# On compactly-fibered coset spaces

Hanfeng Wang Wei He

**Abstract**

Topological properties of compactly-fibered coset spaces are investigated. It is proved that for a compactly-fibered coset space $X$ with $Nag(X) \leq \tau$, the closure of a family of $G_\tau$-sets is also a $G_\tau$-set in $X$. We also show that the equation $\chi(X) = \pi\chi(X)$ holds for any compactly-fibered coset space $X$. A Dichotomy Theorem for compactly-fibered coset spaces is established: every remainder of such a space has the Baire property, or is $\sigma$-compact.

## 1 Introduction

A topological group is a group $G$ with a topology which makes the multiplication and the inversion in $G$ continuous. Algebra and topology, the two fundamental domains of mathematics, play complementary roles in a topological group. Given a topological group $G$ and its closed subgroup $H$, $G/H$ stands for the quotient space of $G$ which consists of left cosets $xH$, where $x \in G$. We call the space $G/H$ so obtained a coset space. One of the main operations on topological groups is that of taking coset spaces. Many non-trivial examples and counterexamples arise as quotients of relatively simple and well-known topological groups. This operation has been the subject of an intensive and thorough study; but there exists still a wealth of interesting open problems related to the behaviour of different topological and algebraic properties under taking quotients. For instance, a natural question for consideration is the following one: which homogeneous spaces can be represented as quotients of topological groups with respect to closed subgroups? So far only partial answers to this question are obtained.

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For a $T_0$ topological group $G$ and its closed subgroup $H$, the coset space $G/H$ has many interesting properties. According to [6, Theorem 1.5.1], $G/H$ is homogeneous, i.e., for any two points $x, y \in G/H$, there is a homeomorphism $f$ of $G/H$ onto itself such that $f(x) = y$. Moreover, $G/H$ is Tychonoff, and hence it is a uniform space. If $G$ is metrizable, then $G/H$ is metrizable too.

If $H$ is compact, then $G/H$ is called a compactly-fibered coset space. Compactly-fibered coset spaces can be regarded as a generalization of topological groups: if $H$ contains only the identity of $G$, then $G/H = G$ is a topological group. It was observed that many classical results on topological groups can be extended to a compactly-fibered coset space. However, a compactly-fibered coset space need't be homeomorphic to a topological group. So the study of such spaces is not trivial. An essential investigation of compactly-fibered coset spaces was given recently by Arhangel'skii [4], who proved among others that the product of any family of pseudocompact compactly-fibered coset spaces is also pseudocompact. Clearly, the similar statement for general topological spaces is no longer true. We can conclude from this result that the Čech-Stone compactification of the product of two pseudocompact compactly-fibered coset spaces is homeomorphic to the product of their Čech-Stone compactifications. In [4], Arhangel'skii also studied the remainders of compactly-fibered spaces. One of his results says that every remainder of such a space is either pseudocompact or metric-friendly. This improves a Dichotomy Theorem for remainders of topological groups [2, Theorem 2.4].

In this paper we show that some results about topological groups still hold in the class of compactly-fibered coset spaces. In Section 3 we investigate some properties of compactly-fibered coset spaces related to Nagami numbers. Theorem 3.5 says that every compactly-fibered coset space $X$ with $Nag(X) \leq \tau$ is $\tau$-cellular. In Theorem 3.8 we show that for a compactly-fibered coset space $X$ with $Nag(X) \leq \tau$, the closure of a family of $G_\tau$-sets is also a $G_\tau$-set in $X$. It also turns out that the *character* of a compactly-fibered coset space coincides with its $\pi$-*character*. In Section 4 some results about remainders of compactly-fibered coset spaces are given. In particular, we prove a dichotomy theorem which says that for such a space, its remainder is either $\sigma$-compact, or has the Baire property. This generalizes a result about remainders on topological groups [3, Theorem 1.1]

## 2 Preliminary results

In this article "a space" always stands for "Tychonoff topological space". By a remainder of a space $X$ we mean the subspace $bX \setminus X$ of a compactification $bX$ of $X$. For a subset $A$ of a space $X$, $\overline{A}^X$ stands for the closure of $A$ in $X$, also denoted as $\overline{A}$ if no confusion is possible.

Recall that a space $X$ is a $p$-space if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of families $\mathcal{U}_n$ of open subsets of the Čech-Stone compactification $\beta X$ of $X$ satisfying the following two conditions: $(a)$ each $\mathcal{U}_n$ covers $X$, and $(b)$ for each $x \in X$, $\bigcap_{n\in\omega} st(x, \mathcal{U}_n) \subset \{x\}$, where $st(x, \mathcal{U}_n) = \bigcup\{U : x \in U \in \mathcal{U}_n\}$. A (Lindelöf) paracompact $p$-space is the preimage of a (separable) metrizable space under a perfect mapping. A mapping is called perfect if it is continuous, closed, and if all its fibers are compact.

