Moduli Spaces of Affine Homogeneous Spaces

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Abstract

Every affine homogeneous space is locally described by its curvature, its torsion and a slightly less tangible object, its connection. Using this description of the local geometry of an affine homogeneous space we construct a variety $\mathfrak{M}(\,\mathfrak{gl}\,V\,)$, which serves as a coarse moduli space for the local isometry classes of affine homogeneous spaces. Infinitesimal deformations of an isometry class of affine homogeneous spaces in this moduli space are described by the Spencer cohomology of a comodule associated to a point in $\mathfrak{M}_\infty(\,\mathfrak{gl}\,V\,)$. In an appendix we discuss the relevance of this construction to the study of locally homogeneous spaces.

1 Introduction

Homogeneous spaces comprise a class of smooth manifolds of particular interest in differential geometry, because many geometric calculations reduce essentially to linear algebra in the presence of a transitively acting Lie group. In particular all of the geometry of a homogeneous space except for its global topology can be represented in terms of converging formal power series on a formal neighborhood of the base point. This representation of the local geometry of a homogeneous space by convergent power series will be used implicitly in this article to construct moduli spaces of isometry classes of and a corresponding deformation theory for affine homogeneous spaces with or without additional geometric structures.

A particular advantage of our approach in comparison to the moduli spaces of locally homogeneous spaces constructed in [Ts] is that the covariant derivatives

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of the curvature and/or the torsion are replaced in the construction by the values of the principal connection associated to the connection in the base point. Locally homogeneous spaces for example as considered in [S] [K1] [O] [Tr] [Ts] can easily be accommodated into this framework by replacing the values of the principal curvature at the base point of a homogeneous space by the varying values of the principal curvature on a locally homogeneous space. In consequence locally homogeneous space fall naturally into the ambit of the deformation theory of globally homogeneous spaces, which is governed by Spencer cohomology akin to the Chevalley–Eilenberg cohomology in the deformation theory of Lie algebras.

Representing the local geometry of an affine homogeneous spaces in terms of converging formal power series obliterates its global topology as a manifold, in this sense we will consider formal affine homogeneous spaces only defined as a pair $\mathfrak{g} \supset \mathfrak{h}$ of Lie algebras endowed with a left invariant connection $A : \mathfrak{g} \longrightarrow \mathrm{End}\, \mathfrak{g}/\mathfrak{h}$. This algebraic simplification is bought at a prize: The integration of a formal affine homogeneous space into a real manifold will fail in general to produce an affine homogeneous space, because the Lie algebra $\mathfrak{h}$ may not integrate to a closed subgroup of the simply connected Lie group G corresponding to the Lie algebra $\mathfrak{g}$. Nevertheless every formal affine homogeneous space corresponds to a homogeneous transitive Lie algebroid over a homogeneous space endowed with a left invariant connection.

In order to construct the moduli spaces of formal affine homogeneous spaces we fix a model vector space V of the appropriate dimension and augment a given formal affine homogeneous space $\mathfrak{g} \supset \mathfrak{h}$ endowed with $A : \mathfrak{g} \longrightarrow \mathrm{End}\, \mathfrak{g}/\mathfrak{h}$ by an isomorphism or frame $F : V \longrightarrow \mathfrak{g}/\mathfrak{h}$ and a not necessarily $\mathfrak{h}$–equivariant split $\mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}$ of the canonical projection. With these two additional pieces of data in place we may associate a connection–curvature–torsion triple

$$(A,\ R,\ T\ ) \ \in\ V^* \otimes \mathrm{End}\, V \ \times\ \Lambda^2 V^* \otimes \mathrm{End}\, V \ \times\ \Lambda^2 V^* \otimes V$$

to an augmented formal affine homogeneous space, which encodes the geometry of $\mathfrak{g}/\mathfrak{h}$ completely. Our main result characterizes the subset $\mathfrak{M}(\,\mathfrak{gl}\, V\,)$ of connection–curvature–torsion triples arising from augmented formal affine homogeneous spaces by means of a formally infinite, but actually finite system of explicit homogeneous algebraic equations:

Theorem 4.8 (Algebraic Variety of Affine Homogeneous Spaces)
A connection–curvature–torsion triple $(A,\ R,\ T)$ on a vector space V represents a formal affine homogeneous space $\mathfrak{g} \supset \mathfrak{h}$ augmented by a frame isomorphism $F : V \longrightarrow \mathfrak{g}/\mathfrak{h}$ and a split $\mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}$ of the canonical projection, if and only if $(A,\ R,\ T)$ satisfies the formal first and second Bianchi identities $d^{(A,T)}T = R \wedge \mathrm{id}$ and $d^{(A,T)}R = 0$ of degrees 2, 3 respectively and for all $r \geq 0$ the following homogeneous equations of degrees $r+3$ and $r+4$

$$\Big(Q(A,T) \ -\ R \Big) \circledast \Big(\ \underbrace{A \circledast (\,A \circledast (\,\ldots\,(\,A}_{r\ \text{times}} \circledast T\,)\,\ldots))\,\Big) \ =\ 0$$
$$\Big(Q(A,T) \ -\ R \Big) \circledast \Big(\ \underbrace{A \circledast (\,A \circledast (\,\ldots\,(\,A}_{r\ \text{times}} \circledast R\,)\,\ldots))\,\Big) \ =\ 0$$

where $Q(A,T)_{x,y} := [A_x, A_y] - A_{A_xy - A_yx - T(x,y)}$ *is the approximate curvature of* (A, R, T).

This result generalizes the well–known description of local isometry classes of symmetric spaces in terms of their curvature R [H] and the classification of manifolds with parallel curvature and torsion by Ambrose and Singer [AS]. Eventually we will introduce the equivalence relation of infinite order contact $\sim_\infty$ on connection–curvature–torsion triples, which describes precisely the effect of changing the frame F and/or the split $\mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}$. The true moduli space of isometry classes of formal affine homogeneous spaces is thus the quotient:

$$\mathfrak{M}_\infty(\,\mathfrak{gl}\,V\,) \;:=\; \mathfrak{M}(\,\mathfrak{gl}\,V\,)/\sim_\infty$$

A subtle invariant of a connection–curvature–torsion triple $(A,\,R,\,T) \in \mathfrak{M}(\,\mathfrak{gl}\,V\,)$ plays a prominent role in the proof of Theorem 4.8, the stabilizer filtration of End V by subalgebras

$$\mathrm{End}\,V \;=\; \ldots \;=\; \mathfrak{h}_{-2} \;=\; \mathfrak{h}_{-1} \;\supsetneq\; \mathfrak{h}_0 \;\supsetneq\; \ldots \;\supsetneq\; \mathfrak{h}_{s-1} \;\supsetneq\; \mathfrak{h}_s \;=\; \mathfrak{h}_{s+1} \;=\; \ldots \;=\; \mathfrak{h}_\infty$$

introduced by Singer [S], in particular the minimal $s \geq -1$ with equality $\mathfrak{h}_s = \mathfrak{h}_{s+1}$ is nowadays called the Singer invariant of $(A,\,R,\,T)$. Our recursive Definition 4.4 of the stabilizer filtration differs significantly from Singer's definition akin to Lemma 4.5 and is certainly easier to use in actual calculations. The graded vector space of successive filtration quotients in the stabilizer filtration associated to a connection–curvature–torsion triple

$$\mathfrak{h}^\bullet \;:=\; \bigoplus_{r\,\in\,\mathbb{Z}} \Big(\,\mathfrak{h}_{r-1}/_{\mathfrak{h}_r}\Big) \;=\; (\,\mathrm{End}\,V/_{\mathfrak{h}_0}\,) \oplus \ldots \oplus (\,\mathfrak{h}_{s-1}/_{\mathfrak{h}_\infty}\,)$$

is naturally a Sym V^*–comodule and thus allows us to associate a cohomology theory to every point $(A,\,R,\,T)$ in the moduli space $\mathfrak{M}(\,\mathfrak{gl}\,V\,)$, namely the Spencer cohomology $H^{\bullet,\circ}(\,\mathfrak{h}\,)$ of the comodule $\mathfrak{h}^\bullet$. The second main result of this article links the special Spencer cohomology spaces $H^{\bullet,1}(\,\mathfrak{h}\,)$ to a geometric filtration on the formal tangent space to the true moduli space $\mathfrak{M}_\infty(\,\mathfrak{gl}\,V\,)$ in the point represented by the connection–curvature–torsion triple $(A,\,R,\,T)$:

$$H^{\bullet,1}(\,\mathfrak{h}\,) \;=\; T_{[A,R,T]}\mathfrak{M}^\bullet_\infty(A,R,T)/_{T_{[A,R,T]}\mathfrak{M}^{\bullet-1}_\infty(A,R,T)} \tag{1}$$

Philosophically this equality reflects the simple fact that a vector field representing an r–jet solution to the affine Killing equation, which can not be lifted to an $r+1$–jet solution, can *not* be an affine Killing field, hence its flow will change the underlying geometry, but gently enough to stay in contact with the original geometry up to order $r-2$. In a rather precise sense equation (1) is a quantitative version of Singer's Theorem [S], which characterizes the globally among the locally Riemannian homogeneous spaces, because the Spencer cohomology spaces are trivial by construction for degrees $\bullet > s$ larger than the Singer invariant so that no further deformations are possible. A paradoxical aspect of the preceding construction is that Spencer cohomology is usually introduced to describe the set of $r+1$–jet solutions as an affine space bundle over the set of r–jet solutions, however it works just as well for the affine Killing equation, where the r–jet solutions are an affine space over the $r+1$–jet solutions.

In Section 2 we recall a couple of well–known facts about connections on homogeneous spaces. Section 3 describes two equivalent ways to encode the local geometry of a formal affine homogeneous space in an algebra endowed with a skew bracket, one of these algebras has been studied in [NT]. In Section 4 we introduce connection–curvature–torsion triples and derive the algebraic equations defining of the coarse moduli space $\mathfrak{M}(\mathfrak{gl}\,V)$ as an algebraic variety. Additional parallel geometric structures are added to the picture in Section 5, whereas the final Section 6 constructs the Spencer cohomology of a formal affine homogeneous space and calculates this cohomology for the family of examples of Riemannian homogeneous spaces with large Singer invariant constructed by C. Meusers [M]. In Appendix A we discuss an alternative construction of the moduli space $\mathfrak{M}(\mathfrak{gl}\,V)$ from the point of view of locally homogeneous spaces and derive a partial differential equation obeyed by these affine manifolds.

The framework of this article was developed in collaboration with C. Meusers, the recursive definition of the stabilizer filtration for example arose directly from these discussion and was subsequently published in [M]. Besides C. Meusers the author would like to thank W. Ballmann for his support and encouragement in writing up this article.

2 Left Invariant Connections

Certainly the most important single concept in the theory of homogeneous spaces is the notion of a homogeneous or equivariant vector bundle. The left regular representation of G on $C^\infty(G/H)$ generalizes to a representation of G on sections of homogeneous vector bundles and a differential operator intertwining with these representation is called a left invariant differential operator. In this section we will discuss a convenient algebraic formalism to study a specific subclass of left invariant differential operators, left invariant connections.

A homogeneous vector bundle on a homogeneous space G/H is a vector bundle on which G acts from the left by vector bundle morphisms covering the left action of G on G/H. Under G–equivariant vector bundle homomorphisms the homogeneous vector bundles on G/H form a category, which is equivalent to the category of linear representations of the isotropy group H. One possible choice for the H–representation in this equivalence is simply the fiber $\Sigma := \Sigma_{eH}(G/H)$ of the homogeneous vector bundle $\Sigma(G/H)$ over the base point $eH \in G/H$, which is a representation of H by restricting the G–action on the total space to this fiber. For the tangent and cotangent bundles these fibers are usually replaced by the representations $\mathfrak{g}/\mathfrak{h}$ and $(\mathfrak{g}/\mathfrak{h})^*$, which are isomorphic to $T_{eH}(G/H)$ and $T^*_{eH}(G/H)$ via:

$$\mathfrak{g}/\mathfrak{h} \xrightarrow{\cong} T_{eH}(G/H), \qquad X + \mathfrak{h} \longmapsto \left.\frac{d}{dt}\right|_0 e^{tX} H$$

The vector space of sections of a homogeneous vector bundle becomes a representation of G

$$G \times \Gamma\Sigma(G/H) \longrightarrow \Gamma\Sigma(G/H), \qquad (\gamma, s) \longmapsto \Big(L_\gamma s : gH \longmapsto \gamma \star s(\gamma^{-1} gH) \Big)$$

generalizing the left regular representation of G on $C^\infty(G/H)$. Sections s satisfying $L_\gamma s = s$ for all $\gamma \in G$ are called left invariant sections. Left invariance entails directly the equality

$$s(gH) = (L_g s)(gH) = g \star s(eH)$$

for all $g \in G$ so that the value $s(eH) \in \Sigma$ of the section s in the base point is necessarily invariant under the isotropy group H and determines the left invariant section completely:

Lemma 2.1 (Characterization of Left Invariant Sections)**.**
Evaluation at the base point provides an isomorphism between the vector space $\Gamma(\Sigma(G/H))^G$ of left invariant sections of a homogeneous vector bundle $\Sigma(G/H)$ on a homogeneous space G/H and the subspace Σ^H of H–invariant vectors in the H–representation $\Sigma := \Sigma_{eH}(G/H)$:

$$\mathrm{ev}_{eH} : \quad [\,\Gamma(\Sigma(G/H))\,]^G \longrightarrow [\,\Sigma\,]^H, \qquad s \longmapsto s(eH)$$

The inverse isomorphism associates to $s_{eH} \in \Sigma^H$ the well defined section $s(gH) := g \star s_{eH}$.

