For $\rho \in \mathfrak{m} \setminus \{0\}$, the following properties hold.*

1. *$\mathcal{I}_\rho \subseteq \mathcal{J}_\rho$.*
2. *$\mathcal{J}_\rho = \mathcal{I}_\rho$ if and only if $\mathcal{J}_\rho$ is divided.*

First, we establish a technical lemma.

Lemma 4.3. *Let R be a domain with linearly ordered prime spectrum and $\mathfrak{m}$ be its maximal ideal. If $\rho \in \mathfrak{m} \setminus \{0\}$, then*

$$\mathcal{J}_\rho = \{x \in R : \text{ for all } n \geq 0, x \text{ does not divide } \rho^n\}.$$

Proof. We let $\mathfrak{p} = \mathcal{J}_\rho$ and $S = \{1, \rho, \rho^2, \rho^3, \dots, \rho^n, \dots\}$. First, we will show that $S^{-1}\mathfrak{p} \cap R = \mathfrak{p}$. Indeed, $\mathfrak{p} \subseteq S^{-1}\mathfrak{p} \cap R$ and $S^{-1}\mathfrak{p} \cap R$ is a prime ideal of R not containing ρ. Hence, as $\mathfrak{p}$ is the largest prime ideal of R not containing ρ, we obtain $S^{-1}\mathfrak{p} \cap R = \mathfrak{p}$. Now, by Gilmer [12, Corollary 5.2], as $S^{-1}\mathfrak{p} \cap R = \mathfrak{p}$, the set $R \setminus \mathfrak{p}$ is the saturation of the multiplicative set S. But the saturation of S is given by

$$\begin{aligned} \overline{S} &= \{x \in R : xy \in S, \text{ for some } y \in R\} \\ &= \{x \in R : x \text{ divides } \rho^n \text{ for some } n \geq 0\}. \end{aligned}$$

It follows that

$$\mathfrak{p} = \{x \in R : \text{ for all } n \geq 0, x \text{ does not divide } \rho^n\}.$$ ■

Proof of Proposition 4.2. Again, we let $\mathfrak{p} = \mathcal{J}_\rho$.
(1) By Proposition 3.1, $R_\mathfrak{p} = R_\rho$. If we let $S = \{1, \rho, \rho^2, \ldots \rho^n, \ldots\}$, then $S^{-1}\mathfrak{p}$ is the maximal ideal of $S^{-1}R = R_\rho$. As $\mathcal{I}_\rho \cap S = \emptyset$, the ideal $S^{-1}\mathcal{I}_\rho$ survives in $S^{-1}R$. Thus $S^{-1}\mathcal{I}_\rho \subseteq S^{-1}P$, this leads to $\mathcal{I}_\rho \subseteq \mathfrak{p}$.
(2) Suppose that $\mathfrak{p}$ is divided; then as $\rho^n \notin \mathfrak{p}$, for all $n \geq 1$, we obtain $\mathfrak{p} \subseteq \rho^n R$. Hence $\mathfrak{p} \subseteq \mathcal{I}_\rho$; and consequently, $\mathfrak{p} = \mathcal{I}_\rho$.

Conversely, assume $\mathfrak{p} = \mathcal{I}_\rho$; and let $x \in R \setminus \mathfrak{p}$. So, by Lemma 4.3, $x \mid \rho^{n_0}$, for some $n_0 \geq 0$.

If $y \in \mathfrak{p}$, then as $\mathfrak{p} = \mathcal{I}_\rho$, $\rho^{n_0} \mid y$. This yields $x \mid y$. Therefore $\mathfrak{p} \subseteq xR$. Thus $\mathfrak{p}$ is comparable with every principal ideal of R. It follows that $\mathfrak{p}$ is divided. ■

The main result in this section is the following

Theorem 4.4. *Let R be a domain; then the following statements are equivalent.*

1. *For all $\rho \in \mathcal{NU}(R)$, R_ρ is quasi-local with maximal ideal $\mathcal{I}_\rho$.*
2. Spec(R) *is linearly ordered with maximal ideal $\mathfrak{m}$, and for each $\rho \in \mathfrak{m} \setminus \{0\}$, $\mathcal{J}_\rho$ is divided.*
3. Spec(R) *is linearly ordered with maximal ideal $\mathfrak{m}$, and for each $\rho \in \mathfrak{m} \setminus \{0\}$, $\mathcal{J}_\rho = \mathcal{I}_\rho$.*
4. *R is a divided domain.*

Proof.
(1) $\Longrightarrow$ (2). Let us show that Spec(R) is linearly ordered. By [3, Theorem 1], it suffices to show that the radicals of any two principal ideals are comparable.