A space $X$ is of countable type if every compact subset $P$ of $X$ is contained in a compact subset $F \subset X$ that has a countable base in $X$. Every $p$-space, as well as each metrizable space, is of countable type.

Let $\mathcal{F}$ be the family of all closed subsets of the Čech-Stone compactification $\beta X$ of a Tychonoff space $X$. The Nagami number $Nag(X)$ of $X$ is defined as follows: $Nag(X) = \min\{|\mathcal{P}| : \mathcal{P} \subset \mathcal{F}, \mathcal{P}$ separates $X$ from $\beta X \setminus X\}$, where $\mathcal{P}$ separating $X$ from $\beta X \setminus X$ means that for any two points $x, y$ such that $x \in X, y \in \beta X \setminus X$, there exists $P \in \mathcal{P}$ such that $x \in P$ and $y \notin P$. A Tychonoff space $X$ with $Nag(X) \leq \omega$ is called a Lindelöf $\Sigma$-space.

For an infinite cardinal $\tau$, a $G_\tau$-set of a topological space $X$ means a subset of $X$ which is the intersection of $\leq \tau$ open subsets in $X$. In particular, a $G_\delta$-set of $X$ is the intersection of a countable family of open subsets in $X$.

A space $X$ is said to be homogeneous if for each $x \in X$ and each $y \in X$, there exists a homeomorphism $f$ of the space $X$ onto itself such that $f(x) = y$. A regular closed subset $F$ of a space $X$ means that $F$ is the closure of some open subset of $X$. A space $X$ is called $\sigma$-compact if it is the union of a countable family of compact subsets of $X$.

Let $\mathcal{P}$ be a topological property. A space $X$ is said to have the property $\mathcal{P}$ locally if, for every $x \in X$, there exists a neighbourhood $O$ (not necessarily open) of $x$ in $X$ such that $O$ has the property $\mathcal{P}$. Further, if $\mathcal{P}$ is closed hereditary, then $O$ can be fixed as a closed neighbourhood of $x$ in $X$.

Let $X$ be a space, $F \subset X$ (or $x \in X$) and $\mathcal{O}$ be a family of non-empty open subsets of $X$. $\mathcal{O}$ is said to be a $\pi$-base for $X$ at $F$ (or $x$) if for every neighbourhood $U$ of $F$ (or $x$) in $X$, there exists an element $O \in \mathcal{O}$ such that $O \subset U$. $\mathcal{O}$ is said to be a $\pi$-base of $X$ if for every non-empty open subset $U$ of $X$, there exists an element $O \in \mathcal{O}$ such that $O \subset U$. The $\pi$-character $\pi\chi(x, X)$ for $X$ at $x$ is a cardinal defined as follows: $\pi\chi(x, X) = \min\{|\mathcal{O}| : \mathcal{O}$ is a $\pi$-base for $X$ at $x\} + \omega$. The $\pi$-character $\pi\chi(X)$ and the $\pi$-weight $\pi w(X)$ of $X$ are defined, respectively, as follows: $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$, $\pi w(X) = \min\{|\mathcal{O}| : \mathcal{O}$ is a $\pi$-base of $X\} + \omega$.

**Theorem 2.1.** *[4, Theorem 4.2] Let $X$ be a compactly-fibered non-locally compact coset space with a remainder $Y$. $X$ is metrizable, and $Y$ is metric-friendly if one of the two following conditions holds.*

$i_1$ *The $\pi$-character of the space $Y$ is countable at each $y \in Y$, and the space $Y$ is not countably compact.*

$i_2$ *The $\pi$-character of the space $X$ (at some point of $X$) is countable.*

It follows from the proof of [4, Theorem 4.2] that the assumption that $X$ is non-locally compact is superfluous if condition $i_2$ holds.

Throughout, we follow the terminology and notation from [6, 7].

## 3 Topological properties of compactly-fibered coset spaces related to Nagami numbers

Lemma 3.1 generalizes a well-known property for topological groups.