In pretty much the same way we may define homogeneous principal instead of vector bundles over G/H, principal bundles endowed with a left action of G on its total space, which covers the canonical left action on G/H, and commutes with the right action characteristic for principal bundles. In particular the group G itself can be considered as a homogeneous principal H–bundle over G/H, where the commuting left and right multiplications in

$$G \times G \times H \longrightarrow G, \qquad (\gamma, g, h) \longmapsto \gamma g h$$

define the left G–action on the total space and the principal H–bundle structure respectively. In turn a left invariant principal connection is a differential form $\omega \in \Gamma(T^*G \otimes \mathfrak{h})$ on G with values in $\mathfrak{h}$ invariant under the left G–action, such that the principal connection axiom

$$\omega_{g_0 h_0}\Big(\frac{d}{dt}\Big|_0 g_t h_t \Big) = \mathrm{Ad}_{h_0^{-1}} \omega_{g_0}\Big(\frac{d}{dt}\Big|_0 g_t \Big) + \frac{d}{dt}\Big|_0 h_0^{-1} h_t \tag{2}$$

is satisfied for all curves $t \longmapsto g_t$ in G and $t \longmapsto h_t$ in H. Because the Maurer–Cartan form $\theta \in \Gamma(T^*G \otimes \mathfrak{g})$ provides a left invariant trivialization of TG, every left invariant connection is necessarily of the form $\omega \circ \theta$ for a linear map $\omega : \mathfrak{g} \longrightarrow \mathfrak{h}$, which is necessarily an H–equivariant section of the inclusion $\mathfrak{h} \subset \mathfrak{g}$ to meet the connection axiom (2):

Definition 2.2 (Left Invariant Principal Connections)**.**
A left invariant principal connection on G considered as a homogeneous principal H–bundle over G/H is an H–equivariant section $\omega : \mathfrak{g} \longrightarrow \mathfrak{h}$ of the short exact sequence:

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0$$

In consequence of this definition the set of left invariant principal connections on a given homogeneous space G/H is either the empty set or an affine space modelled on $\mathrm{Hom}_H(\mathfrak{g}/\mathfrak{h}, \mathfrak{h})$. In fact the set of all left invariant principal connections on G/H is precisely the preimage of $\mathrm{id}_\mathfrak{h} \in \mathrm{Hom}_H(\mathfrak{h}, \mathfrak{h})$ under the map $\mathrm{Hom}_H(\mathfrak{g}, \mathfrak{h}) \longrightarrow \mathrm{Hom}_H(\mathfrak{h}, \mathfrak{h})$ in the long exact sequence

$$0 \longrightarrow \mathrm{Hom}_H(\,\mathfrak{g}/\mathfrak{h},\,\mathfrak{h}\,) \longrightarrow \mathrm{Hom}_H(\,\mathfrak{g},\,\mathfrak{h}\,) \longrightarrow \mathrm{Hom}_H(\,\mathfrak{h},\,\mathfrak{h}\,) \stackrel{\delta}{\longrightarrow} \mathrm{Ext}^1_H(\,\mathfrak{g}/\mathfrak{h},\,\mathfrak{h}\,)$$

associated to the short exact sequence central to Definition 2.2. The unique obstruction against the existence of a left invariant principal connection is thus the extension class $\delta(\,\mathrm{id}_\mathfrak{h}\,) \in \mathrm{Ext}^1_H(\,\mathfrak{g}/\mathfrak{h},\,\mathfrak{h}\,)$, a homogeneous space G/H with vanishing $\delta(\,\mathrm{id}_\mathfrak{h}\,) = 0$ is properly called a reductive homogeneous space. In the literature however this notion implicitly includes a fixed choice for the left invariant principal connection $\omega : \mathfrak{g} \longrightarrow \mathfrak{h}$. Concealing this *arbitrary* choice in the inconspicuous adjective *reductive* it is somewhat duplicitous though to call the induced connections on homogeneous vector bundles *canonical* connections.

Turning from principal connections on G considered as a principal H–bundle over the homogeneous space G/H to general connections we observe that every left invariant connection ∇ on a homogeneous vector bundle $\Sigma(G/H)$ over G/H with base point fiber $\Sigma := \Sigma_{eH}(G/H)$ is associated to a G–invariant principal connection on the homogeneous frame bundle

$$\mathrm{Frame}(\,\Sigma,\,\Sigma(G/H)\,) \;:=\; \{\;(\,gH,\,F\,)\;|\;gH \in G/H \text{ and } F : \Sigma \stackrel{\cong}{\longrightarrow} \Sigma_{gH}G/H\;\} \tag{3}$$

endowed with the natural left action $\gamma \star (gH, F) := (\gamma gH, (\gamma\star) \circ F)$ on its total space. The identification of the model space $\Sigma = \Sigma_{eH}(G/H)$ of the frame bundle with the fiber over the base point $eH \in G/H$ allows us to construct the G–equivariant isomorphism

$$G \times_H \mathbf{GL}\,\Sigma \stackrel{\cong}{\longrightarrow} \mathrm{Frame}(\,\Sigma,\,\Sigma(G/H)\,), \quad [\,g, F\,] \longrightarrow (\;gH,\;F : s \longmapsto g \star Fs\;)$$

of homogeneous principal bundles over G/H, where $[g, F]$ denotes the orbit equivalence class of the tuple $(g, F) \in G \times \mathbf{GL}\,\Sigma$ under the free right H–action $(g, F) \star h := (gh, (h^{-1}\star) \circ F)$ defining the quotient $G \times_H \mathbf{GL}\,\Sigma$. Taking a clue from the formula describing a principal connection from its restriction to a reduction of the underlying principal bundle we find that every G–invariant principal connection on $G \times_H \mathbf{GL}\,\Sigma \cong \mathrm{Frame}(\,\Sigma,\,\Sigma(G/H)\,)$ factorizes over the Maurer–Cartan form θ and a linear map $A : \mathfrak{g} \longrightarrow \mathrm{End}\,\Sigma$, $X \longrightarrow A_X$, via:

$$\omega_{[\,g_0, F_0\,]}\Big(\,\frac{d}{dt}\Big|_0 [\,g_t, F_t\,]\,\Big) \;=\; \mathrm{Ad}_{F_0^{-1}}\, A_{\theta(\,\frac{d}{dt}|_0 g_t\,)} + \frac{d}{dt}\Big|_0 F_0^{-1} F_t \tag{4}$$

With this ansatz the axiom (2) for principal connections is automatically satisfied, whenever ω is well–defined as a differential from on the quotient $G \times_H \mathbf{GL}\,\Sigma$ or equivalently whenever

$$A_Z \;=\; Z\star \qquad\qquad \mathrm{Ad}_{h_0^{-1}}\, A_X \;=\; A_{\mathrm{Ad}_{h_0^{-1}} X}$$

for all $Z \in \mathfrak{h}$, $X \in \mathfrak{g}$ and $h_0 \in H$. Summarizing this argument we conclude:

Definition 2.3 (Left Invariant Connections).
A left invariant connection on a homogeneous vector bundle $\Sigma(G/H)$ is an H–equivariant extension $A : \mathfrak{g} \longrightarrow \mathrm{End}\,\Sigma$ of the infinitesimal representation $\star : \mathfrak{h} \longrightarrow \mathrm{End}\,\Sigma$ of the Lie algebra $\mathfrak{h}$ associated to the representation of H on the base point fiber $\Sigma := \Sigma_{eH}(G/H)$.

Similar to the classification of left invariant principal connections the set of left invariant connections $A : \mathfrak{g} \longrightarrow \mathrm{End}\,\Sigma$ on $\Sigma(G/H)$ is the preimage of the infinitesimal representation $\star : \mathfrak{h} \longrightarrow \mathrm{End}\,\Sigma$ under $\mathrm{Hom}_H(\mathfrak{g}, \mathrm{End}\,\Sigma) \longrightarrow \mathrm{Hom}_H(\mathfrak{h}, \mathrm{End}\,\Sigma)$ in the long exact sequence:

$$\ldots \longrightarrow \mathrm{Hom}_H(\,\mathfrak{g},\,\mathrm{End}\,\Sigma\,) \longrightarrow$$
$$\mathrm{Hom}_H(\,\mathfrak{h},\,\mathrm{End}\,\Sigma\,) \stackrel{\delta}{\longrightarrow} \mathrm{Ext}^1_H(\,\mathfrak{g}/\mathfrak{h},\,\mathrm{End}\,\Sigma\,) \longrightarrow \ldots$$

In consequence the existence of left invariant connections on $\Sigma(G/H)$ is obstructed by the extension class $\delta(\,\star\,) \in \mathrm{Ext}^1_H(\mathfrak{g}/\mathfrak{h}, \mathrm{End}\,\Sigma)$, its vanishing provides us with an affine space of left invariant connections on $\Sigma(G/H)$ modelled on $\mathrm{Hom}_H(\mathfrak{g}/\mathfrak{h}, \mathrm{End}\,\Sigma)$. Every left invariant principal connection $\omega : \mathfrak{g} \longrightarrow \mathfrak{h}$ induces a left invariant connection $A^\omega : \mathfrak{g} \longrightarrow \mathrm{End}\,\Sigma$ on every homogeneous vector bundle by means of $A^\omega(X) := \omega(X)\star$. Due to this universality every non–vanishing obstruction $\delta(\,\star\,) \neq 0$ for some H–representation Σ is an obstruction against the existence of a left invariant principal connection as well.

Lemma 2.4 (Left Invariant Connections and Rigidity).
If there exists a left invariant connection on the tangent bundle of a homogeneous space G/H, then the kernel $\mathfrak{n} := \mathbf{ker}\,\star$ of the isotropy representation $\star : \mathfrak{h} \longrightarrow \mathrm{End}\,\mathfrak{g}/\mathfrak{h}$, $Z \longmapsto Z\star$, is an ideal not only in the isotropy algebra $\mathfrak{h}$, but already in the full Lie algebra $\mathfrak{g}$.

Proof. According to the preceding discussion a left invariant connection on the tangent bundle can be thought of as an H–equivariant extension $A : \mathfrak{g} \longrightarrow \mathrm{End}\,\mathfrak{g}/\mathfrak{h}$ of the adjoint representation $\star$ of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$. We need to show that $[\,N,\,X\,] \in \mathfrak{n}$ for all $N \in \mathfrak{n}$ and $X \in \mathfrak{g}$ or equivalently $[\,[\,N,\,X\,],\,Y\,] \equiv 0 \mod \mathfrak{h}$ for all $X, Y \in \mathfrak{g}$, because $[\,N,\,X\,] \in \mathfrak{h}$ by the very definition of $\mathfrak{n}$. Since the left invariant connection A extends the infinitesimal representation

$$[\,[\,N,\,X\,],\,Y\,] \;\equiv\; A_{[N,X]}(\,Y + \mathfrak{h}\,) \;\equiv\; [\,N,\,A_X(\,Y + \mathfrak{h}\,)\,] - A_X(\,[\,N,\,Y\,] + \mathfrak{h}\,)$$

modulo $\mathfrak{h}$, where the second congruence is the infinitesimal version of H–equivariance for the left invariant connection A under $N \in \mathfrak{h}$. Evidently the right hand side vanishes, because $[\,N,\,Y\,]\;\in\;\mathfrak{h}$ and $[\,N,\,A_X(Y+\mathfrak{h})\,]\;\in\;\mathfrak{h}$ by assumption. ∎

In general the kernel $\mathfrak{n}$ of the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ fails to be an ideal of $\mathfrak{g}$ and thus obstructs the existence of a left invariant connection on the tangent bundle and in turn a left invariant principal connection on a homogeneous space G/H. Homogeneous spaces G/H admitting a left invariant connection on their tangent bundle are really rather special in the class of all homogeneous spaces, in general G is not a subgroup of some affine group.

Corollary 2.5 (Kernel of the Adjoint Representation).
If a homogeneous space G/H carries a left invariant connection on its tangent bundle, then we may assume without loss of generality that the adjoint representation $\star:\;\mathfrak{h}\longrightarrow\mathrm{End}\;\mathfrak{g}/\mathfrak{h}$ is faithful. Namely under this assumption the kernel $\mathfrak{n}$ of the adjoint representation is the Lie algebra of the maximal normal subgroup N of G contained in H so that we may present G/H alternatively as $(G/N)\;/\;(H/N)$ with faithful adjoint representation $\star:\;\mathfrak{h}/\mathfrak{n}\longrightarrow\mathrm{End}\;\mathfrak{g}/\mathfrak{h}$.

Lemma 2.6 (Curvature and Torsion of a Left Invariant Connection).
The curvature of a left invariant connection $A:\;\mathfrak{g}\longrightarrow\mathrm{End}\;\Sigma$ on a homogeneous vector bundle $\Sigma(G/H)$ is the $\mathrm{End}\;\Sigma$–valued 2–form $R\;\in\;[\,\Lambda^2(\mathfrak{g}/\mathfrak{h})^*\otimes\mathrm{End}\;\Sigma\,]^H$ on $\mathfrak{g}/\mathfrak{h}$ defined by

$$R_{X+\mathfrak{h},Y+\mathfrak{h}}\;\;:=\;\;[\,A_X,\;A_Y\,]\;-\;A_{[\,X,Y\,]}$$

for representatives $X,\,Y\;\in\;\mathfrak{g}$. In the same vein the torsion of a left invariant connection $A\;:\;\mathfrak{g}\longrightarrow\mathrm{End}\;\mathfrak{g}/\mathfrak{h}$ on the tangent bundle $T(G/H)$ of the homogeneous space G/H equals the $\mathfrak{g}/\mathfrak{h}$–valued 2–form $T\;\in\;[\,\Lambda^2(\mathfrak{g}/\mathfrak{h})^*\otimes(\mathfrak{g}/\mathfrak{h})\,]^H$ given on representatives $X,\,Y\;\in\;\mathfrak{g}$ by:

$$T(\,X+\mathfrak{h},Y+\mathfrak{h}\,)\;\;\equiv\;\;A_X(\,Y+\mathfrak{h}\,)\;-\;A_Y(\,X+\mathfrak{h}\,)\;-\;[\,X,\,Y\,]\qquad\mathrm{mod}\;\;\mathfrak{h}$$

In light of the description (4) of the left invariant principal connection on the frame bundle of $\Sigma(G/H)$ associated to an algebraic connection $A:\;\mathfrak{g}\longrightarrow\mathrm{End}\;\Sigma$ the formulas for the curvature and the torsion presented in Lemma 2.6 simply reflect the Maurer–Cartan equation and Cartan's structure equations for the curvature and torsion of a principal connection.