If ρ is a nonunit element of R; then as $\mathcal{I}_\rho$ is the unique maximal ideal of R_ρ, $\mathcal{I}_\rho$ is a prime ideal of R and $R_{\mathcal{I}_\rho} = R_\rho$, by [12, Corollary 5.2].

Now, let ρ_1, ρ_2 be nonunit elements of R. Assume $\sqrt{\rho_1 R} \nsubseteq \sqrt{\rho_2 R}$, then $\rho_1 \notin \sqrt{\rho_2 R}$; a fortiori $\rho_1 \notin \mathcal{I}_{\rho_2}$. But as $R_{\mathcal{I}_{\rho_2}} = R_{\rho_2}$ we deduce, by Lemma 3.2 (3), that $\rho_2 \in \sqrt{\rho_1 R}$. Therefore, $\sqrt{\rho_2 R} \subseteq \sqrt{\rho_1 R}$.

We conclude that Spec(R) is linearly ordered. Letting $\mathfrak{m}$ be the maximal ideal of R, we deduce that for each $\rho \in \mathfrak{m} \setminus \{0\}$, we have $R_{J_\rho} = R_\rho$ (by Lemma 3.2). Hence $R_{\mathcal{I}_\rho} = R_{J_\rho} = R_\rho$, and consequently $J_\rho = \mathcal{I}_\rho$. It follows that J_ρ is divided, by Proposition 4.2.
(2) $\Longrightarrow$ (3). Follows immediately from Proposition 4.2.
(3) $\Longrightarrow$ (4). Combining Theorem 3.4, Proposition 4.2 and taking into consideration the fact that any intersection of divided prime ideals is divided, we deduce that every prime ideal of R is divided, and consequently R is a divided domain.
(2) $\Longrightarrow$ (1). Let ρ be a nonunit element of R; then by Kaplansky [15, Theorem 11, page 6] $\mathfrak{p} = \cup[\mathfrak{q} \in \text{Spec}(R) : \rho \notin \mathfrak{q}]$ is a prime ideal of R. As $(\downarrow \mathfrak{p}) = \mathcal{D}(\rho)$, by Lemma 3.2, we get $R_\mathfrak{p} = R_\rho$ and $\mathfrak{p}$ has an immediate successor. So, by our assumption, $\mathfrak{p}$ is divided. Again by Proposition 4.2, we deduce that $\mathfrak{p} = \mathcal{I}_\rho$, this implies that $\mathcal{I}_\rho$ is the maximal ideal of R_ρ.
(3) $\Longrightarrow$ (2). Straightforward.
(4) $\Longrightarrow$ (1). Assume R is a divided ring with maximal ideal $\mathfrak{m}$; then for each $\rho \in \mathfrak{m} \setminus \{0\}$, $\mathfrak{J}_\rho = \mathcal{I}_\rho$ by Proposition 4.2. But as $R_\rho = R_{\mathfrak{J}_\rho} = R_{\mathcal{I}_\rho}$, R_ρ is quasi-local with maximal ideal $\mathcal{I}_\rho R_{\mathcal{I}_\rho} = \mathcal{I}_\rho$. ■

The following theorem gives the structure of prime ideals of a divided domain.

Corollary 4.5 (The Prime Spectrum of Divided Domains)**.** *Let R be a divided domain with maximal ideal $\mathfrak{m}$, then*

$$\mathrm{Spec}(R) = \{\mathfrak{m}\} \bigcup \left\{ \bigcap_{\rho \in T} \mathcal{I}_\rho \,:\, \emptyset \neq T \subset \mathfrak{m} \setminus \{0\} \right\}.$$

Proof. According to Proposition 4.1, we have

$$\mathrm{Spec}(R) \subseteq \{\mathfrak{m}\} \bigcup \left\{ \bigcap_{\rho \in T} \mathcal{I}_\rho \,:\, \emptyset \neq T \subset \mathfrak{m} \setminus \{0\} \right\}.$$

Conversely, as in a divided domain R, every $\mathcal{I}_\rho$ is prime (see Theorem 4.4 (3)) and $\mathrm{Spec}(R)$ is linearly ordered, we deduce, using [15, Theorem 9, page 6], that any intersection of a family of $\mathcal{I}_\rho$ is a prime ideal of R. ■

5 Spectral Properties of Divided Ideals

Recall that a poset $(X, \leq)$ is said to be a *spectral set* if it is order isomorphic to $(\mathrm{Spec}(R), \subseteq)$, for some commutative ring R with identity.

