Some drift exponentially fitted stochastic Runge-Kutta methods for solving Itô SDE systems

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Abstract

In this paper, we introduce a family of drift exponentially fitted stochastic Runge-Kutta (DEFSRK) methods for multi-dimensional Itô stochastic differential equations (SDEs). For the presented class of DEFSRK methods, the regions of mean-square stability (MS-stability) are obtained with reasonable results. Also, general order conditions for the coefficients and the random variables of the DEFSRK methods are extracted. Then, a set of order conditions for a subclass with stochastic weak second order is obtained. Some numerical examples are presented to establish the efficiency and accuracy of the new schemes.

1 Introduction

In many fields like theoretical physics, epidemiology and mathematical finance where random effects are crucial, SDEs appear to be used for mathematical modelling [10, 16]. Since analytical solutions of SDEs are not generally available, numerical methods are an important tool for the calculation of approximate solutions of SDEs. Therefore, the development of numerical methods for the approximation of SDEs is increasing nowadays. There are two approaches to measure the accuracy of a numerical solution of an SDE, namely weak and strong

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approximation. For the approximation of moments of the solution process, numerical methods converging in the weak sense have to be applied [1, 2, 16, 20–22]. On the other hand, in the strong approximation or pathwise approximation, the trajectories of the numerical solutions must be sufficiently close to the exact solution [3,6,9,16,24]. Above mentioned numerical approximations include explicit or implicit schemes. As we know, using implicit methods for stiff SDEs, like stiff ODEs, causes to achieve more suitable stability properties. But, a straightforward formulation of a fully implicit scheme for SDEs often causes the problem of being stochastically unstable [16]. Therefore, in order to develop the region of MS-stability, some studies have been presented for solving SDEs numerically [3,5,12,26]. In looking for effective numerical methods, one of the ideas in ODEs to extract the high order methods with good stability properties is to use the exponentially fitted Runge-Kutta (EFRK) methods [7,8,19,27–29], and the first theoretical principles of this technique was introduced by Gautschi [13] and Lyche [17]. Thus, in this paper by utilizing this idea we use the stochastic Runge-Kutta (SRK) methods of [23] and then construct the new methods. Since we use the exponentially fitting for deterministic part of the scheme, the new methods are called drift exponentially fitted stochastic Runge-Kutta (DEFSRK). For a subclass of proposed methods with stochastic weak second order, suitable parameters of MS-stability will be obtained and those regions will be displayed.

The outline of this study is as follows. In Section 2, some definitions and preliminary requirements will be stated. In Section 3, DEFSRK methods will be formulated and their general order conditions will be obtained. Then, a class of DEFSRK methods with stochastic weak second order for d-dimensional SDEs with multi-dimensional noise will be illustrated in Section 4. Also, in this Section, we discuss the concepts of MS-stability for solutions of SDEs and for numerical approximations and then a stochastic weak second order DEFSRK method which is named DEFSRK5, will be introduced. In Section 5, some numerical examples are considered to justify our theoretical results.

2 Some definitions and preliminary requirements

In this section, we provide a brief explanation for exponentially fitted Runge-Kutta methods and stochastic B-series. In the following, we consider the equidistant discretization $\mathcal{I}_h = \{0 \leq t_0 < t_1 < ... < t_N = \mathcal{T}\}$ of the time interval $[t_0, \mathcal{T}]$ with stepsize $h = \frac{\mathcal{T}-t_0}{N}$ and $t_j = t_0 + jh$ for $j = 0, 1, ..., N$ and time discrete approximation $y_t, t \in \mathcal{I}_h$. We also apply y_n instead of y_{t_n}.

2.1 Exponentially fitted Runge-Kutta methods

If we consider ODE

$$dX_t = g_0(t, X_t)dt, \qquad X_{t_0} = x_0 \in \mathbb{R}, \tag{2.1}$$

then the standard s-stage Runge-Kutta method for solving this ODE can be formulated as follows:

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i g_0(t_n + c_i h, H_i), \tag{2.2}$$

$$H_i = y_n + h\sum_{j=1}^{s} a_{ij} g_0(t_n + c_j h, H_j), i = 1, ..., s. \tag{2.3}$$

The method (2.2)-(2.3) can be associated with the operator

$$(Lz)(t) = z(t+h) - z(t) - h\sum_{i=1}^{s} b_i z'(t + c_i h, Z_i), \tag{2.4}$$

$$Z_i = z(t) + h\sum_{j=1}^{s} a_{ij} z'(t + c_j h, Z_j), i = 1, ..., s. \tag{2.5}$$

in which, z is a continuously differentiable function. We now express the following definition from [4].

Definition 2.1. *The method (2.4)-(2.5) is called exponential of order p if the related linear operator L vanishes for any linear combination of the linearly independent functions $e^{w_0 t}, e^{w_1 t}, ..., e^{w_p t}$ in which $w_i, i = 0, 1, ..., p$ are real or complex numbers.*

So, if we take $z(t) = e^{w_j t}, j = 0, 1, ..., p$ then we have

$$e^{w_j(t+h)} - e^{w_j t} - h\sum_{i=1}^{s} b_i w_j e^{w_j(t+c_i h)} = 0,$$

therefore we obtain

$$e^{w_j h} - 1 - h w_j \sum_{i=1}^{s} b_i e^{w_j c_i h} = 0,$$

now, if one puts $\theta_j = w_j h$ then one obtains

$$e^{\theta_j} - 1 - \theta_j \sum_{i=1}^{s} b_i e^{\theta_j c_i} = 0, \qquad j = 0, 1, ..., p.$$

Remark 2.2. *Generally, one takes $\theta_i = w_i h$ that $w_i, i = 0, 1, ..., p$ are real or complex numbers, but in this paper, to achieve more appropriate methods with suitable deterministic order, we set $\theta_i = w_i h^2$.*

Therefore, different choices of values θ_i cause to create various exponentially fitted Runge-Kutta (EFRK) methods. For more details about EFRK methods see [13,17]. In [18] and [19], the authors have proposed the following generalized RK methods to solve the ODEs

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i(\theta) g_0(t_n + c_i h, H_i),$$

$$H_1 = y_n, \quad H_i = \eta_i(\theta) y_n + h\sum_{j=1}^{i-1} a_{ij}(\theta) g_0(t_n + c_j h, H_j), i = 2, ..., s.$$

To design a new efficient method to solve the SDEs, we utilize the EFRK's idea and the above idea in a general form. As previously mentioned, and also according to the references considered, one of the advantages of using the EFRK method is that it makes it possible to obtain a method with better stability properties. So, in the next sections, we try to extend this idea to SDEs in order to achieve good MS-stability properties.

2.2 Stochastic B-series

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and take $\mathcal{I} = [t_0, \mathcal{T}]$ for some $0 \leq t_0 < \mathcal{T} < \infty$. In the following, for more simplicity, we suppose that $(X_t)_{t\in\mathcal{I}}$ is the solution of the d-dimensional Itô SDE system in autonomous form

$$dX_t = g_0(X_t)dt + \sum_{l=1}^{m} g_l(X_t)dW_t^l, \qquad X_{t_0} = x_0, \tag{2.6}$$

where $g_l : \mathbb{R}^d \to \mathbb{R}^d$ for $l = 0, 1, ..., m$ and $\{W_t = (W_t^1, ..., W_t^m)\}_{t\geq 0}$ is an m-dimensional Wiener process. We assume that the Borel-measurable coefficients g_l fulfill the usual conditions, i.e. those are sufficiently differentiable and satisfy a Lipschitz and a linear growth condition such that the existence and uniqueness theorem [16] are applicable. Also, for convenience, we consider $W_t^0 = t$, then (2.6) can be exhibited as

$$dX_t = \sum_{l=0}^{m} g_l(X_t)dW_t^l, \qquad X_{t_0} = x_0. \tag{2.7}$$

In the following we consider the set of families of measurable mappings denoted by Ξ

$$\Xi := \{\{\varphi(h)\}_{h\geq 0} : \varphi(h) : \Omega \to \mathbb{R} \quad is \quad \mathcal{F} - \mathcal{B} - measurable \quad \forall h \geq 0\}, \tag{2.8}$$

and let Ξ_0 be its subset defined by

$$\Xi_0 := \{\{\varphi(h)\}_{h\geq 0} \in \Xi : \varphi(0) \equiv 0\}. \tag{2.9}$$

We now state some definitions related to colored rooted trees theory from [9].

Definition 2.3. *The set of $m+1$-colored rooted trees*

$$T = \{\emptyset\} \cup T_0 \cup T_1 \cup \cdots \cup T_m$$

is recursively defined by:

i) *The graph $\bullet_l = [\emptyset]_l$ with only one vertex of color l belongs to T_l. Let $\tau = [\tau_1, \tau_2, \ldots, \tau_k]_l$ be the tree formed by joining the subtrees $\tau_1, \tau_2, \ldots, \tau_k$ each by a single branch to a common root of color l.*

ii) *If $\tau_1, \tau_2, \ldots, \tau_k \in T$ then $\tau = [\tau_1, \tau_2, \ldots, \tau_k]_l \in T_l$.*

Therefore, T_l is the set of trees with an l-colored root, and T is the union of these sets.