Remark 2.7 (Covariant Derivatives of Left Invariant Sections).
The covariant derivative of a left invariant section $s\;\in\;\Gamma(\,\Sigma(G/H)\,)^G$ of a homogeneous vector bundle $\Sigma(G/H)$ under a left invariant connection $A\;:\;\mathfrak{g}\longrightarrow\mathrm{End}\;\Sigma$ is a left invariant section $\nabla^A s$ of the homogeneous vector bundle $T^*(G/H)\otimes\Sigma(G/H)$. In particular the identification $\Gamma(\,\Sigma(G/H)\,)^G\;\cong\;[\Sigma]^H$ turns the left invariant connection into the linear map

$$A\circledast:\quad[\,\Sigma\,]^H\;\longrightarrow\;[\,(\mathfrak{g}/\mathfrak{h})^*\otimes\Sigma\,]^H,\qquad s\;\longmapsto\;A\circledast s$$

well–defined by $(A\circledast s)_{X+\mathfrak{h}}\;:=\;A_X s$ for $X\;\in\;\mathfrak{g}$, because $A_X s\;=\;X\star s\;=\;0$ for all $X\;\in\;\mathfrak{h}$.

3 Formal Affine Homogeneous Spaces

Abstracting the concept of affine homogeneous spaces into a purely algebraic concept requires us to ignore the global aspects of a homogeneous space G/H considered as a manifold and to focus on the pair $\mathfrak{g} \supset \mathfrak{h}$ of Lie algebras instead endowed with a connection, this is an $\mathfrak{h}$–equivariant extension $A : \mathfrak{g} \longrightarrow \mathrm{End}\ \mathfrak{g}/\mathfrak{h}$ of the infinitesimal isotropy representation $\star : \mathfrak{h} \longrightarrow \mathrm{End}\ \mathfrak{g}/\mathfrak{h}$. Using this connection we construct a skew algebra $(\mathrm{End}\ \mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$ in this section, which contains (a quotient of) the Lie algebra $\mathfrak{g}$ by construction, and proceed to show that this algebra is isomorphic to a skew algebra $(\mathrm{End}\ \mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h})$ constructed using only the curvature and the torsion of the connection. The failure of the Jacobi identity is a characteristic feature of the algebras $(\mathrm{End}\ \mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g} \cong (\mathrm{End}\ \mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h})$.

Definition 3.1 (Formal Affine Homogeneous Spaces).
A formal affine homogeneous space is a pair $\mathfrak{g} \supset \mathfrak{h}$ of Lie algebras endowed with a formal left invariant connection on its isotropy representation, this is an $\mathfrak{h}$–equivariant linear map $A : \mathfrak{g} \longrightarrow \mathrm{End}\ \mathfrak{g}/\mathfrak{h}$ extending the isotropy representation $\star : \mathfrak{h} \longrightarrow \mathrm{End}\ \mathfrak{g}/\mathfrak{h}$. The curvature $R \in [\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes (\mathrm{End}\ \mathfrak{g}/\mathfrak{h})]^{\mathfrak{h}}$ and the torsion $T \in [\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes (\mathfrak{g}/\mathfrak{h})]^{\mathfrak{h}}$ of a formal affine homogeneous space are defined by the same formulas as for an actual homogeneous space:

$$\begin{aligned} R_{X+\mathfrak{h},Y+\mathfrak{h}} &:= [\,A_X,\ A_Y\,] - A_{[X,Y]} \\ T(X+\mathfrak{h}, Y+\mathfrak{h}) &:= A_X(Y+\mathfrak{h}) - A_Y(X+\mathfrak{h}) - [X,Y] + \mathfrak{h} \end{aligned}$$

Evidently every affine homogeneous space G/H endowed with a left invariant connection $A : \mathfrak{g} \longrightarrow \mathrm{End}\ \mathfrak{g}/\mathfrak{h}$ on its tangent bundle $T(G/H)$ defines a formal affine homogeneous spaces $\mathfrak{g} \supset \mathfrak{h}$ with formal connection A. On the other hand we may encounter serious problems in integrating a formal affine homogeneous space to an actual homogeneous space, because the subset $\exp \mathfrak{h} \subset G$ of the simply connected Lie group G with Lie algebra $\mathfrak{g}$ does not in general generate a closed subgroup $H \subset G$, quite simple and beautiful counterexamples in this direction have been constructed by Kowalski [K1].

In the case H fails to be a closed subgroup of G the formal affine homogeneous space $\mathfrak{g} \supset \mathfrak{h}$ does not integrate to a true affine homogeneous space G/H. In order to define some kind of surrogate we may consider the closure $\overline{H} \supset H$ of H in G, which is a Lie subgroup of G with its own Lie algebra $\overline{\mathfrak{h}} \supset \mathfrak{h}$. Since the adjoint action $\star : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$ of the Lie group G on its Lie algebra $\mathfrak{g}$ is continuous, the H–invariant subspace $\mathfrak{h} \subset \mathfrak{g}$ is actually $\overline{H}$–invariant so that $\overline{\mathfrak{h}}/\mathfrak{h}$ is a Lie algebra. The resulting short exact sequence of representations of $\overline{H}$

$$0 \longrightarrow \overline{\mathfrak{h}}/_{\mathfrak{h}} \longrightarrow \mathfrak{g}/_{\mathfrak{h}} \longrightarrow \mathfrak{g}/_{\overline{\mathfrak{h}}} \longrightarrow 0$$

corresponds to a short exact sequence of homogeneous vector bundles on $G/\overline{H}$ involving the tangent bundle $T(G/\overline{H})$ modelled on $\mathfrak{g}/\overline{\mathfrak{h}}$ and the homogeneous Lie algebra bundle modelled on $\overline{\mathfrak{h}}/\mathfrak{h}$. Hence the homogeneous vector bundle on $G/\overline{H}$ modelled on $\mathfrak{g}/\mathfrak{h}$ is a transitive Lie algebroid bundle "$T(G/H)$" endowed with a left invariant connection ∇, whose curvature R and torsion T equal the formal curvature and torsion defined above. In this context we recall that the torsion is actually defined for linear connections on transitive Lie algebroids.

Definition 3.2 (Skew Algebra associated to Connection)**.**
Consider a left invariant, not necessarily torsion free connection $A : \mathfrak{g} \longrightarrow \mathrm{End}\,\mathfrak{g}/\mathfrak{h}$ *on the isotropy representation* $\mathfrak{g}/\mathfrak{h}$ *of a pair* $\mathfrak{g} \supset \mathfrak{h}$ *of Lie algebras. The quotient of* $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus \mathfrak{g}$

$$(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g} \;:=\; (\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus \mathfrak{g} \,\big/ \{\, (-H\star) \oplus H \;|\; H \in \mathfrak{h} \,\}$$

by the diagonal subspace $\mathfrak{h}$ *can be endowed with a skew symmetric bilinear bracket via:*

$$[\,\mathfrak{X} \oplus_{\mathfrak{h},A} X, \mathfrak{Y} \oplus_{\mathfrak{h},A} Y\,] \;:=\; [\,\mathfrak{X}, \mathfrak{Y}\,] \oplus_{\mathfrak{h},A} [\,X, Y\,] + (\,[\,\mathfrak{X}, A_Y\,] - A_{\mathfrak{X}Y}\,) \oplus_{\mathfrak{h},A} \mathfrak{X}Y \\ - (\,[\,\mathfrak{Y}, A_X\,] - A_{\mathfrak{Y}X}\,) \oplus_{\mathfrak{h},A} \mathfrak{Y}X$$

The notation $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$ *reflects the dependence of the resulting skew algebra on* A.

Some thoughts should be spent on the interpretation of the terms $\mathfrak{X}Y$ and $\mathfrak{Y}X$ in the definition of the bracket above, which are used as if the classes $\mathfrak{X}(\,Y + \mathfrak{h}\,)$ and $\mathfrak{Y}(\,X + \mathfrak{h}\,)$ in $\mathfrak{g}/\mathfrak{h}$ were well defined elements of $\mathfrak{g}$. Nevertheless the resulting ambiguities cancel out in the quotient $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$ of $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus \mathfrak{g}$ by the diagonal $\{\,(-H\star) \oplus H \,|\; H \in \mathfrak{h}\,\}$

$$(\,[\,\mathfrak{X}, A_Y\,] - A_{\mathfrak{X}Y}\,) \oplus_{\mathfrak{h},A} \mathfrak{X}Y \;=\; [\,\mathfrak{X}, A_Y\,] \oplus_{\mathfrak{h},A} 0 + (\,-A_{\mathfrak{X}Y}\,) \oplus_{\mathfrak{h},A} \mathfrak{X}Y$$

as long as we take the same representatives $\mathfrak{X}Y$ and $\mathfrak{Y}X$ in $\mathfrak{g}$ for the classes $\mathfrak{X}(Y + \mathfrak{h})$ and $\mathfrak{Y}(X + \mathfrak{h})$ in $\mathfrak{g}/\mathfrak{h}$ on both sides of $\oplus$. With this proviso the bracket of two elements in $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus \mathfrak{g}$ is well defined in the quotient $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$. The bracket will descend to a skew algebra structure on $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$, if all elements representing 0 in the quotient have vanishing brackets with all other elements of $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus \mathfrak{g}$. Observing that the classes $H\star(Y + \mathfrak{h})$ and $\mathfrak{Y}(H + \mathfrak{h})$ for given $H \in \mathfrak{h}$ are represented in $\mathfrak{g}$ by $[H, Y]$ and 0 we find

$$\begin{aligned} &[\,(-H\star) \oplus H, \mathfrak{Y} \oplus Y\,] \\ &\quad= [\,-H\star, \mathfrak{Y}\,] \oplus_{\mathfrak{h},A} [\,H, Y\,] \\ &\qquad + \Big(\,[\,-H\star, A_Y\,] + A_{[H,Y]} - [\,\mathfrak{Y}, A_H\,]\Big) \oplus_{\mathfrak{h},A} \Big(-[\,H, Y\,]\Big) \\ &\quad= -\Big(\,[\,H\star, A_Y\,] - A_{[H,Y]}\Big) \oplus_{\mathfrak{h},A} 0 \end{aligned}$$

using $A_H = H\star$ in the second line. Due to the characteristic infinitesimal $\mathfrak{h}$–equivariance of the left invariant connection $A : \mathfrak{g} \longrightarrow \mathrm{End}\,\mathfrak{g}/\mathfrak{h}$ the right hand side vanishes so that the bracket is in fact well–defined on the quotient $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$.

The inclusion of the first summand $\mathfrak{X} \longmapsto \mathfrak{X} \oplus_{\mathfrak{h},A} 0$ is evidently an injective skew algebra homomorphism turning $\mathrm{End}\,\mathfrak{g}/\mathfrak{h}$ into a Lie subalgebra of $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$. Things are slightly more complicated for the inclusion $X \longmapsto 0 \oplus_{\mathfrak{h},A} X$ of the second summand, which may well fail to be injective. Its kernel however agrees with the kernel of the isotropy representation $\star : \mathfrak{h} \longrightarrow \mathrm{End}\,\mathfrak{g}/\mathfrak{h}$, $Z \longmapsto Z\star$, which in turn agrees with the maximal ideal $\mathfrak{n} \subset \mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{h}$ according to Lemma 2.4. In consequence the skew algebra $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$ is generated as a vector space by its two Lie subalgebras $\mathrm{End}\,\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{n}$.

Definition 3.3 (Skew Algebra associated to Curvature–Torsion)**.**
Consider a left invariant connection $A : \mathfrak{g} \longrightarrow \mathrm{End}\,\mathfrak{g}/\mathfrak{h}$ on the isotropy representation $\mathfrak{g}/\mathfrak{h}$ of a pair $\mathfrak{g} \supset \mathfrak{h}$ of Lie algebras with associated curvature $R \in [\,\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathrm{End}\,\mathfrak{g}/\mathfrak{h}\,]^{\mathfrak{h}}$ and torsion $T \in [\,\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}/\mathfrak{h}\,]^{\mathfrak{h}}$. The direct sum $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{g}/\mathfrak{h})$ of vector spaces is actually a skew algebra denoted by $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h})$ under the following skew bilinear bracket:

$$[\,\mathfrak{X} \oplus_{R,T} x,\ \mathfrak{Y} \oplus_{R,T} y\,] \ := \ \Big(\,[\,\mathfrak{X}, \mathfrak{Y}\,] \ - \ R_{x,y}\Big) \oplus_{R,T} \Big(\,\mathfrak{X}y \ - \ \mathfrak{Y}x \ - \ T(\,x,\,y\,)\,\Big)$$

Up to the additional terms $-R_{x,y}$ and $-T(x,y)$ the skew bracket on $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h})$ agrees with the Lie bracket on the semidirect product $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{g}/\mathfrak{h})$ of the Lie algebra $\mathrm{End}\,\mathfrak{g}/\mathfrak{h}$ with its representation $\mathfrak{g}/\mathfrak{h}$. The possible failure of the Jacobi identity for the bracket on $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h})$ is thus due to these two additional terms, for the moment however we postpone a detailed analysis of this problem to Section 4.