**Lemma 3.1.** *If a compactly-fibered coset space $X$ has a non-empty compact subset $F$ which has a countable base of open neighbourhoods in $X$, then $X$ is a paracompact $p$-space.*

*Proof.* Fix a topological group $G$, a compact subgroup $H$ of $G$, and the quotient mapping $q: G \to G/H$ such that $X = G/H$. Then $q$ is a perfect mapping. Since $F$ is a non-empty compact subset of $X$ with a countable neighbourhood base in $X$, and $q$ is perfect, it follows that the subset $q^{-1}(F)$ is a non-empty compact subset of $G$ with a countable base of open neighbourhoods in $G$. Since $G$ is a topological group, it follows $G$ is of countable type. Then, by [6, Theorem 4.3.35], $G$ is a paracompact $p$-space. Therefore, $X$ is a paracompact $p$-space, since $X$ is the image of $G$ under the perfect mapping $q$ (see [8]). ∎

We say that a space $X$ has countable tightness if for every point $x \in X$ and every subset $A \subset X$ satisfying $x \in \overline{A}$, there is a countable subset $B \subset A$ such that $x \in \overline{B}$.

**Proposition 3.2.** *If $X$ is a compactly-fibered coset space of countable type and has countable tightness, then $X$ metrizable.*

*Proof.* Fix a non-empty compact subspace $F$ of $X$ such that $X$ has a countable base at $F$. By the assumption, $F$ has countable tightness. According to [12], $F$ has countable $\pi$-character. Since $F$ has a countable neighbourhood base in $X$, it follows that $X$ has countable $\pi$-character at each point of $F$. Then the homogeneity of $X$ implies that $X$ has countable $\pi$-character. Therefore, $X$ metrizable [4, Theorem 4.2] ∎

**Proposition 3.3.** *If a compactly-fibered coset space $X$ is the union of a finite family $\eta$ of metrizable subspaces, then $X$ is metrizable.*

*Proof.* Since $\eta$ is finite, there exists an element $M$ of $\eta$ and a non-empty open subset $U$ of $X$ such that $M \cap U$ is dense in $U$. Since $X$ is regular and $M \cap U$ is metrizable, we conclude that $U$ has a countable base at each point of $M \cap U$. Then it follows from the homogeneity of $X$ and $U$ being open that $X$ is first countable. Therefore, $X$ is metrizable [4, Theorem 4.2]. ∎

**Theorem 3.4.** *If a compactly-fibered coset space $X$ is locally paracompact (locally metacompact, locally subparacompact), then it is paracompact (metacompact, subparacompact).*

*Proof.* Fix a topological group $G$, a compact subgroup $H$ of $G$, and the quotient mapping $q : G \to G/H$ such that $X = G/H$. Then $q$ is an open and perfect mapping. Assume that $X$ is locally paracompact. So we can take a non-empty regular closed subset $F$ of $X$ such that $F$ is paracompact. Clearly, the restriction $f = q\,|_{q^{-1}(F)}: q^{-1}(F) \to F$ of $q$ to $q^{-1}(F)$ is perfect. Since the inverse image of a

paracompact space under a perfect mapping is also paracompact, it follows that $q^{-1}(F)$ is a paracompact subspace of $G$. Let $U$ be the interior of $F$ in $X$. Then $q^{-1}(F) = q^{-1}(\overline{U}) = \overline{q^{-1}(U)}$, since $q$ is open and $F$ is a regular closed subset of $X$. Hence, $q^{-1}(F)$ is a regular closed subset of $G$. Hence, it follows from the homogeneity of $G$ that $G$ is locally paracompact. By [5, Corollary 1.2], $G$ is paracompact. Therefore, $X$ is paracompact, since $X$ is the image of $G$ under the perfect mapping $q$.
The proof in the situation where $X$ is locally metacompact or locally subparacompact is similar. ∎

**Theorem 3.5.** *For an infinite cardinal $\tau$, every compactly-fibered coset space $X$ with $Nag(X) \leq \tau$ is $\tau$-cellular, i.e., for every family $\mathcal{F}$ of $G_\tau$-sets in $X$, there exists a subfamily $\mathcal{P}$ of $\mathcal{F}$ with $|\mathcal{P}| \leq \tau$ such that $\bigcup \mathcal{P}$ is dense in $\bigcup \mathcal{F}$.*

*Proof.* Fix a topological group $G$, a compact subgroup $H$ of $G$, and the quotient mapping $q : G \to G/H$ such that $X = G/H$. Since $q$ is perfect, it follows from [6, Proposition 5.3.6] that $Nag(G) = Nag(X) \leq \tau$. Suppose that $\mathcal{F}$ is a family of $G_\tau$-sets in $X$. Clearly, $\{q^{-1}(F) : F \in \mathcal{F}\}$ is also a family of $G_\tau$-sets in $G$. Since each topological group with the Nagami number $\leq \tau$ is $\tau$-cellular [6, Theorem 5.3.18], there exists a subfamily $\mathcal{T}$ of $\{q^{-1}(F) : F \in \mathcal{F}\}$ such that $\bigcup \mathcal{T}$ is dense in $\bigcup\{q^{-1}(F) : F \in \mathcal{F}\}$ and $|\mathcal{T}| \leq \tau$. Therefore, $\bigcup\{q(T) : T \in \mathcal{T}\}$ is dense in $\bigcup \mathcal{F}$, since $q$ is continuous. ∎