Definition 2.4. *For a tree $\tau \in T$ the elementary differential is a mapping $F(\tau) : \mathbb{R}^d \to \mathbb{R}^d$ defined recursively by:*

i) $F(\emptyset)(x_0) = x_0$

ii) $F(\bullet_l)(x_0) = g_l(x_0)$

iii) *If* $\tau = [\tau_1, \tau_2, \ldots, \tau_k]_l \in T_l$ *then*
$F(\tau)(x_0) = g_l^{(k)}(x_0)\left(F(\tau_1)(x_0), F(\tau_2)(x_0), \ldots, F(\tau_k)(x_0)\right).$

Definition 2.5. *Given a mapping $\phi : T \to \Xi$ satisfying*

$$\phi(\emptyset) \equiv 1 \quad \text{and} \quad \phi(\tau)(0) = 0, \qquad \forall \tau \in T \setminus \{\emptyset\}. \tag{2.10}$$

A (stochastic) B-*series is then a formal series of the form*

$$B(\phi, x_0; h) = \sum_{\tau \in T} \alpha(\tau).\phi(\tau)(h).F(\tau)(x_0), \tag{2.11}$$

where $\alpha : T \to \mathbb{Q}$ is given by

$$\alpha(\emptyset) = 1, \qquad \alpha(\bullet_l) = 1, \qquad \alpha(\tau = [\tau_1, \tau_2, \ldots, \tau_k]_l) = \frac{1}{r_1! r_2! \cdots r_q!} \prod_{j=1}^{k} \alpha(\tau_j), \tag{2.12}$$

where $r_1, r_2, \ldots, r_q$ count equal trees among $\tau_1, \tau_2, \ldots, \tau_k$.

In the next lemma it is established that if $Y(h)$ can be written as a B-series, then $f(Y(h))$ can also be written as a same series, in which the sum is taken over trees with a root of color f and subtrees in T. The lemma is essential for extracting B-series for the exact and the numerical solution. Also, it can be used for obtaining weak convergence results.

Lemma 2.6. *If $Y(h) = B(\phi, x_0; h)$ is some* B-*series with $\phi(\emptyset) \equiv 1$ and $f \in C^\infty(\mathbb{R}^d, \mathbb{R}^{\hat{d}})$ then $f(Y(h))$ can also be expressed as a formal series of the form*

$$f(Y(h)) = \sum_{u \in U_f} \beta(u).\psi_\phi(u)(h).G(u)(x_0), \tag{2.13}$$

where U_f is a set of trees derived from T by

i) $[\emptyset]_f \in U_f$, *and* $\tau_1, \tau_2, \ldots, \tau_k \in T$ *then* $[\tau_1, \tau_2, \ldots, \tau_k]_f \in U_f$,

ii) $G([\emptyset]_f)(x_0) = f(x_0)$ *and*
$G(u = [\tau_1, \tau_2, \ldots, \tau_k]_f)(x_0) = f^{(k)}(x_0)\left(F(\tau_1)(x_0), F(\tau_2)(x_0), \ldots, F(\tau_k)(x_0)\right),$

iii) $\beta([\emptyset]_f) = 1$ *and* $\beta(u = [\tau_1, \tau_2, \ldots, \tau_k]_f) = \frac{1}{r_1! r_2! \cdots r_q!} \prod_{j=1}^{k} \alpha(\tau_j)$ *where* $r_1, r_2, \ldots, r_q$ *count equal trees among* $\tau_1, \tau_2, \ldots, \tau_k$,

iv) $\psi_\phi([\emptyset]_f) \equiv 1$ *and* $\psi_\phi(u = [\tau_1, \tau_2, \ldots, \tau_k]_f)(h) = \prod_{j=1}^{k} \phi(\tau_j)(h)$.

Proof. See Lemma 2.1 of [9] or Lemma 3 in [10]. ∎

As a first result we can utilize Lemma 2.6 to function g_l: if $Y(h) = B(\phi, x_0; h)$ then

$$g_l(Y(h)) = \sum_{\tau \in T_l} \alpha(\tau).\phi_l'(\tau)(h).F(\tau)(x_0) \tag{2.14}$$

where

$$\phi_l'(\tau)(h) = \begin{cases} 1 & \tau = \bullet_l, \\ \prod_{j=1}^{k} \phi(\tau_j)(h) & \tau = [\tau_1, \tau_2, \ldots, \tau_k]_l \in T_l. \end{cases} \tag{2.15}$$

Definition 2.7. *The order of a tree $\tau \in U_f$ is defined by*

$$\rho(\tau) = \begin{cases} 0 & \tau = \emptyset, \\ \sum_{i=1}^{l} \rho(\tau_i) & \tau = [\tau_1, \tau_2, \ldots, \tau_l]_f \\ \sum_{i=1}^{l} \rho(\tau_i) + \chi_{\{k=0\}} + \frac{1}{2}\chi_{\{k \neq 0\}} & \tau = [\tau_1, \tau_2, \ldots, \tau_l]_k \end{cases} \tag{2.16}$$

where χ is the indicator function.

The next theorem expressed that the exact solution X_{t_0+h} can be formulated as B-series.

Theorem 2.8. *The solution X_{t_0+h} of (2.7) can be written as a* B-series *of the form $B(\varphi, x_0; h)$ with*

$$\varphi(\emptyset) \equiv 1, \quad \varphi(\bullet_l)(h) = W_h^l, \quad \varphi(\tau = [\tau_1, \tau_2, \ldots, \tau_k]_l)(h) = \int_0^h \prod_{j=1}^{k} \varphi(\tau_j)(s) dW_s^l. \tag{2.17}$$

Proof. See Theorem 5 of [10] or Theorem 2 of [11]. ∎

3 DEFSRK methods for SDEs

Recently, some effective stochastic Runge-Kutta methods have introduced in [20, 21, 23–25]. In this section, we propose the following drift exponentially fitted stochastic Runge-Kutta methods for solving the SDE (2.7):

$$y_{n+1} = y_n + \sum_{i=1}^{s} \sum_{k=0}^{m} \sum_{\nu=0}^{M} z_i^{(k,\nu)}(\theta) g_k(H_i^{(k,\nu)}) \tag{3.1}$$

for $n = 0, 1, ...N-1$, with the stage values

$$H_i^{(k,\nu)} = \eta_i^{(k,\nu)}(\theta) y_n + \sum_{j=1}^{s} \sum_{r=0}^{m} \sum_{\mu=0}^{M} Z_{ij}^{(k,\nu)(r,\mu)}(\theta) g_r(H_j^{(r,\mu)}) \tag{3.2}$$

for $i = 1, ..., s$, in which $\theta = wh^2, w \in \mathbb{R}$ and $z^{(k,\nu)}(\theta) \in \Xi_0^s, Z^{(k,\nu)(r,\mu)}(\theta) \in \Xi_0^{s,s}$, for $k, r = 0, 1, ..., m$ and $\mu, \nu = 0, 1, ..., M$, moreover, for convenience and obtaining a numerical method with a desired weak order p we put

$$\eta^{(k,\nu)}(\theta) = e_s + \gamma^{(k,\nu)}(\theta)$$

where $e_s = (1, \dots, 1)^\top \in \mathbb{R}^s$, $\gamma^{(k,\nu)}(\theta) \in \Xi_0^s$, and also

$$\Xi^s := \{\{\varphi^i(h)\}_{i=1}^s : h \geq 0, \varphi^i(h) : \Omega \to \mathbb{R} \quad is \quad \mathcal{F} - \mathcal{B} - measurable \quad \forall h \geq 0\},$$

and Ξ_0^s is its subset that is defined by

$$\Xi_0^s := \{\{\varphi^i(h)\}_{i=1}^s \in \Xi^s : h \geq 0, \varphi^i(0) \equiv 0, 1 \geq i \geq s\}.$$

$\Xi_0^{s,s}$ is defined in the same way to include two-dimensional arrays $\{\varphi^i(h), \varphi^j(h)\}_{i,j=1}^s$. With the above notations, for $\theta = 0$, DEFSRK methods (3.1)-(3.2) coincide with those of standard s-stage SRK methods [23].
For the simplicity of using of stochastic B-series theory, one can rewrite (3.1)-(3.2) in a more compact form as below (see [10]):

$$y_{n+1} = y_n + \sum_{k=0}^{m} \sum_{\nu=0}^{M} (z^{(k,\nu)}(\theta)^\top \otimes I_d) g_k(H^{(k,\nu)}) \tag{3.3}$$

for $n = 0, 1, ...N - 1$ in which $I_d \in \mathbb{R}^{d,d}$ is the identity matrix, and

$$H^{(k,\nu)} = \eta^{(k,\nu)}(\theta) \otimes y_n + \sum_{r=0}^{m} \sum_{\mu=0}^{M} (Z^{(k,\nu)(r,\mu)}(\theta) \otimes I_d) g_r(H^{(r,\mu)}) \tag{3.4}$$

where

$$H^{(k,\nu)} := (H_1^{(k,\nu)}, H_2^{(k,\nu)}, \dots, H_s^{(k,\nu)})^\top, \tag{3.5}$$

$$g_k(H^{(k,\nu)}) := \left(g_k(H_1^{(k,\nu)})^\top, \dots, H_s^{(k,\nu)})^\top\right)^\top. \tag{3.6}$$