Lemma 3.4 (Skew Algebra Isomorphism)**.**
Let $A : \mathfrak{g} \longrightarrow \mathrm{End}\,(\mathfrak{g}/\mathfrak{h})$ be a left invariant connection on the isotropy representation $\mathfrak{g}/\mathfrak{h}$ of a pair $\mathfrak{g} \supset \mathfrak{h}$ of Lie algebras with curvature $R \in [\,\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes (\mathrm{End}\,\mathfrak{g}/\mathfrak{h})\,]^{\mathfrak{h}}$ and torsion $T \in [\,\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes (\mathfrak{g}/\mathfrak{h})\,]^{\mathfrak{h}}$. The connection A defines a canonical isomorphism of skew algebras

$$\begin{array}{rccc} \Phi_A : & (\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g} & \stackrel{\cong}{\longrightarrow} & (\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h}) \\ & \mathfrak{X} \oplus_{\mathfrak{h},A} X & \longmapsto & (\mathfrak{X} + A_X) \oplus_{R,T} (X + \mathfrak{h}) \end{array}$$

between the skew algebras associated to A and R, T. In consequence the the quotient $\mathfrak{g}/\mathfrak{n}$ of $\mathfrak{g}$ by the maximal ideal $\mathfrak{n} \subset \mathfrak{g}$ contained in $\mathfrak{h}$ becomes a Lie subalgebra of $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h})$:

$$\mathfrak{g}/\mathfrak{n} \stackrel{\subset}{\longrightarrow} (\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h}), \qquad X + \mathfrak{n} \longmapsto A_X \oplus_{R,T} (X + \mathfrak{h})$$

Proof. A short inspection shows that Φ_A is well defined on the quotient $(\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$ due to $A_H = H\star$ for all $H \in \mathfrak{h}$. Moreover Φ_A is a linear isomorphism with well–defined inverse $\Phi_A^{-1} : (\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h}) \longrightarrow (\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{\mathfrak{h},A} \mathfrak{g}$, $\mathfrak{X} \oplus_{R,T} x \longmapsto (\mathfrak{X} - A_X) \oplus_{\mathfrak{h},A} X$, for an arbitrary representative

$X \in \mathfrak{g}$ of the class $x \in \mathfrak{g}/\mathfrak{h}$. Eventually Φ is a homomorphism

$$\begin{aligned}
&\Big[\, \Phi_A(\,\mathfrak{X} \oplus_{\mathfrak{h},A} X\,),\ \Phi_A(\,\mathfrak{Y} \oplus_{\mathfrak{h},A} Y\,)\,\Big] \\
&\quad = \Big(\, [\,\mathfrak{X}, \mathfrak{Y}\,] \,+\, [\,A_X, \mathfrak{Y}\,] \,+\, [\,\mathfrak{X}, A_Y\,] \,+\, [\,A_X, A_Y\,] \,-\, R_{X+\mathfrak{h}, Y+\mathfrak{h}} \Big) \\
&\qquad\quad \oplus_{R,T}\Big(\, (\,\mathfrak{X} + A_X\,)(\,Y + \mathfrak{h}\,) \,-\, (\,\mathfrak{Y} + A_Y\,)(\,X + \mathfrak{h}\,) \,-\, T(X + \mathfrak{h}, Y + \mathfrak{h})\,\Big) \\
&\quad = \Big(\, [\,\mathfrak{X}, \mathfrak{Y}\,] + [\,\mathfrak{X}, A_Y\,] - [\,\mathfrak{Y}, A_X\,] + A_{[X,Y]} \Big) \\
&\qquad\quad \oplus_{R,T}\Big(\, \mathfrak{X}\,Y \,-\, \mathfrak{Y}\,X \,-\, [X,Y] \,+\, \mathfrak{h} \Big) \\
&\quad = \Phi_A\Big(\, (\,[\mathfrak{X},\mathfrak{Y}] + [\mathfrak{X}, A_Y] - A_{\mathfrak{X}\,Y} - [\mathfrak{Y}, A_X] + A_{\mathfrak{Y}\,X}\,) \\
&\qquad\quad \oplus_{\mathfrak{h},A}(\,\mathfrak{X}\,Y - \mathfrak{Y}\,X + [X,Y]\,)\,\Big)
\end{aligned}$$

of algebras, where $\mathfrak{X}\,Y,\ \mathfrak{Y}\,X \in \mathfrak{g}$ are fixed representatives of the corresponding classes. ■

4 A Parametrization of Formal Affine Spaces

In this section we change our point of view away from a fixed formal affine homogeneous space towards a parametrization of such spaces by connection–curvature–torsion triples. In order to compare different affine homogeneous spaces we choose a linear isomorphism or frame $F : V \longrightarrow \mathfrak{g}/\mathfrak{h}$ to pull back curvature and torsion to V and choose a section $\mathrm{rep} : \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}$ to complement the information contained in $R \in \Lambda^2 V^* \otimes \mathrm{End}\, V$ and $T \in \Lambda^2 V^* \otimes V$ by an additional $A \in V^* \otimes \mathrm{End}\, V$ describing the connection. In this section we make the algebraic equations characterizing a triple (A, R, T) arising from a formal affine homogeneous space $\mathfrak{g} \supset \mathfrak{h}$ explicit, moreover we will briefly discuss the relevance of these algebraic equations for the study of locally homogeneous spaces considered for example in [Tr].

For the time being let us fix a finite dimensional vector space V. Augmenting a formal affine homogeneous space $\mathfrak{g} \supset \mathfrak{h}$ of dimension $\dim \mathfrak{g}/\mathfrak{h} = \dim V$ with a linear isomorphism or frame $F : V \longrightarrow \mathfrak{g}/\mathfrak{h}$ allows us to think of curvature and torsion of the left invariant connection $A : \mathfrak{g} \longrightarrow \mathrm{End}\, \mathfrak{g}/\mathfrak{h}$ as elements $R \in \Lambda^2 V^* \otimes \mathrm{End}\, V$ and $T \in \Lambda^2 V^* \otimes V$. The resulting curvature–torsion tuple $(\,R,\ T\,)$ does not describe the original formal affine homogeneous space completely, hence we choose a section $\mathrm{rep} : \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}$ of the canonical projection to $\mathfrak{g}/\mathfrak{h}$ in order to capture the information contained in the connection in a linear map $A \circ \mathrm{rep} : \mathfrak{g}/\mathfrak{h} \longrightarrow \mathrm{End}\, \mathfrak{g}/\mathfrak{h}$, which becomes an element $A \in V^* \otimes \mathrm{End}\, V$ under F:

Definition 4.1 (Connection–Curvature–Torsion Triples).
A connection–curvature–torsion triple on a vector space V is a triple of the form:

$$(\,A,\ R,\ T\,) \ \in\ (\,V^* \otimes \mathrm{End}\ V\,) \times (\,\Lambda^2 V^* \otimes \mathrm{End}\ V\,) \times (\,\Lambda^2 V^* \otimes V\,)$$

Every such a triple (A, R, T) *endows* $\mathrm{End}\,V \oplus_{R,T} V \;:=\; \mathrm{End}\,V \oplus V$ *with the skew bracket:*

$$[\,\mathfrak{X} \oplus_{R,T} x,\ \mathfrak{Y} \oplus_{R,T} y\,] \;:=\; \Big(\,[\,\mathfrak{X},\ \mathfrak{Y}\,] \;-\; R_{x,y}\Big) \oplus_{R,T} \Big(\,\mathfrak{X}\,y \;-\; \mathfrak{Y}\,x \;-\; T(\,x,\,y\,)\,\Big)$$

The connection–curvature–torsion triples (A, R, T) coming from actual formal affine homogeneous spaces $\mathfrak{g} \supset \mathfrak{h}$ augmented by frames $F : V \longrightarrow \mathfrak{g}/\mathfrak{h}$ and sections $\mathrm{rep} : \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}$ are characterized by the fact that the skew algebra $\mathrm{End}\,V \oplus_{R,T} V$ contains a Lie subalgebra

$$\mathfrak{g}/\mathfrak{n} \stackrel{\subset}{\longrightarrow} (\mathrm{End}\,\mathfrak{g}/\mathfrak{h}) \oplus_{R,T} (\mathfrak{g}/\mathfrak{h}) \stackrel{\cong}{\longrightarrow} \mathrm{End}\,V \oplus_{R,T} V$$

isomorphic to $\mathfrak{g}/\mathfrak{n}$ according to Lemma 3.4, which contains the image of the extension of A

$$A^{\mathrm{ext}} : \quad V \longrightarrow \mathrm{End}\,V \oplus_{R,T} V, \qquad x \longmapsto A_x \oplus_{R,T} x$$

and thus projects onto V under the canonical projection $\mathrm{End}\,V \oplus_{R,T} V \longrightarrow V$. Conversely:

Definition 4.2 (Isotropy Algebra and Tautological Connection)**.**
Consider a Lie subalgebra $\mathfrak{g}$ *of the skew algebra* $\mathrm{End}\,V \oplus_{R,T} V$ *associated to a curvature–torsion tuple* $(R, T) \in \Lambda^2 V^* \otimes \mathrm{End}\,V \times \Lambda^2 V^* \otimes V$*, which projects surjectively onto* V *under:*

$$\mathrm{End}\,V \oplus_{R,T} V \longrightarrow V, \qquad \mathfrak{X} \oplus_{R,T} x \longmapsto x$$

Defining the isotropy algebra $\mathfrak{h} := \mathfrak{g} \cap \mathrm{End}\,V$ *as the kernel of this projection we observe* $\mathfrak{g}/\mathfrak{h} \stackrel{\cong}{\longrightarrow} V$ *as* $\mathfrak{h}$*–representations via the projection to* V*. With this proviso the projection*

$$A^{\mathrm{taut}} : \quad \mathrm{End}\,V \oplus_{R,T} V \longrightarrow \mathrm{End}\,V, \qquad \mathfrak{X} \oplus_{R,T} x \longmapsto \mathfrak{X}$$

becomes the tautological left invariant connection $A^{\mathrm{taut}} : \mathfrak{g} \longrightarrow \mathrm{End}\,V$ *on* $V \cong \mathfrak{g}/\mathfrak{h}$*.*

Although all statements about the tautological connection A^{taut} eventually reduce to tautologies, it is slightly confusing to use it directly in calculations. Nevertheless the tautological connection A^{taut} is $\mathfrak{h}$–equivariant to due $[\mathfrak{X} \oplus_{R,T} 0, \mathfrak{Y} \oplus_{R,T} y] = [\mathfrak{X}.\mathfrak{Y}] \oplus_{R,T} \mathfrak{X}\,y$ and its curvature and torsion agree with $R \in \Lambda^2 V^* \otimes \mathrm{End}\,V$ and $T \in \Lambda^2 V^* \otimes V$ respectively. In consequence every Lie subalgebra $\mathfrak{g} \subset \mathrm{End}\,V \oplus_{R,T} V$ projecting surjectively onto V defines a formal affine homogeneous space $\mathfrak{g} \supset \mathfrak{h}$ with tautological frame $F : V \longrightarrow \mathfrak{g}/\mathfrak{h}$ under the tautological left invariant connection $A^{\mathrm{taut}} : \mathfrak{g} \longrightarrow \mathrm{End}\,V$. Motivated by Remark 2.7 we define for every representation Σ of the Lie algebra $\mathrm{End}\,V$ and every $d \in \mathbb{N}_0$ the bilinear operation $\circledast : \bigotimes^d V^* \otimes \mathrm{End}\,V \times \bigotimes^\bullet V^* \otimes \Sigma \longrightarrow \bigotimes^{\bullet+d} V^* \otimes \Sigma$ by setting

$$(\,Q \circledast s\,)(\,x_1, \ldots, x_d; y_1, y_2, \ldots\,) \;:=\; (\,Q_{x_1,\ldots,x_d} \star s\,)(\,y_1, y_2, \ldots\,)$$

where $\star$ denotes the tensor product representation of $\mathrm{End}\,V$ on $\bigotimes^\bullet V^* \otimes \Sigma$.

Definition 4.3 (Formal Covariant Derivatives of R and T).
For a given connection–curvature–torsion triple (A, R, T) on a finite–dimensional vector space V we define the formal iterated covariant derivatives of R and T by setting:

$$\nabla^r T \;:=\; \underbrace{A \circledast (A \circledast (\ldots (A}_{r \text{ times}} \circledast T) \ldots)) \;\in\; \bigotimes^r V^* \otimes (\Lambda^2 V^* \otimes V)$$

$$\nabla^r R \;:=\; \underbrace{A \circledast (A \circledast (\ldots (A}_{r \text{ times}} \circledast R) \ldots)) \;\in\; \bigotimes^r V^* \otimes (\Lambda^2 V^* \otimes \mathrm{End}\; V)$$

Definition 4.4 (Stabilizer Filtration).
Associated to every connection–curvature–torsion triple is the strictly descending filtration

$$\mathrm{End}\; V \;=\; \ldots \;=\; \mathfrak{h}_{-2} \;=\; \mathfrak{h}_{-1} \;\supsetneq\; \mathfrak{h}_0 \;\supsetneq\; \ldots \;\supsetneq\; \mathfrak{h}_{s-1} \;\supsetneq\; \mathfrak{h}_s \;=\; \mathfrak{h}_{s+1} \;=\; \ldots \;=\; \mathfrak{h}_\infty$$

of End V *defined recursively by* $\mathfrak{h}_0 := \mathfrak{stab}\; R \cap \mathfrak{stab}\; T \subset \mathrm{End}\; V$ *and for all* $r \geq 0$*:*

$$\mathfrak{h}_{r+1} \;:=\; \{\; \mathfrak{X} \in \mathfrak{h}_r \;|\; [\, \mathfrak{X}, A_x \,] \equiv A_{\mathfrak{X} x} \mod \mathfrak{h}_r \textit{ for all } x \in V \;\}$$

The minimal $s \geq -1$ *with equality* $\mathfrak{h}_s = \mathfrak{h}_{s+1}$ *is called the Singer invariant of* (A, R, T)*.*

By its recursive definition this filtration is strictly descending in the sense that it becomes stationary at the first equality $\mathfrak{h}_s = \mathfrak{h}_{s+1}$ of successive filtration steps for some $s \geq 0$, in passing we observe that this argument works even for $s = -1$, because the equality $\mathfrak{h}_{-1} = \mathfrak{h}_0$ renders the stipulated congruences $[\mathfrak{X}, A_x] \equiv A_{\mathfrak{X}x}$ modulo End V trivial. A better way to understand the recursive definition of the stabilizer filtration is to observe that for every subalgebra $\hat{\mathfrak{h}} \subset \mathrm{End}\; V$ the vector space $V^* \otimes (\mathrm{End}\; V/\hat{\mathfrak{h}})$ is actually a representation of $\hat{\mathfrak{h}}$:

$$(\, \mathfrak{X} \star A \,)_x \;:=\; \mathfrak{X} \star A_x \;-\; A_{\mathfrak{X} \star x} \;+\; \hat{\mathfrak{h}} \;=\; [\, \mathfrak{X}, A_x \,] \;-\; A_{\mathfrak{X} x} \;+\; \hat{\mathfrak{h}}$$