**Corollary 3.6.** *If a compactly-fibered coset space $X$ is a Lindelöf $\Sigma$-space, then $X$ is $\omega$-cellular. In particular, $X$ has countable cellularity.*

A sequence $\{\mathcal{U}_n : n \in \omega\}$ of families $\mathcal{U}_n$ of subsets of a space $Y$ is called $\sigma$-disjoint if for each $n \in \omega$, any two elements of $\mathcal{U}_n$ are disjoint as subsets of $Y$.

**Corollary 3.7.** *If a compactly-fibered coset space $X$ is a Lindelöf $\Sigma$-space and has a $\sigma$-disjoint $\pi$-base, then $X$ is separable and metrizable.*

*Proof.* By Corollary 3.6, $X$ has countable cellularity. Then it follows from $X$ having a $\sigma$-disjoint $\pi$-base that $X$ has a countable $\pi$-base. Therefore, $X$ is separable and metrizable. ∎

**Theorem 3.8.** *Let $\Pi = \prod_{i \in I} X_i$ be a product of a family $\{X_i : i \in I\}$ of compactly-fibered coset spaces satisfying $Nag(X_i) \leq \tau$ for each $i \in I$, where $\tau$ is an infinite cardinal. Then $\Pi$ is $\tau$-cellular.*

*Proof.* For each $i \in I$, fix a topological group $G_i$, a compact subgroup $H_i$ of $G_i$, and the quotient mapping $q_i : G_i \to G_i/H_i$ such that $X_i = G_i/H_i$. Since each $q_i$ is perfect and $Nag(X_i) \leq \tau$, it follows that $Nag(G_i) \leq \tau$. According to [6, Theorem 5.3.30], $\prod_{i \in I} G_i$ is $\tau$-cellular. Therefore, $\Pi$ is $\tau$-cellular, since $\Pi$ is the image of $\prod_{i \in I} G_i$ under the perfect mapping $\prod_{i \in I} q_i$. ∎

**Corollary 3.9.** *Suppose that $X = \prod_{i \in I} X_i$ is a product of a family $\{X_i : i \in I\}$ of compactly-fibered coset spaces and each $X_i$ is a Lindelöf $\Sigma$-space, then $X$ has countable cellularity.*

**Theorem 3.10.** *Let $X$ be a compactly-fibered coset space with $Nag(X) \leq \tau$, where $\tau$ is an infinite cardinal. If $\mathcal{F}$ is a family of $G_\tau$-sets in $X$, then the closure of $\bigcup \mathcal{F}$ is also a $G_\tau$-set in $X$.*

*Proof.* Fix a topological group $G$, a compact subgroup $H$ of $G$, and the quotient mapping $q : G \to G/H$ such that $X = G/H$. Suppose that $\mathcal{F}$ is a family of $G_\tau$-sets in $X$. Clearly, $\{q^{-1}(F) : F \in \mathcal{F}\}$ is a family of $G_\tau$-sets in $G$. Since $G$ is a topological group with $Nag(G) \leq \tau$, it follows from [6, Theorem 5.3.26] that the closure $\overline{\bigcup\{q^{-1}(F) : F \in \mathcal{F}\}}$ of $\bigcup\{q^{-1}(F) : F \in \mathcal{F}\}$ is a $G_\tau$-set in $G$. Put $A = \overline{\bigcup\{q^{-1}(F) : F \in \mathcal{F}\}}$. So there is a family of open subsets $\{O_\alpha : \alpha < \tau\}$ of $G$ such that $A = \bigcap_{\alpha<\tau} O_\alpha$. For each $\alpha < \tau$ and $x \in G \setminus O_\alpha$, choose open neighbourhoods $U_{\alpha,x}$ and $V_{\alpha,x}$ of the identity $e$ of $G$ such that $xU_{\alpha,x} \cap A = \emptyset$ and $V^2_{\alpha,x} \subset U_{\alpha,x}$. Clearly, for each $\alpha < \tau$, the family $\{xV_{\alpha,x} : x \in G \setminus O_\alpha\}$ of open subsets of $G$ covers the closed subset $G \setminus O_\alpha$ of $G$. Then there exists a subset $B_\alpha$ of $G \setminus O_\alpha$ such that $G \setminus O_\alpha \subset \bigcup\{xV_{\alpha,x} : x \in B_\alpha\}$ and $|B_\alpha| \leq \tau$, since the Lindelöf number $l(G)$ of $G$ is no larger than the Nagami number $Nag(G)$ of $G$. Put $\nu = \{xV_{\alpha,x} : x \in B_\alpha, \alpha < \tau\}$. Clearly, $|\nu| \leq \tau$ and $\bigcup \nu = G \setminus A$.