Since, Taylor series expansion of the numerical solution includes additional parameters $\gamma^{(k,\nu)}(\theta)$ which are coefficients of y_n in the stage values, we use node τ_y, as an empty tree, for corresponding terms consisting of these coefficients. For instance, the Ith component of ${g_0}'y$ is

$$({g_0}'(y))^I = \sum_{J=1}^{d} \frac{\partial {g_0}^I}{\partial {X_t}^J} y^J$$

that appears in Taylor expansion and corresponds to tree $([\tau_y]_0)$. As we stated above, for computing the order of trees we consider τ_y as an empty tree with weight $\gamma^{(k,\nu)}(\theta)$. The order of τ_y should correspond to the h-order contributed to the weights by τ_y. As the weight of τ_y is given by $\gamma^{(k,\nu)}(\theta)$ with $\gamma^{(k,\nu)}(\theta) = \mathcal{O}(\theta)$ and $\theta = \mathcal{O}(h^2)$, we conclude that $\rho(\tau_y) = 2$. Thus, we reform the definition of the order of tree with slight change:

$$\rho(\tau) = \begin{cases} 0 & \tau = \emptyset \\ 2 & \tau = \tau_y \\ \sum_{i=1}^{l} \rho(\tau_i) & \tau = [\tau_1, \tau_2, \dots, \tau_l]_f \\ \sum_{i=1}^{l} \rho(\tau_i) + \chi_{\{k=0\}} + \frac{1}{2}\chi_{\{k\neq 0\}} & \tau = [\tau_1, \tau_2, \dots, \tau_l]_k \end{cases}$$

For example, the order of tree $[\tau_y, [\tau_y, \tau_0]_2]_0$ is $\frac{13}{2}$ because it has 2 nodes τ_y, 2 deterministic nodes τ_0 and one stochastic node τ_2. Based on what was said, in the Taylor series expansion of the numerical solution we have two terms; the terms which correspond to trees containing at least one node τ_y and second terms associated with other nodes. Thus, we consider two sets of color rooted trees; the first are trees that consist of at least one node τ_y and denoted by T_y and the second are trees which have no node τ_y that was described in Definition 2.3. It should be mentioned that for trees belonging to T_y the node τ_y as a father does not appear in the graphical representation of the tree. Now, for obtaining desired order conditions of numerical method we state the following theorem.

Theorem 3.1. *If the coefficients $Z^{(k,\nu)(r,\mu)}(\theta) \in \Xi_0^{s,s}$ and $z^{(k,\nu)}(\theta), \gamma^{(k,\nu)}(\theta) \in \Xi_0^s$, then the stage values $H^{(k,\nu)}$ and also the numerical solution y_1 can be written as* B-*series as follows:*

$$H^{(k,\nu)} = B(\Phi^{(k,\nu)}, x_0; h) = \sum_{\tau \in \overline{T}} \alpha(\tau).(\Phi^{(k,\nu)}(\tau)(h) \otimes I_d)(e_s \otimes F(\tau)(x_0)),$$
$$y_1 = B(\Phi, x_0; h) = \sum_{\tau \in \overline{T}} \alpha(\tau).\Phi(\tau)(h).F(\tau)(x_0)$$

in which $\overline{T} = T \cup T_y$ and

$$\Phi^{(k,\nu)}(\tau)(h) = \begin{cases} e_s & \tau = \emptyset, \\ \gamma^{(k,\nu)}(\theta) & \tau = \bullet_y \\ \sum_{\mu=0}^{M} Z^{(k,\nu)(r,\mu)}(\theta) \prod_{j=1}^{l} \Phi^{(r,\mu)}(\tau_j)(h) & \tau = [\tau_1, \tau_2, \ldots, \tau_l]_r \end{cases} \tag{3.7}$$

and

$$\Phi(\tau)(h) = \begin{cases} 1 & \tau = \emptyset \\ 0 & \tau = \bullet_y \\ \sum_{\nu=0}^{M} z^{(k,\nu)}(\theta)^{\top} e_s & \tau = \bullet_k, \\ \sum_{\nu=0}^{M} z^{(k,\nu)}(\theta)^{\top} \prod_{j=1}^{l} \Phi^{(k,\nu)}(\tau_j)(h) & \tau = [\tau_1, \tau_2, \ldots, \tau_l]_k. \end{cases} \tag{3.8}$$

Proof. Proof of this theorem is similar to Theorem 5 of [10] or Theorem 3.1 of [1]. Use the relation (2.14) together with (3.4), we conclude that $H^{(r,\mu)}$ can be written as the following series:

$$H^{(r,\mu)} = \sum_{\tau \in \overline{T}} \alpha(\tau).(\Phi^{(r,\mu)}(\tau)(h) \otimes I_d)(e_s \otimes F(\tau)(x_0)). \tag{3.9}$$

Now, it is sufficient to show that $\Phi^{(k,\nu)}(\tau)(h)$ satisfies (3.7). From the DEFSRK method (3.1)-(3.2) and (3.4), we obtain

$$H^{(k,\nu)} = \eta^{(k,\nu)}(\theta) \otimes x_0 + \sum_{r=0}^{m} \sum_{\mu=0}^{M} \sum_{\tau \in \overline{T}_r} \alpha(\tau) \left(\left(Z^{(k,\nu)(r,\mu)}(\theta) \left(\Phi^{(r,\mu)} \right)_r' (\tau)(h) \right) \otimes I_d \right) (e_s \otimes F(\tau)(x_0)),$$

in which $\overline{T}_r$ includes trees in $\overline{T}$ with root τ_r and

$$\left(\Phi^{(r,\mu)}\right)'_r(\tau)(h) = \left(\left(\Phi_1^{(r,\mu)}\right)'_r(\tau)(h), \ldots, \left(\Phi_s^{(r,\mu)}\right)'_r(\tau)(h)\right)^{\top}.$$

Now, Comparing term-by term achieves the relations (3.7). Finally, the second part of proof, i.e., the proof of (3.8) can be demonstrated in the same way. ∎

It should be mentioned that, similar to U_f in Lemma 2.6 we can construct $\overline{U}_f$ in which $\overline{T}$ trees were used. Moreover, we have $\overline{U}_f = U_f \bigcup U_{y_f}$ such that U_{y_f} are the trees with root f and contain at least one node τ_y and U_f does not include node τ_y. Now, we are ready to provide the conditions for extracting the DEFSRK methods of weak order p. For this purpose, from Lemma 2.6, Theorem 2.8 and Theorem 3.1 one can obtain B-series of the function f, evaluated at the exact and the numerical solutions as follows:

$$f(X(t_0+h)) = \sum_{u \in U_f} \beta(u).\psi_\varphi(u)(h).G(u)(x_0),$$
$$f(y_1) = \sum_{u \in U_f} \beta(u).\psi_\Phi(u)(h).G(u)(x_0) + \sum_{u \in U_{y_f}} \beta(u).\psi_\Phi(u)(h).G(u)(x_0),$$

in which

$$\psi_\varphi([\emptyset]_f)(h) \equiv 1, \qquad \psi_\varphi(u = [\tau_1, \ldots, \tau_k]_f)(h) = \prod_{j=1}^{k} \varphi(\tau_j)(h)$$
$$\psi_\Phi([\emptyset]_f)(h) \equiv 1, \qquad \psi_\Phi(u = [\tau_1, \ldots, \tau_k]_f)(h) = \prod_{j=1}^{k} \Phi(\tau_j)(h).$$

If we consider $le_f(h;t,x)$ as weak local error of the numerical method starting at the point (t,x) corresponding to f, with stepsize h, i.e., [10]

$$le_f(h;t,x) = E[f(y_{t+h}) - f(X_{t+h}) \mid y_t = X_t = x]. \tag{3.10}$$

then, we have two terms in the local error as below:

$$le_f(h;t,x) = \sum_{u \in U_f} \beta(u).E[\psi_\Phi(u)(h) - \psi_\varphi(u)(h)].G(u)(x) + \sum_{u \in U_{y_f}} \beta(u).E[\psi_\Phi(u)(h)].G(u)(x_0).$$