By induction it thus follows that all subspaces $\mathfrak{h}_r \subset \mathrm{End}\; V$ of the stabilizer filtration are subalgebras, in fact this is true by definition for $\mathfrak{h}_0 := \mathfrak{stab}\; R \cap \mathfrak{stab}\; T$, whereas $\mathfrak{h}_{r+1}$ for $r \geq 0$ is the stabilizer subalgebra of the class $A + V^* \otimes \mathfrak{h}_r$ represented by A in the $\mathfrak{h}_r$–representation $V^* \otimes (\mathrm{End}\; V/\mathfrak{h}_r)$. A remarkable consequence of this observation is the following very neat interpretation of the stabilizer filtration in terms of covariant derivatives:

Lemma 4.5 (Interpretation of the Stabilizer Filtration).
The strictly descending filtration $\mathfrak{h}_\bullet$ *of* End V *associated to a connection–curvature–torsion triple* (A, R, T) *can be interpreted geometrically as the filtration given by the joint stabilizers*

$$\mathfrak{h}_r \;=\; \mathfrak{stab}\,(\, R \oplus \nabla R \oplus \ldots \oplus \nabla^r R \,) \;\cap\; \mathfrak{stab}\,(\, T \oplus \nabla T \oplus \ldots \oplus \nabla^r T \,)$$

of the iterated covariant derivatives of curvature R *and torsion* T *up to order* $r \geq 0$*.*

Actually this Lemma is a special case of a more general fact, for every representation Σ of the Lie algebra End V and every $s \in \Sigma$ with stabilizer $\mathfrak{stab}\; s \subset \mathrm{End}\; V$ the linear map

$$V^* \otimes (\, \mathrm{End}\; V \,/_{\mathfrak{stab}\; s} \,) \;\longrightarrow\; V^* \otimes \Sigma, \qquad A \;\longmapsto\; A \circledast s$$

defined by $(A \circledast s)_x := A_x \star s$ is well–defined, injective and equivariant under $\mathfrak{stab}\ s$. Since the bilinear operation $\circledast$ is naturally defined and thus equivariant under End V in the sense

$$\mathfrak{X} \star (\, A \circledast s\,) \;=\; (\, \mathfrak{X} \star A\,) \circledast s \,+\, A \circledast (\, \mathfrak{X} \star s\,)$$

for $\mathfrak{X} \in$ End V we find that $\mathfrak{X} \in \mathfrak{stab}\ s$ stabilizes $A \in V^* \otimes (\mathrm{End}\ V/\mathfrak{stab}\ s)$, if and only if:

$$(\, \mathfrak{X} \star A\,) \circledast s \;=0\;=\; \mathfrak{X} \star (\, A \circledast s\,)$$

Using this general argument the statement of the Lemma follows by an easy induction based on the definition $\mathfrak{h}_0 := \mathfrak{stab}\ R \cap \mathfrak{stab}\ T$ as well as on the recursive definition of $\mathfrak{h}_{r+1}$ as the stabilizer of the connection class $A + V^* \otimes \mathfrak{h}_r \in V^* \otimes (\mathrm{End}\ V/\mathfrak{h}_r)$.

For a little interlude in our algebraic considerations let us now recall the twisted exterior derivative of differential forms associated to a connection ∇ on a vector bundle ΣM over a smooth manifold M. The twisted exterior derivative d^∇ is a first order differential operator

$$d^\nabla : \quad \Gamma(\, \Lambda^\bullet T^*M \otimes \Sigma M\,) \;\longrightarrow\; \Gamma(\, \Lambda^{\bullet+1}T^*M \otimes \Sigma M\,), \qquad \omega \;\longmapsto\; d^\nabla \omega$$

on the differential forms on M with values in ΣM. Although the twisted exterior derivative is defined solely in terms of the connection ∇, a standard formula for d^∇ reads

$$\begin{aligned}(\, d^\nabla \omega\,)(\, X_0, \ldots, X_r\,) = \sum_{\mu=0}^{r} (-1)^\mu \Big(\, (\nabla, \nabla^{\mathrm{aux}})_{X_\mu}\omega \,\Big)(\, X_0, \ldots, \widehat{X}_\mu, \ldots, X_r\,) \\ + \sum_{0 \le \mu < \nu \le r} (-1)^{\mu+\nu-1}\, \omega\Big(\, T^{\mathrm{aux}}(X_\mu, X_\nu), X_0, \ldots, \widehat{X}_\mu, \ldots, \widehat{X}_\nu, \ldots, X_r \Big)\end{aligned}$$

for an arbitrary auxiliary connection ∇^{aux} on TM of torsion $T^{\mathrm{aux}} \in \Gamma(\, \Lambda^2 T^*M \otimes TM\,)$. In terms of twisted exterior derivatives the First and Second Bianchi Identity can be written

$$d^{\nabla^{\mathrm{aux}}} T^{\mathrm{aux}} \;=\; R^{\mathrm{aux}} \wedge \mathrm{id} \qquad\qquad d^\nabla R \;=\; 0 \tag{5}$$

respectively, where R and R^{aux} denote the curvatures of ∇ and ∇^{aux}, whereas:

$$(\, R^{\mathrm{aux}} \wedge \mathrm{id}\,)(\, X, Y, Z\,) \;:=\; R^{\mathrm{aux}}_{X,Y}Z \,+\, R^{\mathrm{aux}}_{Y,Z}X \,+\, R^{\mathrm{aux}}_{Z,X}Y$$

Coming back to formal affine homogeneous spaces and connection–curvature–torsion triples (A, R, T) we take the formula expressing d^∇ in terms of the connection $(\nabla, \nabla^{\mathrm{aux}})$ and the torsion T^{aux} as a lead to define the twisted exterior derivative $d^{(A,T)}$ by

$$\begin{aligned}(\, d^{(A,T)}\omega\,)(\, x_0, \ldots, x_r\,) = \sum_{\mu=0}^{r} (-1)^\mu \Big(\, A_{x_\mu} \star \omega \Big)(\, x_0, \ldots, \widehat{x}_\mu, \ldots, x_r\,) \\ + \sum_{0 \le \mu < \nu \le r} (-1)^{\mu+\nu-1} \omega\Big(\, T(x_\mu, x_\nu),\, x_0, \ldots, \widehat{x}_\mu, \ldots, \widehat{x}_\nu, \ldots, x_r\,\Big)\end{aligned}$$

for every r–form $\omega \in \Lambda^r V^* \otimes \Sigma$ with values in a representation Σ of the Lie algebra End V. Specifically for $R \in \Lambda^2 V^* \otimes \mathrm{End}\, V$ the twisted exterior derivative reads

$$(d^{(A,T)} R)_{x,y,z} \;=\; \Big((A_x \star R)_{y,z} + R_{T(y,z),\,x} \Big) + \text{cyclic permutations of } x,\, y,\, z$$

and an almost identical formula is valid for $d^{(A,T)}T \in \Lambda^3 V^* \otimes V$. Last but not least we define the V–valued 3–form $R \wedge \mathrm{id} \in \Lambda^3 V^* \otimes V$ by $(R \wedge \mathrm{id})(x,y,z) := R_{x,y}z + R_{y,z}x + R_{z,x}y$.

Definition 4.6 (Approximate Curvature).
The approximate curvature tensor of a connection–curvature–torsion triple (A, R, T) *on a vector space* V *is defined as an* End V*–valued* 2*–form* $Q(A,T) \in \Lambda^2 V^* \otimes \mathrm{End}\, V$ *on* V *by:*

$$Q(\, A,\, T\,)_{x,y} \;:=\; [\, A_x,\, A_y\,] \;-\; A_{A_x y \,-\, A_y x \,-\, T(x,y)}$$

In order to study the failure of the Jacobi identity for the bracket of the skew algebra End $V \oplus_{R,T} V$ associated to a connection–curvature–torsion triple (A, R, T) we consider the trilinear standard Jacobiator defined as an alternating 3–form **Jac** on End $V \oplus_{R,T} V$ by:

$$\begin{aligned}\mathbf{Jac}(\, \mathfrak{X} \oplus_{R,T} x,\, \mathfrak{Y} \oplus_{R,T},\, \mathfrak{Z} \oplus_{R,T} z\,) \;:=\; &+ [\, \mathfrak{X} \oplus_{R,T} x,\, [\, \mathfrak{Y} \oplus_{R,T} y,\, \mathfrak{Z} \oplus_{R,T} z\,]\,] \\ &+ [\, \mathfrak{Y} \oplus_{R,T} y,\, [\, \mathfrak{Z} \oplus_{R,T} z,\, \mathfrak{X} \oplus_{R,T} x\,]\,] \\ &+ [\, \mathfrak{Z} \oplus_{R,T} z,\, [\, \mathfrak{X} \oplus_{R,T} x,\, \mathfrak{Y} \oplus_{R,T} y\,]\,]\end{aligned}$$

Observing that the bracket with $\mathfrak{X} \in \mathrm{End}\, V$ reproduces the infinitesimal representation

$$[\, \mathfrak{X} \oplus_{R,T} 0,\, \mathfrak{Y} \oplus_{R,T} y\,] \;=\; [\, \mathfrak{X},\, \mathfrak{Y}\,] \oplus_{R,T} \mathfrak{X} y \;=\; \mathfrak{X} \star \Big(\mathfrak{Y} \oplus_{R,T} y \Big)$$

of End V on End $V \oplus_{R,T} V$ we may calculate the Jacobiator for the special choice

$$\begin{aligned}&\mathbf{Jac}(\, \mathfrak{X} \oplus_{R,T} 0,\, 0 \oplus_{R,T} y,\, 0 \oplus_{R,T} z\,) \\ &\quad= [\, \mathfrak{X} \oplus_{R,T} 0,\, (\, - R_{x,y}\,) \oplus_{R,T} (\, - T(\, x,\, y\,)\,)\,] \\ &\qquad - [\, 0 \oplus_{R,T} y,\, 0 \oplus_{R,T} \mathfrak{X} z\,] + [\, 0 \oplus_{R,T} z,\, 0 \oplus_{R,T} \mathfrak{X} y\,] \\ &\quad= (\, - [\, \mathfrak{X},\, R_{y,z}\,] + R_{\mathfrak{X} y,\, z} + R_{y,\, \mathfrak{X} z}\,) \\ &\qquad \oplus_{R,T} (\, - \mathfrak{X}\, T(\, y,\, z\,) + T(\, \mathfrak{X} y,\, z\,) + T(\, y,\, \mathfrak{X} z\,)\,) \\ &\quad= (\, - (\, \mathfrak{X} \star R\,)_{y,z}\,) \oplus_{R,T} (\, - (\, \mathfrak{X} \star T\,)(\, y,\, z\,)\,)\end{aligned}$$

of arguments $\mathfrak{X} \in \mathrm{End}\, V$ and $y,\, z \in V$. Similarly we obtain for all three arguments in V:

$$\begin{aligned}&\mathbf{Jac}(\, 0 \oplus_{R,T} x,\, 0 \oplus_{R,T} y,\, 0 \oplus_{R,T} z\,) \\ &= [\, 0 \oplus_{R,T} x,\, (\, - R_{y,z}\,) \oplus_{R,T} (\, - T(\, y,\, z\,)\,)\,] + \text{cyclic permutations of } x,\, y,\, z \\ &= R_{x,\, T(\, y,\, z\,)} \oplus_{R,T} (\, R_{y,z} x + T(\, x,\, T(\, y,\, z\,)\,)\,) + \text{cyclic permutations of } x,\, y,\, z \\ &= \Big(- d^{(\,0,\,T\,)} R \Big)_{x,y,z} \oplus_{R,T} \Big(R \wedge \mathrm{id} - d^{(\,0,\,T\,)} T \Big)(\, x,\, y,\, z\,)\end{aligned}$$

The latter two results feature prominently in the proof of the following lemma:

Lemma 4.7 (Lie Subalgebras of End $V \oplus_{R,T} V$).
The skew algebra End $V \oplus_{R,T} V$ *associated to a connection–curvature–torsion triple* (A, R, T) *on a vector space* V *allows no Lie subalgebra* $\mathfrak{g} \supset \{ A_x \oplus_{R,T} x \mid x \in V \}$ *unless the connection–curvature–torsion triple* (A, R, T) *satisfies the First and Second Bianchi Identity:*

$$d^{(A,T)}T = R \wedge \mathrm{id} \qquad\qquad d^{(A,T)}R = 0$$

In case the First and Second Bianchi Identity are both satisfied the isotropy algebra association $\mathfrak{g} \longmapsto \mathfrak{g} \cap \mathrm{End}\, V$ *induces a bijection between the Lie subalgebras* $\mathfrak{g} \subset \mathrm{End}\, V \oplus_{R,T} V$ *containing* $\{ A_x \oplus_{R,T} x \,|\, x \in V \}$ *and Lie subalgebras* $\mathfrak{h} \subset \mathrm{End}\, V$ *satisfying*

$$\begin{aligned} \mathfrak{h} &\subset \mathfrak{stab}\, R \cap \mathfrak{stab}\, T \\ \mathfrak{h} \star A &\subset V^* \otimes \mathfrak{h} \\ Q(A, T) &\equiv R \mod \Lambda^2 V^* \otimes \mathfrak{h} \end{aligned}$$

where $\mathfrak{h} \star A \subset V^* \otimes \mathfrak{h}$ *is a shorthand for* $[\mathfrak{X}, A_x] - A_{\mathfrak{X}x} \equiv 0$ *modulo* $\mathfrak{h}$ *for* $x \in V$, $\mathfrak{X} \in \mathfrak{h}$.