A poset $(X, \leq)$ is said to satisfy the conditions

(K_1) if every chain in X has a supremum (sup) and an infimum (inf) [first Kaplansky's condition];

(K_2) if for each $x < y$ in X, there exist two consecutive elements $x_1 < x_2$ of X such that $x \leq x_1 < x_2 \leq y$ [second Kaplansky's condition].

In [15, Theorems 9, 11, page 6] Kaplansky showed that $(\mathrm{Spec}(R), \subseteq)$ satisfies conditions (K_1) and (K_2). The converse does not hold, as shown by Lewis-Ohm in [18, Example 2.1].

Definition 5.1. Let R be a ring; we denote by $\mathcal{D}\mathrm{iv}(R)$ the divided spectrum of R; that is the set of all divided prime ideals of R. By a *divspectral set* we mean a poset which is isomorphic to $(\mathcal{D}\mathrm{iv}(R), \subseteq)$, for some commutative ring R with identity.

The following result gives a complete characterization of divspectral sets.

Theorem 5.2. *Let $(X, \leq)$ be a poset. Then the following statements are equivalent.*

1. *$(X, \leq)$ is divspectral.*
2. *$(X, \leq)$ is linearly ordered and satisfies (K_1) and (K_2).*
3. *$(X, \leq)$ is linearly ordered spectral set.*

4. *There exists a valuation domain V such that $(X, \leq)$ is order isomorphic to $(\mathrm{Spec}(V), \subseteq)$.*

Before showing this theorem, let us recall a result due to Lewis [17].

Lemma 5.3 ([17, Corollary 3.6]). *Let $(X, \leq)$ be a poset; then $(X, \leq)$ is a linearly ordered set satisfying conditions (K_1) and (K_2) if and only if there is a valuation ring V such that $(\mathrm{Spec}(V), \subseteq)$ is order isomorphic to $(X, \leq)$.*

Proof of Theorem 5.2. As $(4) \implies (1)$ is clear and $(2), (3), (4)$ are equivalent (by Lewis [17, Corollary 3.6], it is enough to show the implication $(1) \implies (2)$.

As $(\mathcal{D}\mathrm{iv}(R), \subseteq)$ is linearly ordered, it suffices to show that it satisfies (K_1) and (K_2).

Indeed, let $(\mathfrak{p}_i, i \in I)$ be a linearly ordered family of divided ideals of R; then by [15, Theorems 9, 11, page 6], the ideals $\mathfrak{p} = \bigcap_{i \in I} \mathfrak{p}_i$ and $\mathfrak{q} = \bigcup_{i \in I} \mathfrak{p}_i$ are primes. It remains to show that they are divided.

For, let $x \in R \setminus \mathfrak{p}$; then there exists $i_0 \in I$ such that $x \notin \mathfrak{p}_{i_0}$. As $\mathfrak{p}_{i_0}$ is divided, we deduce that $\mathfrak{p}_{i_0} \subseteq xR$. Thus, a fortiori, $\mathfrak{p} \subseteq xR$, showing that $\mathfrak{p}$ is divided.

Now, let $x \in R \setminus \mathfrak{q}$; then for all $i \in I$ $x \notin \mathfrak{p}_i$. As $\mathfrak{p}_i$ is divided, we deduce that $\mathfrak{p}_i \subseteq xR$. Thus, $\mathfrak{q} \subseteq xR$, and consequently, showing that $\mathfrak{q}$ is divided.

The proof of (K_2) is similar to that of Kaplansky[15, Theorem 11, page 6]. ∎

Now, we will investigate topological properties of the divided spectrum of a ring.

Definition 5.4. A *divspectral space* is a topological space which is homeomorphic to some $\mathcal{D}\mathrm{iv}(R)$ endowed with the topology inherited by the Zariski topology on $\mathrm{Spec}(R)$.

Next, we provide a topological characterization of divspectral spaces.