Hence, with the above notation we get weak consistency of order p if and only if

$$\textbf{i)} E[\psi_\Phi(u)(h)] = E[\psi_\varphi(u)(h)] + \mathcal{O}(h^{p+1}), \qquad \forall u \in U_f \ \ with \ \ \rho(u) \leq p + \frac{1}{2}, \tag{3.11}$$

$$\textbf{ii)} E[\psi_\Phi(u)(h)] = \mathcal{O}(h^{p+1}), \qquad \forall u \in U_{y_f} \ \ with \ \ \rho(u) \leq p + \frac{1}{2}. \tag{3.12}$$

On the other hand, if we apply numerical method (3.1)-(3.2) for corresponding ODE (2.1) then we have

$$y_{n+1} = y_n + \sum_{i=1}^{s} z_i^{(0,0)}(\theta) g_0(t_n + c_i^{(0)} h, H_i^{(0,0)}), n = 0, 1, ...N-1, \tag{3.13}$$

$$H_i^{(0,0)} = \eta_i^{(0,0)}(\theta) y_n + \sum_{j=1}^{s} Z_{ij}^{(0,0)(0,0)}(\theta) g_0(t_n + c_j^{(0)} h, H_j^{(0,0)}), i = 1, ..., s. \tag{3.14}$$

Furthermore, for exponentially-fitting in the case of ODE (2.1) we suppose that $w_0 = w, w_i = 0$ for $i = 1, 2, ..., s$ and also take $\theta = wh^2$. Therefore, according to relations (2.4)-(2.5) we have the following extra order conditions;

$$e^{\theta} - 1 - \frac{\theta}{h} \sum_{i=1}^{s} z_i^{(0,0)}(\theta) e^{\theta c_i^{(0)}} = 0, \tag{3.15}$$

$$e^{\theta c_i^{(0)}} - \eta_i^{(0,0)}(\theta) - \frac{\theta}{h} \sum_{j=1}^{s} Z_{i,j}^{(0,0)(0,0)}(\theta) e^{\theta c_j^{(0)}} = 0. \tag{3.16}$$

Hence, we call SRK method (3.1)-(3.2) drift exponentially fitted if its coefficients satisfy additional conditions (3.15)-(3.16).

Remark 3.2. *We say that the DEFSRK method (3.1)-(3.2) is of the stochastic weak order p if the conditions (3.11)-(3.12) are fulfilled.*

4 A subclass of DEFSRK methods of stochastic weak second order and its MS-stability analysis

In this section, we introduce some subclasses of DEFSRK methods of stochastic weak second order and investigate their MS-stability analysis. In searching for more effective SRK methods, we develop the SRK methods in [12] introduced by Rößler that have good stability properties and need less computational effort and running time compared with the other implicit methods. Let us consider the following stochastic weak second order DEFSRK methods

$$\begin{aligned} y_{n+1} = y_n + h \sum_{i=1}^{s} \alpha_i(\theta) g_0(H_i^{(0)}) &+ \sum_{i=1}^{s} \sum_{k=1}^{m} \left(\beta_i^{(1)}(\theta) \hat{I}_{(k)} + \beta_i^{(2)}(\theta) \frac{\hat{I}_{(k,k)}}{\sqrt{h}} \right) g_k(H_i^{(k)}) \\ &+ \sum_{i=1}^{s} \sum_{k=1}^{m} \left(\beta_i^{(3)}(\theta) \hat{I}_{(k)} + \beta_i^{(4)}(\theta) \sqrt{h} \right) g_k(\widehat{H}_i^{(k)}), \end{aligned} \tag{4.1}$$

for $n = 0, 1, ...N-1$ with the stage values,

$$\begin{aligned}
H_i^{(0)} &= y_n + \gamma_i^{(0)}(\theta) y_n + h \sum_{j=1}^{s} A_{ij}^{(0)}(\theta) g_0(H_j^{(0)}) + \sum_{j=1}^{s} \sum_{l=1}^{m} \hat{I}_{(l)} B_{ij}^{(0)}(\theta) g_l(H_j^{(l)}), \\
H_i^{(k)} &= y_n + \gamma_i^{(1)}(\theta) y_n + h \sum_{j=1}^{s} A_{ij}^{(1)}(\theta) g_0(H_j^{(0)}) + \sqrt{h} \sum_{j=1}^{s} B_{ij}^{(1)}(\theta) g_k(H_j^{(k)}), \\
\widehat{H}_i^{(k)} &= y_n + \gamma_i^{(2)}(\theta) y_n + h \sum_{j=1}^{s} A_{ij}^{(2)}(\theta) g_0(H_j^{(0)}) + \sum_{j=1}^{s} \sum_{\substack{l=1 \\ l \neq k}}^{m} \frac{\hat{I}_{(k,l)}}{\sqrt{h}} B_{ij}^{(2)}(\theta) g_l(H_j^{(l)}).
\end{aligned} \tag{4.2}$$

for $i = 1, \ldots, s$ and $k = 1, \ldots, m$. Furthermore, the random variables $\hat{I}_{(i,j)}$ are determined as below:

$$\hat{I}_{(i,j)} = \begin{cases} \frac{1}{2}\left(\hat{I}_{(i)}\hat{I}_{(j)} - \sqrt{h}\tilde{I}_{(i)}\right) & for\ \ i < j \\ \frac{1}{2}\left(\hat{I}_{(i)}\hat{I}_{(j)} + \sqrt{h}\tilde{I}_{(j)}\right) & for\ \ j < i \\ \frac{1}{2}\left(\hat{I}_{(i)}^2 - h\right) & for\ \ i = j \end{cases} \tag{4.3}$$

where $\tilde{I}_{(i)}$ is defined by a two point distribution with $P(\tilde{I}_{(i)} = \pm\sqrt{h}) = \frac{1}{2}$ and we also select $\hat{I}_{(i)}$ as three-point distributed random variables with $P(\hat{I}_{(i)} = \pm\sqrt{3h}) = \frac{1}{6}$ and $P(\hat{I}_{(i)} = 0) = \frac{4}{6}$ (see [12] for more details). To indicate the weighting coefficients and the coefficient matrices, we use $\alpha = (\alpha_i(\theta))$, $\gamma^{(l)} = (\gamma_i(\theta)), l = 1, 2, 3, \beta^{(k)} = (\beta_i^{(k)}(\theta)), k = 1, ..., 4$ for corresponding vector notations and $A^{(k)} = (A_{ij}^{(k)}(\theta)), B^{(k)} = (B_{ij}^{(k)}(\theta)), k = 0, 1, 2$ for corresponding matrix notations. Then, with the above notations, the coefficients of the DEFSRK methods (4.1)-(4.2) can be displayed in an extended Butcher array, see Table 1.

$\gamma^{(0)}$	$A^{(0)}$	$B^{(0)}$	
$\gamma^{(1)}$	$A^{(1)}$	$B^{(1)}$	
$\gamma^{(2)}$	$A^{(2)}$	$B^{(2)}$	
	α^T	${\beta^{(1)}}^T$	${\beta^{(2)}}^T$
		${\beta^{(3)}}^T$	${\beta^{(4)}}^T$

Table 1: Butcher tableau for DEFSRK methods (4.1)-(4.2)

In the class of DEFSRK methods (4.1)-(4.2) if we take

$$\begin{aligned}
&z^{(0,0)}(\theta) = \alpha h, && z^{(k,0)}(\theta) = \beta^{(1)}\hat{I}_{(k)} + \beta^{(2)}\frac{\hat{I}_{(k,k)}}{\sqrt{h}}, && z^{(k,1)}(\theta) = \beta^{(3)}\hat{I}_{(k)} + \beta^{(4)}\sqrt{h}, \\
&\gamma^{(0,0)}(\theta) = \gamma^{(0)}, && Z^{(0,0)(0,0)}(\theta) = A^{(0)}h, && Z^{(0,0)(r,0)}(\theta) = B^{(0)}\hat{I}_{(r)}, \\
&\gamma^{(k,0)}(\theta) = \gamma^{(1)}, && Z^{(k,0)(0,0)}(\theta) = A^{(1)}h, && Z^{(k,0)(r,0)}(\theta) = B^{(1)}\sqrt{h}\mathbf{1}_{\{k=r\}}, \\
&\gamma^{(k,1)}(\theta) = \gamma^{(2)}, && Z^{(k,1)(0,0)}(\theta) = A^{(2)}h, && Z^{(k,1)(r,0)}(\theta) = B^{(2)}\frac{\hat{I}_{(k,r)}}{\sqrt{h}}\mathbf{1}_{\{k\neq r\}}
\end{aligned} \tag{4.4}$$

and $H_i^{(0,0)} = H_i^{(0)}, H_i^{(k,0)} = H_i^{(k)}, H_i^{(k,1)} = \widehat{H}_i^{(k)}$ and $H_i^{(k,r)} = 0$ for $1 \leq i \leq s$, $1 \leq k \leq m$, $1 < r \leq m$, it is observed that the class of DEFSRK methods (4.1)-(4.2) belong to the very general class (3.1)-(3.2). Hence, with the above setting and the colored rooted tree analysis we can extract the stochastic weak second order conditions for the coefficients of the DEFSRK methods (4.1)-(4.2) by employing relations (3.11)-(3.12) to all rooted trees up to order 2.5 and conditions (3.15)-(3.16). Also, in the following, we refer the stochastic weak order of convergence of methods (4.1)-(4.2) for SDE (2.7) by p_S and in the deterministic case by p_D and for their pair by (p_D, p_S) such that $p_D \geq p_S$. Furthermore, according to arguments in [20], we denote $C_P^{\alpha}(\mathbb{R}^d, \mathbb{R})$ as the space of functions $u \in C^{\alpha}(\mathbb{R}^d, \mathbb{R})$ for which all the partial derivatives up to order $\alpha \in \mathbb{N}$ fulfilling a polynomial growth condition [16].