Proof. Every Lie subalgebra $\mathfrak{g} \subset \mathrm{End}\, V \oplus_{R,T} V$ containing $\{ A_x \oplus_{R,T} x \mid x \in V \}$ is certainly determined by its isotropy algebra $\mathfrak{h} = \mathfrak{g} \cap \mathrm{End}\, V$. Conversely suppose that $\mathfrak{h} \subset \mathrm{End}\, V$ is the isotropy algebra of the Lie subalgebra $\mathfrak{g}$ of End $V \oplus_{R,T} V$ given by:

$$\mathfrak{g} := \mathfrak{h} + \mathrm{span}\{ A_x \oplus_{R,T} x \mid x \in V \} \tag{6}$$

In particular then $\mathfrak{g}$ is closed under the skew bracket on End $V \oplus_{R,T} V$ so that

$$[\mathfrak{X} \oplus_{R,T} 0,\ A_x \oplus_{R,T} x] = \Big([\mathfrak{X}, A_x] - A_{\mathfrak{X}x} \Big) \oplus_{R,T} 0 + A_{\mathfrak{X}x} \oplus_{R,T} \mathfrak{X}x$$

is necessarily an element of $\mathfrak{g}$ proving the congruence $[\mathfrak{X}, A_x] - A_{\mathfrak{X}x} \equiv 0$ modulo $\mathfrak{h}$ for all $\mathfrak{X} \in \mathfrak{h}$ and $x \in V$. Similarly the result of the following calculation is an element of $\mathfrak{g}$

$$\begin{aligned} [A_x \oplus_{R,T} x,\ A_y \oplus_{R,T} y] = &\Big([A_x, A_y] - R_{x,y} - A_{A_x y - A_y x - T(x,y)} \Big) \oplus_{R,T} 0 \\ &+ A_{A_x y - A_y x - T(x,y)} \oplus_{R,T} \Big(A_x y - A_y x - T(x,y) \Big) \end{aligned}$$

and thus requires $Q(A,T)_{x,y} \equiv R_{x,y}$ modulo $\mathfrak{h}$ for all $x, y \in V$. In passing we remark that $[\mathfrak{X} \oplus_{R,T} 0, \mathfrak{Y} \oplus_{R,T} 0] = [\mathfrak{X}, \mathfrak{Y}] \oplus_{R,T} 0$ lies in $\mathfrak{g}$ for $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{h}$ without further ado.

In consequence the subspace $\mathfrak{g}$ of equation (6) is a skew subalgebra of End $V \oplus_{R,T} V$ as soon as $\mathfrak{h} \star A \in V^* \otimes \mathfrak{h}$ and $Q(A,T) - R \in \Lambda^2 V^* \otimes \mathfrak{h}$. On the other hand the skew bracket on End $V \oplus_{R,T} V$ agrees with the Lie bracket of the semidirect product End $V \oplus V$ of End V with its representation V up to terms quadratic in V. Hence the Jacobiator on every skew subalgebra $\mathfrak{g} \subset \mathrm{End}\, V \oplus_{R,T} V$ vanishes automatically for two or three arguments in the isotropy subalgebra $\mathfrak{h} \subset \mathrm{End}\, V$. In light of this observation the Jacobiator satisfies

$$\begin{aligned}\mathbf{Jac}(\, \mathfrak{X} \oplus_{R,T} 0,\ A_y \oplus_{R,T} y,\ A_z \oplus_{R,T} z\,) &= \mathbf{Jac}(\, \mathfrak{X} \oplus_{R,T} 0,\ 0 \oplus_{R,T} y,\ 0 \oplus_{R,T} z\,) \\ &= -\Big(\, (\,\mathfrak{X} \star R\,)_{y,z} \oplus_{R,T} (\,\mathfrak{X} \star T\,)(y,z)\,\Big)\end{aligned}$$

for all $X \in \mathfrak{h}$ and $y, z \in V$, thus the isotropy algebra of a Lie subalgebra $\mathfrak{g} \subset \mathrm{End}\, V \oplus_{R,T} V$ projecting onto V is necessarily a subalgebra $\mathfrak{h} \subset \mathfrak{stab}\, R \cap \mathfrak{stab}\, T$ of the joint stabilizer of R and T in $\mathrm{End}\, V$. Eventually we calculate along a similar line of argument

$$\begin{aligned}&\mathbf{Jac}(\, A_x \oplus_{R,T} x,\ A_y \oplus_{R,T} y,\ A_z \oplus_{R,T} z\,) \\ &\quad= +\,\mathbf{Jac}(\, A_x \oplus_{R,T} 0,\ 0 \oplus_{R,T} y,\ 0 \oplus_{R,T} z\,) \\ &\qquad +\,\mathbf{Jac}(\, A_y \oplus_{R,T} 0,\ 0 \oplus_{R,T} z,\ 0 \oplus_{R,T} x\,) \\ &\qquad +\,\mathbf{Jac}(\, A_z \oplus_{R,T} 0,\ 0 \oplus_{R,T} x,\ 0 \oplus_{R,T} y\,) \\ &\qquad +\,\mathbf{Jac}(\ \ 0\ \ \oplus_{R,T} x,\ 0 \oplus_{R,T} y,\ 0 \oplus_{R,T} z\,) \\ &\quad= \Big(\, -\, d^{(\,A,T\,)}R\,\Big)_{x,y,z} \oplus_{R,T} \Big(\, R \wedge \mathrm{id}\, -\, d^{(\,A,T\,)}T\,\Big)(\,x,y,z\,)\end{aligned}$$

using trilinearity, cyclic invariance and the explicit formulas calculated above for **Jac**. ■

Although the preceding lemma is reasonably explicit, it is certainly not satisfactory in that we would prefer conditions on the triple (A, R, T) alone, which guarantee the existence of a Lie subalgebra $\mathfrak{g}$ of $\mathrm{End}\, V \oplus_{R,T} V$ satisfying $\mathfrak{g} \supset \{\ A_x \oplus_{R,T} x \mid\ x \in V\ \}$. In order to achieve such a reformulation of Lemma 4.7 let us have another look at the stabilizer filtration

$$\mathrm{End}\, V \;=\; \ldots \;=\; \mathfrak{h}_{-1} \supsetneq \mathfrak{h}_0 \supsetneq \ldots \supsetneq \mathfrak{h}_{s-1} \supsetneq \mathfrak{h}_s \;=\; \mathfrak{h}_{s+1} \;=\; \ldots \;=\; \mathfrak{h}_\infty \supseteq \mathfrak{h} \tag{7}$$

constructed in Definition 4.4. Evidently the connection component $A \in V^* \otimes \mathrm{End}\, V$ of the triple (A, R, T) allows us to define for every subalgebra $\hat{\mathfrak{h}} \subset \mathrm{End}\, V$ the derived subalgebra:

$$\hat{\mathfrak{h}}' \;:=\; \{\, \mathfrak{X} \in \hat{\mathfrak{h}} \mid [\,\mathfrak{X}, A_x\,] \equiv A_{\mathfrak{X}x} \bmod \hat{\mathfrak{h}} \text{ for all } x \in V\ \}$$

For every subalgebra $\hat{\mathfrak{h}} \subset \mathrm{End}\, V$ the quotient $\mathrm{End}\, V/\hat{\mathfrak{h}}$ is naturally a representation of $\hat{\mathfrak{h}}$ and the derived subalgebra $\hat{\mathfrak{h}}'$ is nothing else but the stabilizer of the class $A + V^* \otimes \hat{\mathfrak{h}}$ represented by A in $V^* \otimes (\mathrm{End}\, V/\hat{\mathfrak{h}})$. In particular the derived subalgebra is monotone

$$\hat{\mathfrak{h}}_{\mathrm{small}} \subset \hat{\mathfrak{h}}_{\mathrm{large}} \qquad\Longrightarrow\qquad \hat{\mathfrak{h}}'_{\mathrm{small}} \subset \hat{\mathfrak{h}}'_{\mathrm{large}}$$

because the canonical projection $V^* \otimes (\mathrm{End}\, V/\hat{\mathfrak{h}}_{\mathrm{small}}) \longrightarrow V^* \otimes (\mathrm{End}\, V/\hat{\mathfrak{h}}_{\mathrm{large}})$ is equivariant under every subalgebra $\hat{\mathfrak{h}}_{\mathrm{small}} \subset \hat{\mathfrak{h}}_{\mathrm{large}}$ so that $\mathfrak{X} \in \hat{\mathfrak{h}}_{\mathrm{small}}$ stabilizing the class represented by A in $V^* \otimes (\mathrm{End}\, V/\hat{\mathfrak{h}}_{\mathrm{small}})$ still stabilizes its image in $V^* \otimes (\mathrm{End}\, V/\hat{\mathfrak{h}}_{\mathrm{large}})$.

Thinking of the derived subalgebra construction as a dynamical system $\hat{\mathfrak{h}} \longmapsto \hat{\mathfrak{h}}'$ on the set of subalgebras $\hat{\mathfrak{h}} \subset \mathrm{End}\, V$ we observe that Lemma 4.7 is actually asking for the isotropy algebra $\mathfrak{h} = \mathfrak{g} \cap \mathrm{End}\, V$ of a Lie subalgebra $\mathfrak{g} \subset \mathrm{End}\, V \oplus_{R,T} V$ to be a fixed point $\mathfrak{h} = \mathfrak{h}'$ contained in $\mathfrak{stab}\, R \cap \mathfrak{stab}\, T$ while at the same time containing all the values of $Q(A,T) - R \in \Lambda^2 V^* \otimes \mathfrak{h}$. The latter condition however becomes ever the more restrictive the smaller is $\mathfrak{h}$, hence the best we can hope for is that the unique maximal fixed point $\mathfrak{h}_{\max}$ of the dynamical system $\hat{\mathfrak{h}} \longmapsto \hat{\mathfrak{h}}'$ contained in $\mathfrak{stab}\, R \cap \mathfrak{stab}\, T$ is a sufficiently large subalgebra of End V to satisfy $Q(A,T) - R \in \Lambda^2 V^* \otimes \mathfrak{h}_{\max}$.

The existence of this unique maximal fixed point $\mathfrak{h}_{\max} \subset \mathfrak{stab}\, R \cap \mathfrak{stab}\, T$ is guaranteed simply by the monotonicity of the dynamical system $\hat{\mathfrak{h}} \longmapsto \hat{\mathfrak{h}}'$, to wit $\mathfrak{h}_{\max} = \mathfrak{h}_\infty$ is necessarily equal to the limit of the filtration sequence (7) starting in $\mathfrak{h}_0 := \mathfrak{stab}\, R \cap \mathfrak{stab}\, T$ and iterating $\mathfrak{h}_{r+1} := \mathfrak{h}_r'$ the derived subalgebra construction for $r \geq 0$. Depending on the curvature–torsion tuple (R,T) this unique maximal fixed point $\mathfrak{h}_{\max}$ may or may not satisfy $Q(A,T) - R \in \Lambda^2 V^* \otimes \mathfrak{h}_{\max}$. Provided the maximal fixed point $\mathfrak{h}_{\max}$ of the dynamical system $\hat{\mathfrak{h}} \longmapsto \hat{\mathfrak{h}}'$ contained in $\mathfrak{stab}\, R \cap \mathfrak{stab}\, T$ satisfies $Q(A,T) - R \in \Lambda^2 V^* \otimes \mathfrak{h}_{\max}$ there may of course exist other fixed points $\mathfrak{h} \subset \mathfrak{h}_{\max}$ still satisfying $Q(A,T) - R \in \Lambda^2 V^* \otimes \mathfrak{h}$. In geometric terms these additional fixed points correspond to subgroups $G \subset G_{\max}$ still acting transitively on the affine homogeneous space $G_{\max}/H_{\max}$.

In any case the unique maximal fixed point $\mathfrak{h}_{\max} = \mathfrak{h}_\infty$ contained in $\mathfrak{stab}\, R \cap \mathfrak{stab}\, T$ agrees with the joint stabilizer of all iterated covariant derivatives $\nabla^r T$ and $\nabla^r R$ for all $r \geq 0$ together according to Lemma 4.5. In consequence the decisive congruence $Q(A,T)_{x,y} \equiv R_{x,y}$ modulo $\mathfrak{h}_{\max}$ is equivalent to the following set of algebraic equations:

Theorem 4.8 (Algebraic Variety of Affine Homogeneous Spaces)**.**
For a given connection–curvature–torsion triple $(A,\, R,\, T)$ on a vector space V there exists a Lie subalgebra $\mathfrak{g} \subset \mathrm{End}\, V \oplus_{R,T} V$ of the skew algebra associated to (R,T) satisfying

$$\mathfrak{g} \;\supset\; \mathbf{im}\Big(\, A^{\mathrm{ext}} :\; V \;\longrightarrow\; \mathrm{End}\; V \oplus_{R,T} V, \qquad x \;\longmapsto\; A_x \oplus_{R,T} x \,\Big)$$

if and only if (A,R,T) satisfies the first and second Bianchi identities of degrees 2 *and* 3

$$d^{(A,T)}T \;=\; R \wedge \mathrm{id} \qquad\qquad d^{(A,T)}R \;=\; 0$$

as well as the following homogeneous equations of degrees $r+3$ and $r+4$ for all $r \geq 0$:

$$\Big(\, Q(A,T) \;-\; R \,\Big) \circledast \Big(\, \underbrace{A \circledast (\, A \circledast (\, \ldots (\, A}_{r\ \text{times}} \circledast T\,) \ldots)\,)\,\Big) \;=\; 0$$

$$\Big(\, Q(A,T) \;-\; R \,\Big) \circledast \Big(\, \underbrace{A \circledast (\, A \circledast (\, \ldots (\, A}_{r\ \text{times}} \circledast R\,) \ldots)\,)\,\Big) \;=\; 0$$

The set of all solution triples (A,R,T) to these algebraic equations will be denoted $\mathfrak{M}(\,\mathfrak{gl}\, V\,)$.

Although the algebraic variety $\mathfrak{M}(\mathfrak{gl}\,V)$ is formally defined by an infinite set of algebraic equations, only a finite number of these equations can actually be relevant, after all polynomial rings are noetherian rings. Taking the argument leading to these equations into account we observe that at least the equations parametrized by $r \geq (\dim V)^2$ have to be *algebraic* consequences of the equations up to $(\dim V)^2$. In this observation $(\dim V)^2$ enters simply as a trivial upper bound for the maximal length s of a sequence $\hat{\mathfrak{h}}_0 \supsetneq \hat{\mathfrak{h}}_1 \supsetneq \ldots \supsetneq \hat{\mathfrak{h}}_s$ of subalgebras of End V with $\hat{\mathfrak{h}}_{r+1} = \hat{\mathfrak{h}}'_r$ for some $A \in V^* \otimes \mathrm{End}\,V$, which probably grows more like $\dim V$ than $(\dim V)^2$. On the other hand the family of examples constructed in Section 6 shows that there is no universal bound independent of $\dim V$ for the number of algebraic equations needed in order to define $\mathfrak{M}(\mathfrak{gl}\,V)$.