Theorem 5.5. *Let $(X, \mathcal{T})$ be a topological space and $\leq$ be the quasi-order defined by $\mathcal{T}$ (i.e., $x \leq y$ iff $y \in \overline{\{x\}}$); then the following statements are equivalent.*

1. *X is a divspectral space.*
2. *$\mathcal{T}$ is spectral and $\leq$ is a linear order.*
3. *$(X, \mathcal{T})$ is homeomorphic to the prime spectrum of valuation ring.*
4. *$\mathcal{T}$ is compact and $\leq$ is a linear order and X is totally disconnected in its order topology.*

We break the proof into a sequence of lemmata.

Lemma 5.6. *Let R be a ring with linearly ordered spectrum. Then*

$$\mathcal{B} := \{(\downarrow \mathfrak{p}) : \mathfrak{p} \in \mathrm{Gold}(R)\}$$

is a basis of the of the Zariski topology on $\mathrm{Spec}(R)$.

Proof. Let $\mathfrak{m}$ be the maximal ideal of R and $x \in \mathfrak{m} \setminus \{0\}$; then $\mathcal{D}(x) = (\downarrow \mathcal{J}_x)$. So as $\mathfrak{m}$ and $\mathcal{J}_x$ belong to Gold(R), we deduce that $\{\mathcal{D}(x) : x \in R - \{0\}\} \subset \{(\downarrow \mathfrak{p}) : \mathfrak{p} \in \text{Gold}(R)\}$.

Conversely, let $\mathfrak{p} \in \text{Gold}(R)$.

– If $\mathfrak{p} = \mathfrak{m}$, then $(\downarrow \mathfrak{p}) = \text{Spec}(R) = \mathcal{D}(1)$.

– If $\mathfrak{p} \subset \mathfrak{m}$, then by Theorem 3.4, there exists $x \in \mathfrak{m}$ such that $\mathfrak{p} = \mathcal{J}_x$ and so $(\downarrow \mathfrak{p}) = \mathcal{D}(x)$.

It follows that

$$\mathcal{B} := \{(\downarrow \mathfrak{p}) : \mathfrak{p} \in \text{Gold}(R)\}$$

is a basis of the Zariski topology on Spec(R). ∎

Corollary 5.7 ([13]). *If $(X, \leq)$ is a linearly ordered poset, then there exists at most one spectral topology on X which is compatible with the order.*

Proof. By Lemma 5.6, if there is a spectral topology $\mathcal{T}$ on X compatible with the order, then

$$\mathcal{B} := \{(\downarrow x) : x \text{ is a locally closed in } X\}$$

is a basis of $\mathcal{T}$. But, by Lemma 2.3, locally closed points x of X are such that x is maximal or has an immediate successor.

It follows that the topology $\mathcal{T}$ is determined by the order. ∎

Lemma 5.8. *Let R be a ring, then $\mathcal{D}$iv(R) endowed with the topology inherited by the Zariski topology on* Spec(R) *is a spectral space.*

Proof. By Theorem 5.2, $\mathcal{D}$iv(R) is a linearly ordered set satisfying (K_1) and (K_2); so it has greatest element $\mathfrak{m}$. Therefore, $\mathcal{D}$iv(R) is a T_0-compact space, as the topology is compatible with the order.

• Clearly, the collection $\mathcal{B} := \{\mathcal{D}(x) \cap \mathcal{D}\text{iv}(R) : x \in R\}$ is a basis of open sets of $\mathcal{D}$iv(R) closed under finite intersections.

Let us verify that $\mathcal{D}(x) \cap \mathcal{D}\text{iv}(R)$ is compact, one may assume that it is nonempty. We let

$$\mathfrak{q} = \bigcup [\mathfrak{p} \in \mathcal{D}\text{iv}(R) : x \notin \mathfrak{p}\},$$

then $\mathfrak{q}$ is the greatest element of $\mathcal{D}(x) \cap \mathcal{D}\text{iv}(R)$. It follows that $\mathcal{D}(x) \cap \mathcal{D}\text{iv}(R)$ is compact.

• It remains to show that $\mathcal{D}$iv(R) is sober.

Let C be an irreducible closed set of $\mathcal{D}$iv(R); then $C = \mathcal{D}\text{iv}(R) \cap F$, where F is a closed set of Spec(R). We denote by $\mathfrak{p} = \bigcup [\mathfrak{q} : \mathfrak{q} \in C]$. Then $\mathfrak{p} \in \mathcal{D}\text{iv}(R)$.