We can summarize the above listed discussion in the following theorem to provide conditions of stochastic weak second order convergence.

Theorem 4.1. *Suppose that the coefficient functions of SDE (2.7) satisfy $g_k \in C_P^6(\mathbb{R}^d, \mathbb{R})$ for $k = 0, 1, \cdots, m$. Then DEFSRK methods (4.1)-(4.2) achieve order two for the weak approximation of the solution of the Itô SDE (2.7) if the following order conditions are fulfilled:*

1. $\alpha^\top e_s - 1 = \mathcal{O}(h^2)$
2. ${\beta^{(4)}}^\top e_s = \mathcal{O}(h^{\frac{5}{2}})$
3. ${\beta^{(3)}}^\top e_s = \mathcal{O}(h^{\frac{3}{2}})$
4. $({\beta^{(1)}}^\top e_s)^2 - 1 = \mathcal{O}(h^2)$
5. ${\beta^{(2)}}^\top e_s = \mathcal{O}(h^{\frac{3}{2}})$
6. ${\beta^{(1)}}^\top (B^{(1)} e_s) = \mathcal{O}(h^{\frac{3}{2}})$
7. ${\beta^{(4)}}^\top (A^{(2)} e_s) = \mathcal{O}(h^{\frac{3}{2}})$
8. ${\beta^{(3)}}^\top (B^{(2)} e_s) = \mathcal{O}(h^{\frac{3}{2}})$
9. ${\beta^{(4)}}^\top (B^{(2)} e_s)^2 = \mathcal{O}(h^{\frac{3}{2}})$
10. $\alpha^\top (A^{(0)} e_s) - \frac{1}{2} = \mathcal{O}(h)$
11. $\alpha^\top (B^{(0)} e_s)^2 - \frac{1}{2} = \mathcal{O}(h)$
12. $({\beta^{(1)}}^\top e_s)(\alpha^\top (B^{(0)} e_s)) - \frac{1}{2} = \mathcal{O}(h)$
13. $({\beta^{(1)}}^\top e_s)({\beta^{(1)}}^\top (A^{(1)} e_s)) - \frac{1}{2} = \mathcal{O}(h)$
14. ${\beta^{(3)}}^\top (A^{(2)} e_s) = \mathcal{O}(h)$
15. ${\beta^{(2)}}^\top (B^{(1)} e_s) - 1 = \mathcal{O}(h)$
16. ${\beta^{(4)}}^\top (B^{(2)} e_s) - 1 = \mathcal{O}(h)$
17. $({\beta^{(1)}}^\top e_s)({\beta^{(1)}}^\top (B^{(1)} e_s)^2) - \frac{1}{2} = \mathcal{O}(h)$
18. $({\beta^{(1)}}^\top e_s)({\beta^{(3)}}^\top (B^{(2)} e_s)^2) - \frac{1}{2} = \mathcal{O}(h)$
19. ${\beta^{(1)}}^\top (B^{(1)} (B^{(1)} e_s)) = \mathcal{O}(h)$
20. ${\beta^{(3)}}^\top (B^{(2)} (B^{(1)} e_s)) = \mathcal{O}(h)$
21. ${\beta^{(3)}}^\top (B^{(2)} (B^{(1)} (B^{(1)} e_s))) = \mathcal{O}(\sqrt{h})$
22. ${\beta^{(1)}}^\top (A^{(1)} (B^{(0)} e_s)) = \mathcal{O}(h)$
23. ${\beta^{(3)}}^\top (A^{(2)} (B^{(0)} e_s)) = \mathcal{O}(h)$
24. ${\beta^{(4)}}^\top (A^{(2)} e_s)^2 = \mathcal{O}(\sqrt{h})$
25. ${\beta^{(4)}}^\top (A^{(2)} (A^{(0)} e_s)) = \mathcal{O}(\sqrt{h})$
26. $\alpha^\top (B^{(0)} (B^{(1)} e_s)) = \mathcal{O}(\sqrt{h})$
27. ${\beta^{(2)}}^\top (A^{(1)} e_s) = \mathcal{O}(\sqrt{h})$
28. ${\beta^{(1)}}^\top ((A^{(1)} e_s)(B^{(1)} e_s)) = \mathcal{O}(\sqrt{h})$
29. ${\beta^{(3)}}^\top ((A^{(2)} e_s)(B^{(2)} e_s)) = \mathcal{O}(\sqrt{h})$
30. ${\beta^{(4)}}^\top (A^{(2)} (B^{(0)} e_s)) = \mathcal{O}(\sqrt{h})$
31. ${\beta^{(2)}}^\top (A^{(1)} (B^{(0)} e_s)) = \mathcal{O}(\sqrt{h})$
32. ${\beta^{(4)}}^\top ((B^{(2)} e_s)^2 (A^{(2)} e_s)) = \mathcal{O}(\sqrt{h})$
33. ${\beta^{(4)}}^\top (A^{(2)} (B^{(0)} e_s)^2) = \mathcal{O}(\sqrt{h})$
34. ${\beta^{(2)}}^\top (A^{(1)} (B^{(0)} e_s)^2) = \mathcal{O}(\sqrt{h})$
35. ${\beta^{(1)}}^\top (B^{(1)} (A^{(1)} e_s)) = \mathcal{O}(\sqrt{h})$

36. $\beta^{(3)^\top}(B^{(2)}(A^{(1)}e_s)) = \mathcal{O}(\sqrt{h})$
37. $\beta^{(2)^\top}(B^{(1)}e_s)^2 = \mathcal{O}(\sqrt{h})$
38. $\beta^{(4)^\top}(B^{(2)}(B^{(1)}e_s)) = \mathcal{O}(\sqrt{h})$
39. $\beta^{(2)^\top}(B^{(1)}(B^{(1)}e_s)) = \mathcal{O}(\sqrt{h})$
40. $\beta^{(1)^\top}(B^{(1)}e_s)^3 = \mathcal{O}(\sqrt{h})$
41. $\beta^{(3)^\top}(B^{(2)}e_s)^3 = \mathcal{O}(\sqrt{h})$
42. $\beta^{(1)^\top}(B^{(1)}(B^{(1)}e_s)^2) = \mathcal{O}(\sqrt{h})$
43. $\beta^{(3)^\top}(B^{(2)}(B^{(1)}e_s)^2) = \mathcal{O}(\sqrt{h})$
44. $\beta^{(4)^\top}(B^{(2)}e_s)^4 = \mathcal{O}(\sqrt{h})$
45. $\beta^{(4)^\top}(B^{(2)}(B^{(1)}e_s))^2 = \mathcal{O}(\sqrt{h})$
46. $\beta^{(4)^\top}((B^{(2)}e_s)(B^{(2)}(B^{(1)}e_s))) = \mathcal{O}(h)$
47. $\alpha^\top((B^{(0)}e_s)(B^{(0)}(B^{(1)}e_s))) = \mathcal{O}(\sqrt{h})$
48. $\beta^{(1)^\top}((A^{(1)}(B^{(0)}e_s))(B^{(1)}e_s)) = \mathcal{O}(\sqrt{h})$
49. $\beta^{(3)^\top}((A^{(2)}(B^{(0)}e_s))(B^{(2)}e_s)) = \mathcal{O}(\sqrt{h})$
50. $\beta^{(1)^\top}(A^{(1)}(B^{(0)}(B^{(1)}e_s))) = \mathcal{O}(\sqrt{h})$
51. $\beta^{(3)^\top}(A^{(2)}(B^{(0)}(B^{(1)}e_s))) = \mathcal{O}(\sqrt{h})$
52. $\beta^{(4)^\top}((B^{(2)}(A^{(1)}e_s))(B^{(2)}e_s)) = \mathcal{O}(\sqrt{h})$
53. $\beta^{(1)^\top}(B^{(1)}(A^{(1)}(B^{(0)}e_s))) = \mathcal{O}(\sqrt{h})$
54. $\beta^{(3)^\top}(B^{(2)}(A^{(1)}(B^{(0)}e_s))) = \mathcal{O}(\sqrt{h})$
55. $\beta^{(1)^\top}((B^{(1)}e_s)(B^{(1)}(B^{(1)}e_s))) = \mathcal{O}(\sqrt{h})$
56. $\beta^{(1)^\top}((B^{(2)}e_s)(B^{(2)}(B^{(1)}e_s))) = \mathcal{O}(\sqrt{h})$
57. $\beta^{(1)^\top}(B^{(1)}(B^{(1)}(B^{(1)}e_s))) = \mathcal{O}(\sqrt{h})$
58. $\beta^{(4)^\top}((B^{(2)}e_s)(B^{(2)}(B^{(1)}(B^{(1)}e_s)))) = \mathcal{O}(\sqrt{h})$
59. $\beta^{(4)^\top}((B^{(2)}e_s)(B^{(2)}(B^{(1)}e_s)^2)) = \mathcal{O}(\sqrt{h})$
60. $\beta^{(4)^\top}\gamma^{(2)} = \mathcal{O}(h^{\frac{5}{2}})$
61. $e^\theta - 1 - \theta\sum_{i=1}^{s}\alpha_i(\theta)e^{\theta c_i^{(0)}} = 0$
62. $e^{\theta c_i^{(0)}} - (1+\gamma_i^{(0)}(\theta)) - \theta\sum_{j=1}^{s} A_{i,j}^{(0)}(\theta)e^{\theta c_j^{(0)}} = 0, 1 \le i \le s.$