Claim 1: For each $\lambda \subset \nu$, $|\lambda| < \omega$, $\overline{\bigcup \lambda} \cap A = \emptyset$.

Since $\lambda$ is finite, there exists $n \in \omega$ such that $\lambda = \{x_i V_{\alpha_i,x_i} : i < n\}$. Choose a symmetric open neighbourhood $V$ of the identity $e$ of $G$ such that $V \subset \bigcap_{i<n} V_{\alpha_i,x_i}$. Then $\overline{\bigcup \lambda} \cap A \subset (\bigcup \lambda)V^{-1} \cap A = (\bigcup_{i<n} x_i V_{\alpha_i,x_i} V) \cap A \subset (\bigcup_{i<n} x_i V^2_{\alpha_i,x_i}) \cap A \subset (\bigcup_{i<n} x_i U_{\alpha_i,x_i}) \cap A = \emptyset$. The claim is verified.

Put $\mu = \{G \setminus \overline{\bigcup \lambda} : \lambda \subset \nu, |\lambda| < \omega\}$. Clearly, $|\mu| \leq \tau$ and $\bigcap \mu = A$. Since $q$ is open, $\bigcap \mu = A = \overline{\bigcup\{q^{-1}(F) : F \in \mathcal{F}\}} = q^{-1}(\overline{\bigcup \mathcal{F}})$. Thus, $q(\bigcap \mu) = q(A) = \overline{\bigcup \mathcal{F}}$. Now, it remains to prove the following equation.

Claim 2: $q(\bigcap \mu) = \bigcap_{O\in\mu} q(O)$.

Clearly, $q(\bigcap \mu) \subset \bigcap_{O\in\mu} q(O)$ holds, so it suffices to show the converse. Take a point $x$ of $G$ such that $x \notin A$. Clearly, $xH \cap A = \emptyset$, since $A$ is the union of some left cosets of $H$ in $G$. Since $xH$ is a compact subset of $G$ and $\nu$ covers $G \setminus A$, there exists a finite set $\lambda$ of $\nu$ such that $xH \subset \bigcup \lambda$. Thus, $q(x) \notin q(G \setminus \overline{\bigcup \lambda})$. Since $G \setminus \overline{\bigcup \lambda} \in \mu$, it follows that $q(x) \notin \bigcap_{O\in\mu} q(O)$. The claim is verified. ∎

A subset $F$ of a space $X$ is called a zero-set in $X$, if there exists a real-valued function $f$ on $X$ such that $F = f^{-1}(0)$.

**Corollary 3.11.** *Let $X$ be a compactly-fibered coset space. If $X$ is a Lindelöf $\Sigma$-space, then every regular closed subset of $X$ is a zero-set in $X$.*

*Proof.* Fix a regular closed subset $F$ of $X$, i.e., $F$ is the closure of some open subset of $X$. Clearly, every open subset of $X$ is a $G_\delta$-set of $X$. Then, by Theorem 3.8, $F$ is a $G_\delta$-set of $X$. Since $X$ is a normal space, it follows that $F$ is a zero-set in $X$. ∎

**Theorem 3.12.** *For every compactly-fibered coset space $X$, $\chi(X) = \pi\chi(X)$.*

*Proof.* Fix a topological group $G$, a compact subgroup $H$ of $G$, and the quotient mapping $q : G \to G/H$ such that $X = G/H$.

Clearly, $\pi\chi(X) \leq \chi(X)$. It suffices to prove the converse. Assume that $\kappa = \pi\chi(X)$ and take a family $\{U_\alpha : \alpha < \kappa\}$ of non-empty open subsets of $X$ such that $\{U_\alpha : \alpha < \kappa\}$ is a $\pi$-base for $X$ at $q(e)$, where $e$ is the identity of $G$. Since $q$ is perfect, it follows that $\lambda = \{q^{-1}(U_\alpha) : \alpha < \kappa\}$ is a $\pi$-base for $G$ at $H$.

Claim: The family $\mu = \{q(OO^{-1}) : O \in \lambda\}$ is a base for $X$ at $q(e)$.