Assume $\mathfrak{p} \notin F$. Then, as every element of F is comparable to $\mathfrak{p}$, we get $F \subset (\mathfrak{p} \uparrow)$. We claim that F is irreducible in Spec(R). For, let U, V be two open sets of Spec(R) such that $U \cap F$ and $V \cap F$ are nonempty. Pick $\mathfrak{p}_1 \in U \cap F$ and $\mathfrak{p}_2 \in V \cap F$, then $\mathfrak{p} \subset \mathfrak{p}_1$ and $\mathfrak{p} \subset \mathfrak{p}_2$. As $\mathfrak{p}$ is the intersection of all elements of C, we deduce that there are $\mathfrak{q}_1, \mathfrak{q}_2$ in C such that $\mathfrak{p} \subset \mathfrak{q}_1 \subseteq \mathfrak{p}_1$ and $\mathfrak{p} \subset \mathfrak{q}_2 \subseteq \mathfrak{p}_2$. Comparing $\mathfrak{q}_1$ and $\mathfrak{q}_2$, one may assume for instance that $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$; but as U, V are downsets, we deduce that $\mathfrak{q}_1 \in U \cap V \cap F$. This shows that every nonempty set of F is dense, consequently F is irreducible. Therefore F has a generic point $\mathfrak{t}$; that is to say $F = (\mathfrak{t} \uparrow)$. It follows that $\mathfrak{p} \subseteq \mathfrak{t}$. Thus as $C \subseteq F = \mathfrak{t}$, and $\mathfrak{p}$ is the intersection of all elements of C, we get $\mathfrak{p} = \mathfrak{t}$, a contradiction.

We conclude that $\mathfrak{p} \notin F$, and consequently, $F = (\mathfrak{p}\uparrow)$, and $C = \overline{\{\mathfrak{p}\}} \cap \mathcal{D}\mathrm{iv}(R)$, showing that C has a generic point. ∎

Lemma 5.9. *Let R_1, R_2 be rings with linearly ordered spectra; and $\varphi : \mathrm{Spec}(R_1) \longrightarrow \mathrm{Spec}(R_2)$ be an order isomorphism, then φ is a homeomorphism when the spectra are endowed with their Zariski topologies.*

Proof. As φ^{-1} is also an isomorphism, it is enough to show that φ is continuous. Let $\mathfrak{m}_i$ be the maximal ideal of R_i.

We will show that for each $\mathfrak{q} \in \mathrm{Gold}(R_2)$, $\varphi^{-1}(\mathfrak{q}) \in \mathrm{Gold}(R_1)$. Indeed, we consider two cases.

– If $\mathfrak{q} = \mathfrak{m}_2$, then $\varphi^{-1}(\mathfrak{q}) = \mathfrak{m}_1 \in \mathrm{Gold}(R_1)$.

– If $\mathfrak{q} \subset \mathfrak{m}_2$, then by Theorem 2.3 $\mathfrak{q}$ has an immediate successor $\mathfrak{q}_1$. As φ is also an isomorphism, $\varphi^{-1}(\mathfrak{q}_1)$ is the immediate successor of $\varphi^{-1}(\mathfrak{q})$. Hence $\varphi^{-1}(\mathfrak{q}) \in \mathrm{Gold}(R_1)$, by Theorem 2.3.

Now, since $\varphi^{-1}(\downarrow \mathfrak{q}) = (\downarrow \varphi^{-1}(\mathfrak{q}))$, we deduce, thanks to Lemma 5.6, that φ is continuous. ∎

Proof of 5.5. Firstly, note that by Hochster [13, Proposition 13], we have $(2) \Longleftrightarrow (4)$.

$(1) \Longrightarrow (2)$. Follows from Lemma 5.8.

$(2) \Longrightarrow (3)$. Assumption (2) implies that $(X, \leq)$ is a spectral linearly ordered set. Hence by Lewis [17, Corollary 3.6], there exists a valuation ring V such that X is order isomorphic to $\mathrm{Spec}(V)$. So, according to Lemma 5.9, X is homeomorphic to $\mathrm{Spec}(V)$ equipped with the Zariski topology.

$(3) \Longrightarrow (1)$. Straightforward, as a valuation domain is divided. ∎

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