Proof. The proof of this theorem is performed by using Remark 3.2. Consequently, applying relation (3.11) for all rooted trees up to order 2.5, we reach order conditions 1 to 59 that are the same as 59 obtained order conditions in Theorem 5.1 of [25], then by using relation (3.12) for all trees $\tau \in U_{y_f}$ with $\rho(\tau) \le 2.5$, we can obtain order condition 60 and ultimately from relations (3.15)-(3.16), we can attain other order conditions. It should be mentioned that for all trees $\tau \in U_{y_f}$, we have $\rho(\tau) \ge 2$. So, it is sufficient to use relation (3.12) for trees in U_{y_f} with order 2.5. Therefore, we only apply this relation for tree $[[\tau_y]_{j_1}]_f \in U_{y_f}$. Thus, we get $\psi_\Phi([[\tau_y]_{j_1}]) = \Phi([\tau_y]_{j_1}) = z^{(j_1,0)^\top}\gamma^{(j_1,0)} + z^{(j_1,1)^\top}\gamma^{(j_1,1)}$. So, $\Phi([\tau_y]_{j_1}) = (\beta^{(1)^\top}\hat{I}_{(j_1)} + \beta^{(2)^\top}\frac{\hat{I}_{(j_1,j_1)}}{\sqrt{h}})\gamma^{(1)} + (\beta^{(3)^\top}\hat{I}_{(j_1)} + \beta^{(4)^\top}\sqrt{h})\gamma^{(2)}$, and consequently we have $E(\psi_\Phi(\mathrm{t}_{0.5,1})) = \sqrt{h}(\beta^{(4)^\top}\gamma^{(2)})$. On the other hand, we have order condition $\beta^{(4)^\top}\gamma^{(2)} = \mathcal{O}(h^{\frac{5}{2}})$. ∎

4.1 Construction of DEFSRK method of order $(3,2)$

Considering the order conditions obtained from Theorem 3.1, in this section we introduce some specific methods with the reasonable region of MS-stability. To make degrees of freedom in choosing the coefficients of order two DEFSRK schemes, we take $s = 3$. Also, for obtaining better convergence properties of a DEFSRK scheme, particularly in the case of ODE and for SDEs with small noise, we select $p_D > p_S$. Therefore, with the help of the ideas of diagonally drift implicit stochastic Runge-Kutta methods DDIRDI4 and DDIRDI5 in [12] that are

SRK schemes with appropriate computational cost and stability property for stiff SDEs, we construct new schemes with the following corresponding coefficient matrices:

$$A^{(0)} = \begin{bmatrix} c_1 & 0 & 0 \\ A_{21}^{(0)} & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B^{(0)} = \begin{bmatrix} 0 & 0 & 0 \\ B_{21}^{(0)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ c_3^2 & 0 & 0 \\ c_3^2 & 0 & 0 \end{bmatrix}, B^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ c_3 & 0 & 0 \\ -c_3 & 0 & 0 \end{bmatrix},$$
$$B^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ c_4 & 0 & 0 \\ -c_4 & 0 & 0 \end{bmatrix}, \gamma^{(0)^\top} = \left[\gamma_1^{(0)}, \gamma_2^{(0)}, 0\right], \gamma^{(1)^\top} = \left[\gamma_1^{(1)}, 0, 0,\right], \gamma^{(2)^\top} = \left[\gamma_1^{(2)}, 0, 0,\right],$$
$$\beta^{(1)^\top} = \left[1 - \frac{1}{2c_3^2}, \frac{1}{4c_3^2}, \frac{1}{4c_3^2}\right], \beta^{(2)^\top} = \left[0, \frac{1}{2c_3}, -\frac{1}{2c_3}\right], \beta^{(3)^\top} = \left[-\frac{1}{2c_4^2}, \frac{1}{4c_4^2}, \frac{1}{4c_4^2}\right], \beta^{(4)^\top} = \left[0, \frac{1}{2c_4}, -\frac{1}{2c_4}\right],$$
$$\alpha^\top = [\alpha_1, \alpha_2, 0], c^{(0)^\top} = [c_1, c_2, 1 - c_1 - c_2].$$

Also, we take $A^{(2)} \equiv 0$ and $c_3 = c_4 = 1$. Therefore, with $s = 3$ and from the order conditions obtained in Theorem 4.1 we calculate the following coefficients:

$$\begin{aligned} \alpha_1 &= -\frac{\theta e^{c_2\theta} - e^\theta + 1}{\theta\left(e^{c_1\theta} - e^{c_2\theta}\right)}, & \alpha_2 &= \frac{\theta e^{c_1\theta} - e^\theta + 1}{\theta\left(e^{c_1\theta} - e^{c_2\theta}\right)}, \\ \gamma_1^{(0)} &= e^{c_1\theta}(1 - c_1\theta) - 1, & \gamma_2^{(0)} &= (1 - c_1\theta)e^{c_2\theta} - A_{21}^{(0)}\theta e^{c_1\theta} - 1, \\ A_{21}^{(0)} &= -\frac{(2c_1 - 1)\theta\left(e^{c_1\theta} - e^{c_2\theta}\right)}{2\theta e^{c_1\theta} - 2e^\theta + 2}, \end{aligned} \tag{4.5}$$

and then for simplicity's sake we put:

$$\gamma_1^{(1)} = 1 - e^\theta, \quad \gamma_1^{(2)} = 1 - e^\theta, \quad B_{21}^{(0)} = \frac{1}{2\alpha_2}. \tag{4.6}$$

It should be mentioned that for the method of order $(3, 2)$ we need extra order conditions

$$\alpha^\top \left(A^{(0)}e_s\right)^2 - \frac{1}{3} = \mathcal{O}(h), \quad \alpha^\top \left(A^{(0)}(A^{(0)}e_s)\right) - \frac{1}{6} = \mathcal{O}(h),$$
$$\alpha^\top \left(A^{(0)}\gamma^{(0)}\right)^2 = \mathcal{O}(h), \quad \alpha^\top \left(A^{(0)}(A^{(0)}\gamma^{(0)})\right) = \mathcal{O}(h).$$

So, with the above coefficients, if we apply DEFSRK (4.1)-(4.2) to the corresponding ODE then we obtain the following values for c_1 and c_2 for DEFSRK method with $p_D = 3$:

$$c_1 = \frac{1}{2} \pm \frac{\sqrt{3}}{6}, \qquad c_2 = \frac{3c_1 - 2}{6c_1 - 3}.$$