Clearly, $OO^{-1}$ is an open neighbourhood of $e$ in $G$, for each $O \in \lambda$. Since $q$ is open, each $q(OO^{-1})$ is an open neighbourhood of $q(e)$ in $X$. For any neighbourhood $V$ of $q(e)$, $q^{-1}(V)$ is a neighbourhood of $H$ in $G$. Since $G$ is a topological group, the mapping $f : G \times G \to G$ of the product $G \times G$ to $G$ defined by $f(u,v) = u^{-1}v$ is continuous. Observe that $H$ is compact and $f(H,H) = H^{-1}H = H \subset q^{-1}(V)$. By the Wallace Theorem, we can find two open subsets $O_1, O_2$ of $G$ such that $H \times H \subset O_1 \times O_2 \subset f^{-1}(q^{-1}(V))$. Therefore, $H \subset O_1^{-1}O_2 \subset q^{-1}(V)$. Since the family $\lambda$ is a $\pi$-base for $G$ at $H$, we can find $O \in \lambda$ such that $O \subset O_1 \cap O_2$. Thus, $OO^{-1} \subset q^{-1}(V)$, which implies that $q(OO^{-1}) \subset V$. Therefore, $\mu$ is a base for $X$ at $q(e)$. The claim is verified.

Clearly, $|\mu| \leq \kappa$. Since $X$ is homogeneous, we have that $\chi(X) \leq \pi\chi(X)$. ■

## 4 Remainders of compactly-fibered coset spaces

The following result extends a Dichotomy Theorem for remainders of topological groups to compactly-fibered coset spaces. The technique used in the proof is due to Arhangel'skii.

**Theorem 4.1.** *For a compactly-fibered coset space $X$, each remainder of $X$ either has the Baire property, or is $\sigma$-compact.*

*Proof.* If $X$ is locally compact, then each remainder of $X$ is compact having the Baire property and also being $\sigma$-compact.

Assume now that $X$ is non-locally compact. Since $X$ is homogeneous, we have that $X$ is nowhere locally-compact. Let $bX$ be a compactification of $X$ such that the remainder $Y = bX \setminus X$ does not have the Baire property. Then, there exists a sequence $\{O_n : n \in \omega\}$ of open subsets of $Y$ such that each $O_n$ is dense in $Y$ and $\bigcap_{n\in\omega} O_n$ is not dense in $Y$. Observe that $Y$ is dense in $bX$, so there is a sequence $\{U_n : n \in \omega\}$ of open subsets of $bX$ and a non-empty open subset $V$ of $bX$ such that $V \cap \overline{\bigcap_{n\in\omega} O_n}^{bX} = \emptyset$ and $U_n \cap Y = O_n$, for each $n \in \omega$. It is easy to see that $\bigcap_{n\in\omega} U_n$ is Čech-complete and is dense in $bX$. Clearly, $V \cap (\bigcap_{n\in\omega} U_n) = (V \cap X) \cap (\bigcap_{n\in\omega} U_n)$. Hence, $V \cap (\bigcap_{n\in\omega} U_n)$ is a non-empty Čech-complete subspace of $X$ and is dense in $V \cap X$. Since $X$ is homogeneous, it follows that, for each non-empty open subset $U$ of $X$, we can take a non-empty open subset $W$ contained in $U$ and a Čech-complete subspace $S$ of $W$ such that $S$ is dense in $W$. By Zorn's Lemma, there exists a maximal disjoint family $\eta$ of non-empty open subsets of $X$ such that each element of $\eta$ contains a dense Čech-complete subspace. Clearly, $\bigcup \eta$ is dense in $X$. For each $U \in \eta$ fix an open subset $bU$ of $bX$ and a Čech-complete subspace $F_U$ of $U$ such that $bU \cap X = U$ and $F_U$ is dense in $U$. Note that $F_U$ is dense in $bU$, so there exists a countable family $\{O_n(U) : n \in \omega\}$ of open subsets $O_n(U)$ of $bU$ such that $F_U = \bigcap_{n\in\omega} O_n(U)$. Put $F = \bigcup_{U\in\eta} F_U$ and

$W_n = \bigcup_{U\in\eta} O_n(U)$, for $n \in \omega$. Then it follows from $\{bU : U \in \eta\}$ being disjoint that $F = \bigcap_{n\in\omega} W_n$. Since each $W_n$ is dense in $\bigcup_{U\in\eta} bU$ and $\bigcup_{U\in\eta} bU$ is dense in $bX$, each $W_n$ is dense in $bX$. Therefore, $F$ is a dense Čech-complete subspace of $bX$ and is contained in $X$.