Somewhat more interesting from the theoretical point of view is the observation that the algebraic variety $\mathfrak{M}(\mathfrak{gl}\,V)$ is actually a cone over a projective algebraic variety due to the homogeneity of the algebraic equations defining $\mathfrak{M}(\mathfrak{gl}\,V)$. More precisely the group $\mathbb{R}$ acts via $\lambda \star (A, R, T) := (e^\lambda A, e^{2\lambda} R, e^\lambda T)$ on the set of all connection–curvature–torsion triples and thus on the invariant subset $\mathfrak{M}(\mathfrak{gl}\,V) \subset V^* \otimes \mathrm{End}\,V \times \Lambda^2 V^* \otimes \mathrm{End}\,V \times \Lambda^2 V^* \otimes V$. The same observation applies to the algebraic varieties $\mathfrak{M}^{\mathrm{tf}}(\mathfrak{gl}\,V)$ and $\mathfrak{M}^{\mathrm{ref}}(\mathfrak{gl}\,V)$ of torsion free and reductive connection–curvature–torsion triples (A, R, T) defined in:

Corollary 4.9 (Torsion Free Affine Homogeneous Spaces).
Torsion free connection–curvature tuples (A, R) are connection–curvature–torsion triples with vanishing torsion $T = 0$. Their algebras End $V \oplus_{R,0} V$ *allow Lie subalgebras* $\mathfrak{g}$ *with*

$$\mathrm{End}\,V \oplus_{R,0} V \;\supset\; \mathfrak{g} \;\supset\; \mathbf{im}\Big(A^{\mathrm{ext}}:\; V \longrightarrow \mathrm{End}\,V \oplus_{R,0} V, \quad x \longmapsto A_x \oplus_{R,0} x\Big)$$

if and only if the curvature $R \in \Lambda^2 V^* \otimes \mathrm{End}\,V$ and the connection $A \in V^* \otimes \mathrm{End}\,V$ satisfy the formally infinite system of algebraic equations of degrees 2, 3 *and* $r+4$, $r \geq 0$*:*

$$R \wedge \mathrm{id} = 0 \qquad d^{(A,0)}R = 0$$
$$\Big(Q(A,0) - R\Big) \circledast \Big(\underbrace{A \circledast (A \circledast (\ldots (A}_{r\ \text{times}} \circledast R)\ldots))\Big) = 0$$

The set of all solutions (A, R) to these equations will be denoted by $\mathfrak{M}^{\mathrm{tf}}(\mathfrak{gl}\,V) \subset \mathfrak{M}(\mathfrak{gl}\,V)$.

Corollary 4.10 (Reductive Affine Homogeneous Spaces).
Reductive curvature–torsion tuples (R, T) are connection–curvature–torsion triples with vanishing connection $A = 0$. The skew algebra End $V \oplus_{R,T} V$ *allows Lie subalgebras* $\mathfrak{g} \supset V$ *extending V, if and only if the curvature $R \in \Lambda^2 V^* \otimes \mathrm{End}\,V$ and torsion $T \in \Lambda^2 V^* \otimes V$ satisfy the following system of algebraic equations of degrees* 2, 3, 3 *and* 4*:*

$$d^{(0,T)}T = R \wedge \mathrm{id} \qquad d^{(0,T)}R = 0 \qquad R \circledast T = 0 \qquad R \circledast R = 0$$

The notation $\mathfrak{M}^{\mathrm{red}}(\mathfrak{gl}\,V) \subset \mathfrak{M}(\mathfrak{gl}\,V)$ will refer to the set of solutions to these equations.

5 Additional Parallel Geometric Structures

In this short intermediate section we want to discuss variations of the algebraic varieties $\mathfrak{M}(\,\mathfrak{gl}\,V\,)$ parametrizing formal affine homogeneous spaces, which take into account additional parallel geometric structures like Riemannian metrics and almost complex structures. Because the affine geometry determined by the existence of a linear connection of the tangent bundle is intrinsically a first order geometry, the condition of parallelity severely restricts the order of additional geometric structures we may consider, in any case we will restrict ourselves to a discussion of first order geometries usually called G–structures. In order to avoid a clash of notation we prefer to call them K–structures for the time being.

In order to define K–structures on vector spaces and consequently on smooth manifolds M we fix once and for all a closed subgroup $K \subset \mathbf{GL}\,V$ of the general linear group of a vector space V called the model space. By definition a K–structure on a vector space T of the same dimension as V is an equivalence class Ω of linear isomorphisms $F:\;V \longrightarrow T$ under

$$F \;\sim_K\; \tilde{F} \qquad\Longleftrightarrow\qquad F^{-1}\circ\tilde{F} \;\in\; K$$

conversely the linear isomorphisms $F\;:\;V\;\longrightarrow\;T$ representing Ω are called K–frames. The general linear group $\mathbf{GL}\,V$ acts simply transitively from the right on the set $\mathrm{Frame}(\,V,\,T\,)$ of linear isomorphisms $V \longrightarrow T$, thus K–structures Ω correspond bijectively to points in:

$$\Omega \;\in\; \mathrm{Frame}(\,V,\,T\,)/_{K}$$

With K being by assumption a closed subgroup of $\mathbf{GL}\,V$ the quotient $\mathrm{Frame}(\,V,\,T\,)/K$ is actually a manifold so that we can define a smooth K–structure on a manifold M of the same dimension as V as a smooth section of the quotient bundle

$$\Omega \;\in\; \Gamma(\,\mathrm{Frame}(\,V,\,TM\,)/_{K}\,)$$

where $\mathrm{Frame}(\,V,\,TM\,)$ is the principal $\mathbf{GL}\,V$–bundle of tangent bundle frames defined in analogy with (3). Such a smooth K–structure Ω is parallel for an affine connection ∇ on TM, if and only if the parallel transport $\mathrm{PT}^{\nabla}_{\gamma}\;:\;T_{\gamma(0)}M\;\longrightarrow\;T_{\gamma(1)}M$ along curves $\gamma\,:\,[0,1]\longrightarrow M$ maps K–frames to K–frames in the sense $\mathrm{PT}^{\nabla}_{\gamma}\circ F \in \Omega_{\gamma(1)}$ for every $F \in \Omega_{\gamma(0)}$.

Whereas the preceding definition of parallel K–structures on a manifold M for a closed subgroup $K \subset \mathbf{GL}\,V$ does not comprise the most general geometric structures, nevertheless there are quite a number of interesting examples. Consider for example $V = \mathbb{C}^n$ as a real vector space of dimension $2n$, which inherits from $\mathbb{C}^n$ the complex structure $I \in \mathrm{End}\;V$, the real part $g \in \mathrm{Sym}^2_+V^*$ of the standard hermitean form $(\cdot,\cdot)$ and the n–form $\psi \in \Lambda^nV^*$:

$$\psi(\,v_1,\,\ldots,\,v_n\,) \;:=\; \mathrm{Re}\,\det\nolimits_{\mathbb{C}}(\,v_1,\,\ldots,\,v_n\,)$$

The common stabilizer of the triple $(\,g,\,I,\,\psi\,)$ agrees with the subgroup $\mathbf{SU}(n) \subset \mathbf{GL}\,V$ so that an $\mathbf{SU}(n)$–structure Ω on a $2n$–dimensional vector space defines a corresponding triple

$$g_\Omega \ := \ g(F^{-1}\cdot, F^{-1}\cdot) \qquad I_\Omega \ := \ F\circ I\circ F^{-1} \qquad \psi_\Omega \ := \ \psi(F^{-1}\cdot,\ldots,F^{-1}\cdot)$$

on T for any representative $F\in\Omega$. On the other hand the model triple (g, I, ψ) satisfies

$$I^2 \ = \ -\mathrm{id}_V \qquad g(I\cdot, I\cdot) \ = \ g \qquad \mathrm{Der}_I^2\,\psi \ = \ -n^2\,\psi \qquad g^{-1}(\psi,\psi) \ = \ 2^{n-1} \tag{8}$$

and it can be shown straightforwardly that $\mathbf{GL}\,V$ acts transitively on the set of solutions to these algebraic equations in $\mathrm{Sym}^2_+V^*\times\mathrm{End}\,V\times\Lambda^nV^*$. In consequence $\mathbf{SU}(n)$–structures Ω on a vector space T are in bijection to triples (g, I, ψ) in $\mathrm{Sym}^2_+T^*\times\mathrm{End}\,T\times\Lambda^nT^*$ satisfying the algebraic equations (8). Similarly the tuple (g, I) defines the underlying $\mathbf{U}(n)$–structure and $g\in\mathrm{Sym}^2_+T^*$ only the underlying $\mathbf{O}(2n)$–structure on T.

Coming back to homogeneous spaces G/H we define a left invariant K–structure as a left invariant section of the homogeneous quotient bundle $\mathrm{Frame}(V, TG/H)/K$ of the homogeneous frame bundle. By the classification of left invariant sections such a section Ω is completely determined by its H–invariant value in the base point $eH\in G/H$

$$\Omega_{eH} \ \in \ \Big[\,\mathrm{Frame}(V,\ \mathfrak{g}/\mathfrak{h})/_K\,\Big]^H$$

where the condition of H–invariance reads for a representative $F: V\longrightarrow\mathfrak{g}/\mathfrak{h}$ of Ω_{eH}

$$(h\star)\circ F \ \sim_K \ F \qquad\Longleftrightarrow\qquad F^{-1}\circ(h\star)\circ F \ \in \ K$$

for every $h\in H$, equivalently the image $\mathrm{Ad}\,H\subset\mathbf{GL}\,\mathfrak{g}/\mathfrak{h}$ of the isotropy group H must be conjugated to a subgroup of K. Similarly a left invariant K–structure on G/H is parallel for a left invariant connection $A:\mathfrak{g}\longrightarrow\mathrm{End}\,\mathfrak{g}/\mathfrak{h}$ provided $F^{-1}\circ A_X\circ F$ is an element of the Lie algebra $\mathfrak{k}$ of K for every $X\in\mathfrak{g}$.

In consequence we may define a formal affine homogeneous space with parallel K–structure as a pair of Lie algebras $\mathfrak{g}\supset\mathfrak{h}$ endowed with a formal left in variant connection $A:\mathfrak{g}\longrightarrow\mathrm{End}\,\mathfrak{g}/\mathfrak{h}$ and a K–structure Ω on $\mathfrak{g}/\mathfrak{h}$ such that $F^{-1}\circ(H\star)\circ F\subset\mathfrak{k}$ and $F^{-1}\circ A_X\circ F\in\mathfrak{k}$ for all $H\in\mathfrak{h}$ and $X\in\mathfrak{g}$. On the other hand we recall that we need to choose a frame $F: V\longrightarrow\mathfrak{g}/\mathfrak{h}$ anyhow in order to associate to such a formal homogeneous space a point in the algebraic variety $\mathfrak{M}(\mathfrak{gl}\,V)$, making the straightforward choice of a frame F representing Ω we arrive at the algebraic variety of formal affine homogeneous spaces

$$\mathfrak{M}(\mathfrak{k}) \ := \ \mathfrak{M}(\mathfrak{gl}\,V)\cap(V^*\otimes\mathfrak{k}\times\Lambda^2V^*\otimes\mathfrak{k}\times\Lambda^2V^*\otimes V) \tag{9}$$

with parallel K–structure. Passing through the arguments used in the construction of the algebraic variety $\mathfrak{M}(\mathfrak{gl}\,V)$ we conclude that the connection–curvature–torsion triples (A, R, T) in $\mathfrak{M}(\mathfrak{k})$ correspond to maximal Lie subalgebras $\mathfrak{g}\subset\mathfrak{k}\oplus_{R,T}V$ of the skew algebras $\mathfrak{k}\oplus_{R,T}V$, which contain the image of the extended connection $V\longrightarrow\mathfrak{k}\oplus_{R,T}V$, $x\longmapsto A_x\oplus_{R,T}x$. Similar considerations apply of course to the algebraic varieties $\mathfrak{M}^{\mathrm{tf}}(\mathfrak{k})$ and $\mathfrak{M}^{\mathrm{red}}(\mathfrak{k})$ of torsion free and reductive formal affine homogeneous spaces with parallel K–structures.

6 Contact Order and Formal Tangent Space

The algebraic variety $\mathfrak{M}(\,\mathfrak{gl}\,V\,)$ of affine homogeneous spaces and its variants are only coarse moduli spaces, because they parametrize isometry classes of formal affine homogeneous spaces $\mathfrak{g} \supset \mathfrak{h}$ augmented by a frame isomorphism $F : V \longrightarrow \mathfrak{g}/\mathfrak{h}$ and a split $\mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}$ of the canonical projection. The effect of changing the frame and/or the split introduces an equivalence relation on $\mathfrak{M}(\,\mathfrak{gl}\,V\,)$, the equivalence relation $\sim_\infty$ of contact to all orders approximated by the equivalence relations $\sim_d$ of contact up to order $d \in \mathbb{N}_0$. In order to describe the resulting filtration of the formal tangent space to the true moduli space $\mathfrak{M}_\infty(\,\mathfrak{gl}\,V\,)$ we associate a Spencer cohomology $H^{\bullet,\circ}(\,\mathfrak{h}\,)$ to every connection–curvature–torsion triple $(A,\,R,\,T) \in \mathfrak{M}(\,\mathfrak{gl}\,V\,)$ and identify it with the successive filtration quotients. Eventually we illustrate this interpretation of the Spencer cohomology by explicit calculations for the family of Riemannian homogeneous spaces with large Singer invariant constructed by Meusers [M].