Now, we proceed on the stability analysis of the DEFSRK (4.1)-(4.2). In the following, providing the possibility of introducing some specific schemes with appropriate stability properties for stiff SDEs will be our purpose. Let us consider the known scalar linear SDE test with multiplicative noise,

$$dX_t = \lambda X_t dt + \mu X_t dW_t, \quad X_{t_0} = x_0 \tag{4.7}$$

such that $X_t = x_0 \exp\left((\lambda - 1/2\mu^2)t + \mu W_t\right)$ is the exact solution when $x_0 \neq 0$ with probability one. In this paper, we restrict our studies to the MS-stability which refers to the analysis w.r.t. the second moment of the solution process of SDE (4.7) and the corresponding approximation process, respectively [12, 15, 16, 26]. The solution of SDE (4.7) is said to be (asymptotically) MS-stable if we have [15]

$$\lim_{t\to\infty} E[|X_t|^2] = 0, \Leftrightarrow 2\Re(\lambda) + |\mu|^2 < 0, \tag{4.8}$$

for $\lambda, \mu \in \mathbb{C}$. As a consequence of the left hand side of the equivalence in (4.8), we say that a numerical method is asymptotically MS-stable if the numerical solution y_n, generated by the method satisfies

$$\lim_{n\to\infty} E[|y_n|^2] = 0. \tag{4.9}$$

Applying a one-step stochastic numerical method by stepsize $h > 0$ to the linear test (4.7), with notation $x = h\lambda$ and $y = \sqrt{h}\mu$, we obtain the following recurrence formula

$$y_{n+1} = R_n(x, y)y_n, \tag{4.10}$$

in which $R_n(x, y)$ is called stability function. According to (4.9), the domain of MS-stability of the numerical method is the subset of $\mathbb{C}^2$ such as $R_{MS} = \{(x, y) \in \mathbb{C}^2 : \hat{R}_n(x, y) < 1\}$ in which $\hat{R}_n(x, y) = E[|R_n(x, y)|^2]$. Since, in practice the domain of stability for $\lambda, \mu \in \mathbb{C}$ is not easy to visualize, in the following for plotting the areas of MS-stability we suppose that $\lambda, \mu \in \mathbb{R}$. In addition, as a result of the right hand side of the equivalence in (4.8), for $\lambda, \mu \in \mathbb{R}$, the region of MS-stability for SDE (4.7) reduces to the area of the $x - y$ plane with $2x + y^2 < 0$. Furthermore, the numerical method is said to be A-stable if the domain of stability of SDE (4.7) is a subset of R_{MS}. On the other hand, to guarantee A-stability property of the DEFSRK method (4.1)-(4.2), when applied to ODEs, we take $c_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}$ (for more details see [14]).
Now, from the above discussion we can obtain the stability function $\Gamma_{DEFSRK}(x, y)$ of DEFSRK method (4.1)-(4.2) as follows:

$$\Gamma_{DEFSRK}(x,y) = \left(1 + \frac{x}{1-c_1x} + \frac{\alpha_2(1+\gamma_1^{(0)})c_1x^2}{(1-c_1x)^2}\right)^2 + \left(g_1 + g_2\frac{x}{1-c_1x}\right)^2 y^2 + \frac{1}{2}\left(1+\gamma_1^{(1)}\right)^2 y^4, \tag{4.11}$$

in which

$$g_1 = 1 + \frac{1}{2}\gamma_1^{(1)} + \frac{1}{4}(\gamma_2^{(1)} + \gamma_3^{(1)}), \quad g_2 = \alpha_2 B_{21}^{(0)}(1+\gamma_1^{(1)}) + \frac{1}{2}(1+\gamma_1^{(0)}).$$

To decrease the magnitude of the term that includes y^4 in the stability function (4.11), it will be more appropriate that the condition $\left(1+\gamma_1^{(1)}\right)^2 = 0$ is fulfilled.

Therefore we select $\gamma_1^{(1)} = -1$ and the corresponding DEFSRK scheme with $\theta = \log(2)$ is denoted by DEFSRK5. We consider various values for θ and then plot the MS-stability region of corresponding DEFSRK schemes. To exhibit the MS-stability region of the methods, we use Mathematica software. The regions of MS-stability of the methods are presented in Fig. 1. As we observe, for the value of $\gamma_1^{(1)} = -1$, the better stability properties will be obtained. Also, this figure

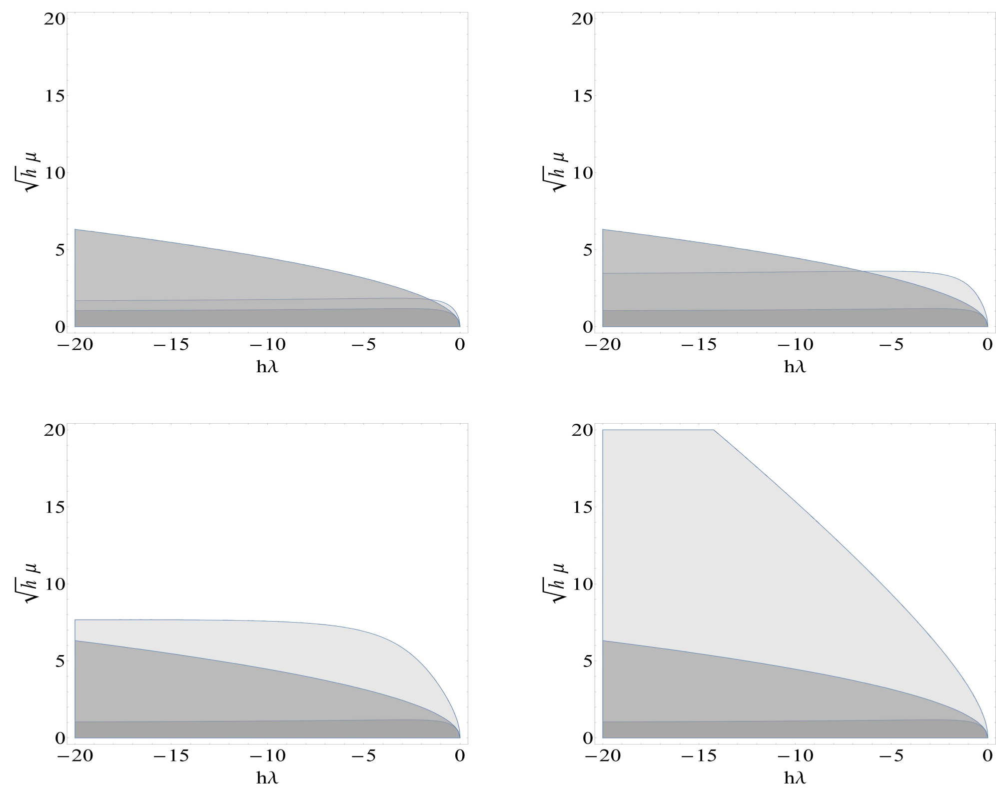

Figure 1: **MS-stability regions.** (left: top $\theta = \log(1.6)$, bottom $\theta = \log(1.98)$, right: top $\theta = \log(1.9)$, bottom $\theta = \log(2)$) DEFSRK method (light gray), DDIRDI5 method (dark) and SDE (4.7) (dark gray).

illustrates the significant improvement of the region of stability of the proposed DEFSRK method DEFSRK5 in comparison with that of DDIRDI5. To indicate the enclosed ability of the region of stability of the SDE (4.7) by DEFSRK5 method, we replace y^2 with $-2x$ in the stability function $\Gamma_{DEFSRK}(x,y) - 1$ and then plot $\hat{f}(x) = \Gamma_{DEFSRK}(x, \sqrt{-2x}) - 1$. Similarly, we carry out this process for DDIRDI5 method for displaying $\hat{g}(x) = \Gamma_{DDIRDI5}(x, \sqrt{-2x}) - 1$ in which $\Gamma_{DDIRDI5}(x,y)$ is the stability function of this method. The plotted figures in Fig. 2 show that the region of MS-stability of the SDE (4.7) is surrounded by the region of MS-stability of DEFSRK5 method in a reasonably large domain, whereas it seems that the region of MS-stability of the SDE (4.7) is not surrounded by the region of the MS-stability of DDIRDI5 method, experimentally.

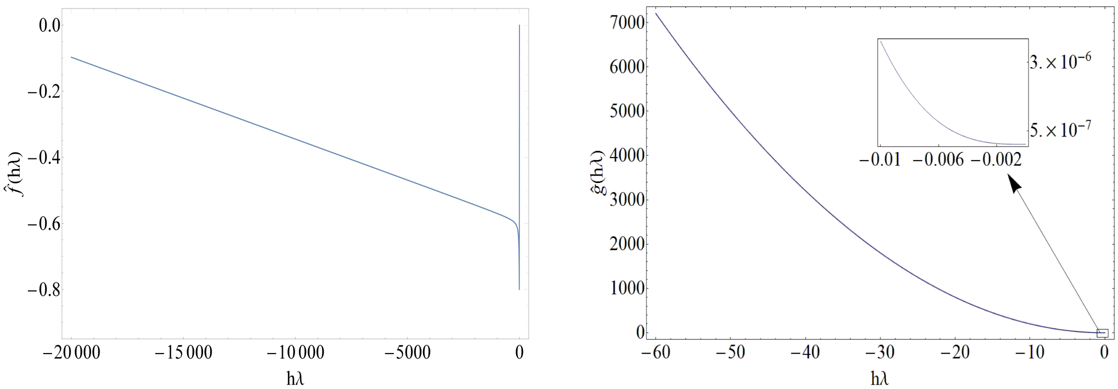

Figure 2: **Stability functions $\hat{f}(x)$ and $\hat{g}(x)$.** DEFSRK5 method (left), DDIRDI5 method (right).