Fix a topological group $G$, a compact subgroup $H$ of $G$, and the quotient mapping $q : G \to G/H$ such that $X = G/H$. Since $q$ is a continuous open mapping, $\overline{q^{-1}(F)} = q^{-1}(\overline{F}) = q^{-1}(X) = G$. Hence, $q^{-1}(F)$ is dense in $G$. Since $q$ is perfect, $q^{-1}(F)$ is Čech-complete. Then, by [3, Theorem 1.2], $G$ is Čech-complete. Thus, $X$ is Čech-complete, since $X$ is the image of $G$ under the perfect mapping $q$. Therefore, $Y$ is $\sigma$-compact. ∎

Recall that a regular space $X$ is developable if there exists a sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ of $X$ such that, for every $x \in X$, $\{st(x,\mathcal{U}_n) : n \in \omega\}$ is a base for $X$ at $x$, where $st(x,\mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n : x \in U\}$. A base $\mathcal{B}$ of a space $X$ is called point-countable if, for every $x \in X$, the set $\{B \in \mathcal{B} : x \in B\}$ is at most countable.

**Theorem 4.2.** *If $X$ is a non-locally compact compactly-fibered coset space with a compactification $bX$ such that $Y = bX \setminus X$ is the union of a finite collection $\eta$ of subspaces each of which is metrizable (or is developable, or has a point-countable base), then $X$ is metrizable.*

*Proof.* Clearly, both $X$ and $Y$ are dense in $bX$, and $Y$ is nowhere locally-compact. Since $\eta$ is finite, it follows that there exists $M \in \eta$ and a non-empty regular closed subset $F$ of $Y$ such that $M \cap F$ is dense in $F$. Put $B = \bigcup\{\overline{S}^{bX} : S \subset M \cap F, |S| \leq \omega\}$. It is easy to see that $B$ is a countably compact subspace of $bX$.

Claim: $B \cap X \neq \emptyset$. Assume the contrary. Then $B$ is a dense subset of $F$. Clearly, $B$ is a union of a finite collection of metrizable subspaces. Since $B$ is countably compact, by [11], $B$ is compact. It follows that $F = B$. This contradicts the fact that $Y$ is nowhere locally-compact.

Take a point $x \in B \cap X$. Then there is a countable subset $S$ of $M \cap F$ such that $x \in \overline{S}^{bX}$. Since $M \cap F$ is first countable and is dense in $F$, we can fix, for each $y \in S$, a countable base $\{W_{y,n} : n \in \omega\}$ for $F$ at $y$. Let $U$ be the interior of $F$ in $Y$ and $O_{y,n} = W_{y,n} \cap U$, $y \in S, n \in \omega$. Clearly, each $O_{y,n}$ is a non-empty open subset of $Y$. Then it follows, for $y \in S$, that $\{O_{y,n} : n \in \omega\}$ is a countable $\pi$-base for $Y$ at $y$. Observe that $Y$ is dense in $bX$, so $bX$ has a countable $\pi$-base $\lambda_y$ at each $y \in S$. Put $\lambda = \{O \cap X : O \in \bigcup_{y\in S} \lambda_y\}$. It is easy to verify that $\lambda$ is a countable $\pi$-base for $X$ at $x$. Therefore, $X$ is metrizable by Theorem 4.2 of [4].

The proof for the case that $Y$ is the union of a finite family of developable subspaces (or subspaces with point-countable bases) is similar. ∎

Recall that an open neighbourhood assignment for a space $X$ is a function $g$ from $X$ to the topology of $X$ such that $x \in g(x)$, for every $x \in X$. A space $X$ is called a $D$-space if, for every open neighbourhood assignment $g$ for $X$, there exists a closed discrete subset $D$ of $X$ such that $\bigcup\{g(y) : y \in D\} = X$.

**Theorem 4.3.** *If $X$ is a non-locally compact compactly-fibered coset space with a compactification $bX$ such that $Y = bX \setminus X$ is the union of a countable collection $\eta$ of subspaces each of which is metrizable (or is developable, or has a point-countable base), then either $X$ is metrizable, or $X$ is a Čech-complete paracompact $p$-space.*

*Proof.* By Theorem 4.1, either $Y$ has the Baire property, or $Y$ is $\sigma$-compact.

If $Y$ has the Baire property, then there exists $M \in \eta$ such that $M$ is somewhere dense in $Y$. So we can take a non-empty regular closed subset $F$ of $Y$ such that $M \cap F$ is dense in $F$. Put $B = \bigcup\{\overline{S}^{bX} : S \subset M \cap F, |S| \leq \omega\}$. Clearly, $B$ is a countably compact subspace of $bX$. Observe that every metrizable space, every space with a point-countable base as well as every developable space is a first-countable hereditarily $D$-space. Also, a countably compact space which is the union of a countable family of $D$-spaces is compact [9]. Then, a similar proof as Theorem 4.2 shows that $X$ is metrizable.