5 Numerical experiments

In the following numerical examples, we illustrate the performance of the DEFSRK5 scheme developed in the previous sections. The efficiency of the proposed second order DEFSRK scheme DEFSRK5 will be indicated in comparison with the second order DDIRDI4 and DDIRDI5 schemes. In the following numerical examples we consider stepsizes $h = \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \frac{1}{256}$ for Test problems 1, 2, $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ for Test problem 3 and 10^5 simulated trajectories.

5.1 Test problem 1

As the first example we consider the scalar linear SDE test (4.7) with $x_0 = 1$, $\mathcal{T} = 2, \lambda = -300, \mu = 20$. In order to analyze the numerical MS-stability, we approximated the $E[|X_t|^2]$ by Monte Carlo simulation. The computed results are shown in Fig. 3. From this figure, MS-stability of the method DEFSRK5 for all stepsizes is clear whereas the method DDIRDI5 is MS-stable only for $h = \frac{1}{256}$. Also, in this figure, $E[|X_t|^2]$ is plotted as reference red line for comparison.

5.2 Test problem 2

As the second example, we consider the nonlinear SDE

$$dX_t = 2\lambda^2 (X_t)^7 dt + \lambda (X_t)^4 dW_t, \quad x_0 = 0.2, t \in [0,1], \tag{5.1}$$

The exact solution of this SDE is $X_t = \left(x_0^{-3} - 3\lambda W_t\right)^{-\frac{1}{3}}$. In Table 2, we report the error $|E[X_\mathcal{T}] - E[y_\mathcal{T}]|$ of the proposed methods. This table illustrates that the results of DEFSRK5 method are acceptably better than those of others. Especially, in the case of $\lambda = 12$, we observe the MS-stability of the DEFSRK5 method for all stepsizes, but not for DDIRDI4 and DDIRDI5 methods.

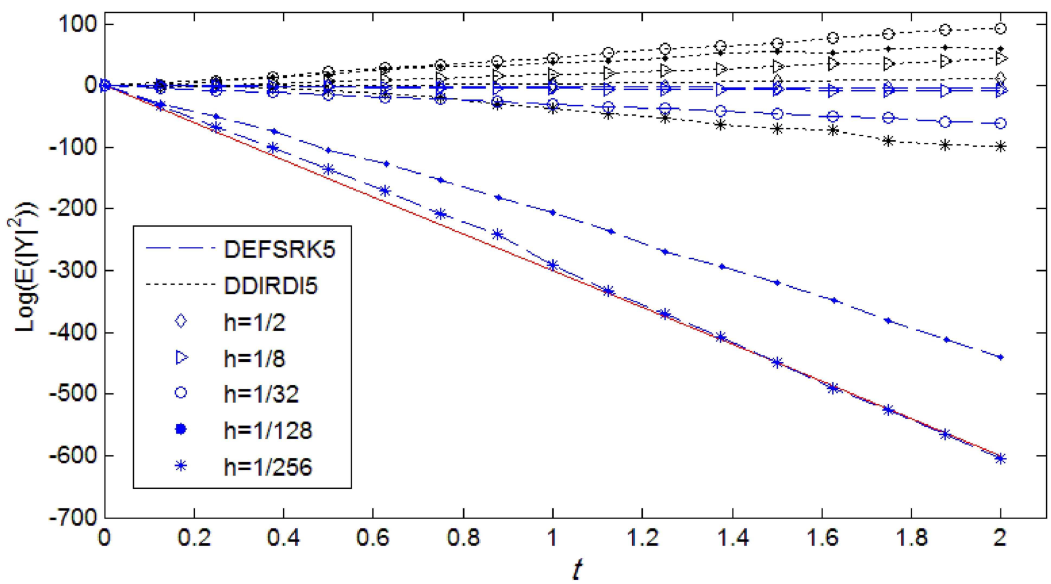

Figure 3: Numerical performance of DEFSRK5 and DDIRDI5 methods for Test problem 1

Table 2: Error of proposed methods at $\mathcal{T} = 1$ for Test problem 2

	$\lambda = 9$			$\lambda = 10$			$\lambda = 12$		
Stepsize	DEFSRK5	DDIRDI5	DDIRDI4	DEFSRK5	DDIRDI5	DDIRDI4	DEFSRK5	DDIRDI5	DDIRDI4
$\frac{1}{2}$	9.3237e-3	4.1701e-04	8.3247e-3	8.1727e-1	3.1960e-1	8.1642e-1	3.5477e-1	3.0600e-1	1.4515e-1
$\frac{1}{8}$	5.4612e-4	1.7176e-4	1.1888e-4	6.3495e-2	**unst.**	**unst.**	1.2709e-2	**unst.**	**unst.**
$\frac{1}{32}$	4.0302e-4	3.4250e-5	3.2610e-5	1.0376e-4	**unst.**	**unst**	8.5303e-3	**unst.**	**unst.**
$\frac{1}{128}$	1.9824e-5	2.4232e-6	1.1652e-5	6.9203e-4	**unst.**	**unst.**	1.7958e-3	**unst.**	**unst.**
$\frac{1}{256}$	5.3987e-7	7.0877e-7	2.8424e-6	5.7142e-4	**unst.**	**unst.**	5.6209e-4	**unst.**	**unst.**

5.3 Test problem 3

In this example, we consider two-dimensional SDE with non-commutative noise

$$dX_t = \begin{bmatrix} \lambda X_t^1 \\ \lambda X_t^2 \end{bmatrix} dt + \begin{bmatrix} \sigma X_t^1 & \epsilon X_t^2 \\ -\sigma X_t^2 & \epsilon X_t^1 \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}, \tag{5.2}$$

in which $X_0^1 = X_0^2 = 1$ and $\mathcal{T} = 4$. In this example, we plotted the estimated mean-square norm of the first component X_t^1 of the numerical solution X_t, which is approximated pointwise by [5]

$$\left(E[(X_{t_i}^1)^2]\right)^{\frac{1}{2}} \approx \left(\frac{1}{M}\sum_{\ell=1}^{M}(y_{\ell,i})^2\right)^{\frac{1}{2}}, \tag{5.3}$$

in which $y_{\ell,i}$ refers to the numerical solution at step point t_i in ℓth simulation. From [5] we conclude that the zero solutions of SDE (5.2) are asymptotically

MS-stable if and only if $\lambda + \frac{\sigma^2}{2} + \frac{\epsilon^2}{2} < 0$. Therefore we consider $\lambda = -10, \sigma = 3$ and $\epsilon = 3$. Again, from Fig. 4, we can see that the appropriate results for DEFSRK5 method is achieved relatively.

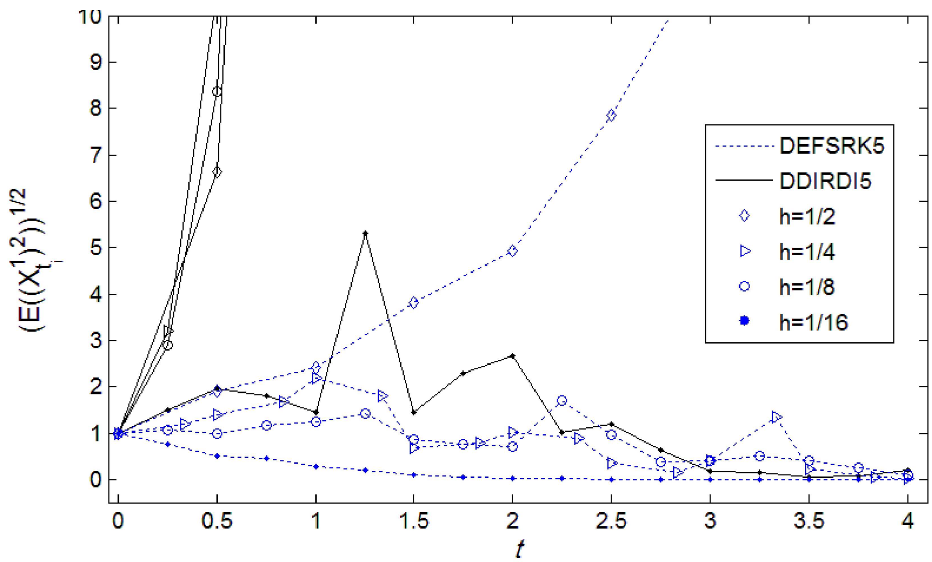

Figure 4: MS-norm of X_t^1 for Test problem 3 with $\lambda = -10, \sigma = 3$ and $\epsilon = 3$

6 Conclusions

In this paper, we have introduced a family of DEFSRK methods for the weak approximation of the systems of SDEs with multiplicative noise in the Itô sense, with arbitrary order p. The desired stochastic weak order conditions of the method have been obtained by the colored rooted trees analysis. With some special coefficient matrices that led us the methods appropriate for stiff SDEs, the MS-stability function was derived. Consequently, by finding appropriate values of the parameters of the proposed methods, we introduced DEFSRK method with weak order two, denoted by DEFSRK5, with suitable stability properties.

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