If $Y$ is $\sigma$-compact, then $X$ is Čech-complete. By Lemma 3.1, $X$ is also a paracompact $p$-space. ∎

In [1, Theorem 4.5], Arhangel'skii proved that for a non-locally compact topological group $G$, $G$ has a remainder that is a $p$-space if and only if either $G$ is $\sigma$-compact, or $G$ is a Lindelöf $p$-space. Now we extend this result to compactly-fibered coset spaces.

**Theorem 4.4.** *Suppose that $X$ is a non-locally compact compactly-fibered coset space and $bX$ is a compactification of $X$. Then the remainder $Y = bX \setminus X$ is locally a $p$-space if and only if either $X$ is $\sigma$-compact, or $X$ is a Lindelöf $p$-space.*

*Proof.* Necessity. Clearly, $Y$ is dense and nowhere locally compact in $bX$. By the assumption, we can take a non-empty regular closed subset $F$ of $Y$ such that $F$ is a $p$-space. Let $K$ be the closure of $F$ in $bX$. Then $K$ is a compactification of $F$, and $K \setminus F$ is dense in $K$. By [1, Corollary 3.7], there exists a $G_\delta$-set $P$ of $K$ such that $F \subset P$ and every $x \in P \setminus F$ is separated from $F$ by a $G_\delta$-set $P_x$ of $P$, i.e., for every $x \in P \setminus F$, there exists a $G_\delta$-set $P_x$ of $P$ such that $x \in P_x \subset P \setminus F$. Let $O$ be the interior of $K \setminus F$ in $X$. Clearly, $O$ is not empty. We consider the following two cases.

Case 1: $P = F$. Then $K \setminus F$ is a $\sigma$-compact subset of $X$. Since the interior of $K \setminus F$ in $X$ is not empty, it follows from the homogeneity of $X$ that $X$ is locally $\sigma$-compact. Since each $p$-space is of countable type, it follows that $Y$ is locally of countable type. Then, by [13, Lemma 2.2], $Y$ is of countable type. Hence, according to [10] $X$ is Lindelöf. Therefore, $X$ is $\sigma$-compact.

Case 2: $P \setminus F \neq \emptyset$. Now we have to consider two subcases.

Subcase 2(a): $P \cap O \neq \emptyset$. Then we can take a point $x \in P \cap O$ and a $G_\delta$-set $P_x$ of $P$ such that $x \in P_x \subset P \setminus F$. Clearly, $P_x$ is a $G_\delta$-set of $K$. Let $U$ be the interior of $K$ in $bX$ and $B = P_x \cap U$. Then $B$ is a $G_\delta$-set of $bX$. Since $U \cap X = O$ and $P_x \cap Y = \emptyset$, it follows that $x \in B \subset O$. Hence, the $G_\delta$-set $B$ of $bX$ is contained in $X$. Let $B = \bigcap_{n \in \omega} W_n$, where each $W_n$ is an open subset of $bX$. Then one can construct by induction a sequence $\{V_n : n \in \omega\}$ of open neighbourhoods $V_n$ of $x$ in $bX$ such that $\overline{V_{n+1}}^{bX} \subset V_n \cap W_n$, for every $n \in \omega$. Put $K_1 = \bigcap_{n \in \omega} V_n$. Then

$K_1$ is a non-empty compact subset of $bX$ such that $K_1 \subset B$ and $K_1$ has a countable neighbourhood base in $bX$. Hence, $K_1$ has a countable neighbourhood base in $X$. Then it follows from Lemma 3.1 that $X$ is a paracompact $p$-space. Therefore, $X$ is a Lindelöf $p$-space, since $X$ is Lindelöf.

Subcase 2(b): $P \cap O = \emptyset$. Clearly, $K \setminus P$ is $\sigma$-compact and $O \subset K \setminus P \subset X$. Take a non-empty regular closed subset $T$ of $X$ such that $T \subset O$. Since $T$ is a closed subset of $K \setminus P$, $T$ is $\sigma$-compact. Then it follows from the homogeneity of $X$ that $X$ is locally $\sigma$-compact. Therefore, $X$ is $\sigma$-compact, since $X$ is Lindelöf.

Sufficiency. If $X$ is $\sigma$-compact, then $Y$ is Čech-complete, and hence is a $p$-space. If $X$ is a Lindelöf $p$-space, then $Y$ is a Lindelöf $p$-space by [1, Theorem 2.1] ■

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Department of Mathematics,
Shandong Agricultural University,
Taian 271018, China
email: whfeng@sdau.edu.cn

Institute of Mathematics,
Nanjin Normal University,
Nanjin 210046, China
email : weihe@njnu.edu.cn