Definition 6.1 (Contact Relation for Connection–Curvature–Torsion Triples).
Two connection–curvature–torsion triples (A, R, T) and $(\tilde{A}, \tilde{R}, \tilde{T})$ on a finite–dimensional vector space V are said to be in contact to order $d \geq 0$ written $(A, R, T) \sim_d (\tilde{A}, \tilde{R}, \tilde{T})$, if there exists a linear automorphism $F \in \mathbf{GL}\,V$ of V pulling the formal covariant derivatives of $\tilde{R}$ and $\tilde{T}$ of all orders $r = 0, \ldots, d$ back to the formal covariant derivatives of R and T:

$$F^*\Big(\underbrace{\tilde{A} \circledast (\tilde{A} \circledast \ldots (\tilde{A}}_{r\ \text{times}} \circledast \tilde{T}\,)\ldots)\Big) \;=\; \underbrace{A \circledast (\,A \circledast \ldots (\,A}_{r\ \text{times}} \circledast T\,)\ldots)$$

$$F^*\Big(\underbrace{\tilde{A} \circledast (\tilde{A} \circledast \ldots (\tilde{A}}_{r\ \text{times}} \circledast \tilde{R}\,)\ldots)\Big) \;=\; \underbrace{A \circledast (\,A \circledast \ldots (\,A}_{r\ \text{times}} \circledast R\,)\ldots)$$

Similarly the notation $(A, R, T) \sim_\infty (\tilde{A}, \tilde{R}, \tilde{T})$ indicates triples in contact to all orders $d \geq 0$.

The naturality of the operation $\circledast$ implies of course for the iterated covariant derivatives

$$F^*(\,A \circledast (\,A \circledast \ldots (\,A \circledast T\,)\ldots)\Big) \;=\; F^*A \circledast (\,F^*A \circledast \ldots (\,F^*A \circledast F^*T\,)\ldots)$$

of T or similarly of R. The main argument of Lemma 4.5 may thus be varied to prove:

Lemma 6.2 (Explicit Form of the Contact Relation).
Two connection–curvature–torsion triples (A, R, T) and $(\tilde{A}, \tilde{R}, \tilde{T})$ on a finite–dimensional vector space V are in contact $(A, R, T) \sim_d (\tilde{A}, \tilde{R}, \tilde{T})$ to order $d \geq 0$, if and only if there exists a linear automorphism $F \in \mathbf{GL}\,V$ satisfying $F^*\tilde{T} = T$ and $F^*\tilde{R} = R$ as well as:

$$F^*\tilde{A} \;\equiv\; A \qquad \mathrm{mod} \quad V^* \otimes \mathfrak{h}_{d-1}$$

In consequence two triples $(A, R, T) \sim_d (\tilde{A}, \tilde{R}, \tilde{T})$ in contact to an order $d >$ Singer(A, R, T) exceeding the Singer invariant of one are in contact $(A, R, T) \sim_\infty (\tilde{A}, \tilde{R}, \tilde{T})$ to all orders.

The infinite order contact relation $\sim_\infty$ reflects exactly the dependence of the connection–curvature–torsion triple (A, R, T) associated to a formal affine homogeneous space $\mathfrak{g} \supset \mathfrak{h}$ on the additional choice of frame $F : V \longrightarrow \mathfrak{g}/\mathfrak{h}$ and split $\mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}$ of the canonical projection. The moduli space of isometry classes of formal affine homogeneous spaces may thus be defined as the set of equivalence classes of points $(A, R, T) \in \mathfrak{M}(\mathfrak{gl}\, V)$ under $\sim_\infty$:

$$\mathfrak{M}_\infty(\, \mathfrak{gl}\, V\,) \;:=\; \mathfrak{M}(\, \mathfrak{gl}\, V\,)/\sim_\infty$$

Similar definitions can be made for the moduli spaces of torsion free $\mathfrak{M}^{\mathrm{tf}}_\infty(\,\mathfrak{gl}\, V\,)$ or reductive formal affine homogeneous spaces $\mathfrak{M}^{\mathrm{red}}_\infty(\,\mathfrak{gl}\, V\,)$ with or without additional left invariant parallel K–structures. It turns out that the deformation theory of isometry classes of formal affine homogeneous spaces considered as points in $\mathfrak{M}_\infty(\,\mathfrak{gl}\, V\,)$ is governed by a suitable version of the Spencer cohomology associated to the purely algebraic concept of a graded comodule:

Definition 6.3 (Comodules over Symmetric Coalgebras)**.**
A comodule over the symmetric coalgebra Sym V^* *of a finite–dimensional vector space V is a $\mathbb{Z}$–graded vector space $\mathscr{A}^\bullet$ together with a bilinear map called directional derivative $V \times \mathscr{A}^\bullet \longrightarrow \mathscr{A}^{\bullet-1}$, $(y, \mathfrak{X}) \longmapsto \frac{\partial \mathfrak{X}}{\partial y}$, homogeneous of degree -1 with respect to $\mathscr{A}$ such that*

$$\frac{\partial}{\partial y}\Big(\,\frac{\partial \mathfrak{X}}{\partial z}\,\Big) \;=\; \frac{\partial}{\partial z}\Big(\,\frac{\partial \mathfrak{X}}{\partial y}\,\Big)$$

for all $y, z \in V$. In particular the iterated directional derivatives like $\frac{\partial^2 \mathfrak{X}}{\partial y\,\partial z}$ are well defined.

Definition 6.4 (Spencer Complex and Cohomology)**.**
The Spencer complex associated to a comodule $\mathscr{A}$ over the symmetric coalgebra Sym V^* *of a finite–dimensional vector space V is the bigraded complex of alternating multilinear forms $\Lambda^\circ V^* \otimes \mathscr{A}^\bullet$ on V with values in the comodule $\mathscr{A}$*

$$\ldots \stackrel{B}{\longrightarrow} \Lambda^{\circ-1}V^* \otimes \mathscr{A}^{\bullet+1} \stackrel{B}{\longrightarrow} \Lambda^{\circ}V^* \otimes \mathscr{A}^{\bullet} \stackrel{B}{\longrightarrow} \Lambda^{\circ+1}V^* \otimes \mathscr{A}^{\bullet-1} \stackrel{B}{\longrightarrow} \ldots$$

endowed with the Spencer coboundary operator B defined for $\eta \in \Lambda^k V^* \otimes \mathscr{A}^\bullet$ by:

$$(\, B\,\eta\,)(\, x_0, \ldots, x_k\,) \;:=\; \sum_{\mu=0}^{k} (\,-1\,)^\mu \,\frac{\partial}{\partial x_\mu} B(\, x_0, \ldots, \widehat{x_\mu}, \ldots, x_k\,)$$

The corresponding bigraded cohomology theory for comodules is called Spencer cohomology:

$$H^{\bullet,\circ}(\,\mathscr{A}\,) \;:=\; \frac{\mathbf{ker}(\; B:\;\; \mathscr{A}^{\bullet} \;\otimes\; \Lambda^{\circ}V^* \;\longrightarrow\; \mathscr{A}^{\bullet-1}\otimes \Lambda^{\circ+1}V^*)}{\mathbf{im}(\; B:\;\; \mathscr{A}^{\bullet+1}\otimes \Lambda^{\circ-1}V^* \longrightarrow\;\; \mathscr{A}^{\bullet} \;\otimes\; \Lambda^{\circ}V^* \;)}$$

A detailed description of the general properties of the Spencer cohomology of comodules is certainly beyond the scope of this article, for more information see for example [BCG], [W]. Nevertheless we want to point out that the Spencer cohomology $H^{\bullet,\circ}(\,\mathscr{A}\,)$ of a comodule $\mathscr{A}^\bullet$ is naturally a graded right module over the exterior algebra $\Lambda^\circ V^*$. For a comodule $\mathscr{A}$ constant in the directions of a subspace $W \subset V$ in the sense $\frac{\partial \mathfrak{X}}{\partial x} = 0$ for all $\mathfrak{X} \in \mathscr{A}$ and all $x \in W$ for example the right $\Lambda^\circ V^*$–module structure turns out to be convenient to prove

$$H^{\bullet,\circ}(\,\mathscr{A}\,) \;\cong\; H^{\bullet,\circ}_{V/W}(\,\mathscr{A}\,) \otimes \Lambda^\circ W^* \tag{10}$$

where $H^{\bullet,\circ}_{V/W}(\,\mathscr{A}\,)$ refers to the Spencer cohomology of $\mathscr{A}^\bullet$ considered as a comodule over the coalgebra Sym $(V/W)^*$ and the repeated grading symbol $\circ$ indicates the product grading.

Definition 6.5 (Formal Directional Derivatives)**.**
Consider the strictly decreasing filtration (7) associated to a connection–curvature–torsion triple (A, R, T) in the variety $\mathfrak{M}(\,\mathfrak{gl}\,V\,)$. The direct sum of successive filtration quotients

$$\mathfrak{h}^\bullet \;=\; \bigoplus_{r\in\mathbb{Z}} \Big(\,\mathfrak{h}_{r-1}/_{\mathfrak{h}_r}\Big) \;=\; (\,\mathrm{End}\ V/_{\mathfrak{h}_0}\,) \oplus \ldots \oplus (\,\mathfrak{h}_{s-1}/_{\mathfrak{h}_\infty}\,)$$

is a comodule over the symmetric coalgebra Sym V^* *under directional derivatives defined by*

$$\frac{\partial \mathfrak{X}}{\partial x} \;:=\; [\,\mathfrak{X},\,A_x\,] \;-\; A_{\mathfrak{X}x} \;+\; \mathfrak{h}_{r-1}$$

for $x \in V$ and all representatives $\mathfrak{X} \in \mathfrak{h}_{r-1}$ of a class $\mathfrak{X} + \mathfrak{h}_r$ in the quotient $\mathfrak{h}^r := \mathfrak{h}_{r-1}/\mathfrak{h}_r$.

Of course we should not forget to verify the axiomatic commutation of the formal directional derivatives for a comodule over the symmetric coalgebra Sym V^*, which is quite surprising in view of the complicated definition of the comodule $\mathfrak{h}^\bullet$ and its formal directional derivatives. Straightforward calculation of iterated formal derivatives results in the not too pleasant

$$\begin{aligned}\frac{\partial}{\partial y}\frac{\partial \mathfrak{X}}{\partial z} \;&\equiv\; [\,[\,\mathfrak{X},\,A_z\,] \,-\, A_{\mathfrak{X}z},\,A_y\,] \;-\; A_{(\,[\,\mathfrak{X},A_z\,]\,-\,A_{\mathfrak{X}z}\,)\,y}\\ &\equiv\; -\,[\,A_y,\,[\,\mathfrak{X},\,A_z\,]\,] \,+\, [\,A_y,\,A_{\mathfrak{X}z}\,] \,-\, A_{\mathfrak{X}(\,A_zy\,)} \,+\, A_{A_z(\,\mathfrak{X}y\,)} \,+\, A_{A_{\mathfrak{X}z}y}\end{aligned}$$

modulo $\mathfrak{h}_{r-2}$ for a given representative $\mathfrak{X} \in \mathfrak{h}_{r-1}$ of a class $\mathfrak{X} + \mathfrak{h}_r \in \mathfrak{h}_{r-1}/\mathfrak{h}_r$, in consequence

$$\begin{aligned}&\frac{\partial}{\partial z}\frac{\partial \mathfrak{X}}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \mathfrak{X}}{\partial z}\\ &\quad\equiv\; +\,[\,\mathfrak{X},\,[\,A_y,\,A_z\,] \,-\, A_{A_yz} \,+\, A_{A_zy} \,+\, A_{T(y,z)}\,] \,-\, A_{\mathfrak{X}\,T(y,z)}\\ &\qquad\; -\,[\,A_{\mathfrak{X}y},\,A_z\,] \,+\, A_{A_{\mathfrak{X}y}z} \,-\, A_{A_z(\,\mathfrak{X}y\,)} \,-\, A_{T(\mathfrak{X}y,z)} \,+\, A_{T(\mathfrak{X}y,z)}\\ &\qquad\; -\,[\,A_y,\,A_{\mathfrak{X}z}\,] \,+\, A_{A_y(\,\mathfrak{X}z\,)} \,-\, A_{A_{\mathfrak{X}z}y} \,-\, A_{T(y,\mathfrak{X}z)} \,+\, A_{T(y,\mathfrak{X}z)}\\ &\qquad\; +\,[\,\mathfrak{X},\,A_{A_yz} \,-\, A_{A_zy} \,-\, A_{T(y,z)}\,] \,-\, A_{\mathfrak{X}(\,A_yz\,-\,A_zy\,-\,T(y,z)\,)}\\ &\quad\equiv\; \Big[\,\mathfrak{X}\star Q(\,A,\,T\,)\Big]_{y.z} \,+\, \frac{\partial \mathfrak{X}}{\partial(\,A_yz \,-\, A_zy \,-\, T(y,\,z)\,)} \,-\, A_{[\,\mathfrak{X}\star T\,](\,y,\,z\,)}\end{aligned}$$

modulo $\mathfrak{h}_{r-2}$. For $r \le 1$ there is actually nothing to prove, because the right hand side vanishes for trivial reasons due to $\mathfrak{h}_{-1} = \mathrm{End}\ V$. On the other hand

$\mathfrak{X} \in \mathfrak{h}_{r-1} \subset \mathfrak{h}_0$ for $r \geq 2$ so that $\mathfrak{X} \star T = 0$ by the very definition of $\mathfrak{h}_0 = \mathfrak{stab}\, R \cap \mathfrak{stab}\, T$. The directional derivative term on the right hand side vanishes by definition modulo $\mathfrak{h}_{r-2}$ leaving us with

$$\frac{\partial}{\partial z}\frac{\partial \mathfrak{X}}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \mathfrak{X}}{\partial z} \equiv \Big[\, \mathfrak{X} \star (\, Q(\, A,\, T\,) - R\,)\,\Big]_{y,z} \qquad \text{mod } \mathfrak{h}_{r-2}$$

in view of $\mathfrak{X} \in \mathfrak{stab}\, R$. For a connection–curvature–torsion triple $(A, R, T) \in \mathfrak{M}(\,\mathfrak{gl}\, V\,)$ however the difference $Q(A, T) - R$ is an $\mathfrak{h}_\infty$–valued 2–form on V so that $\mathfrak{X} \star (\, Q(A, T) - R\,)$ is actually $\mathfrak{h}_{r-1}$–valued for all $\mathfrak{X} \in \mathfrak{h}_{r-1} \supset \mathfrak{h}_\infty$. In consequence the formal directional derivatives of the comodule $\mathfrak{h}^\bullet$ defined above do in fact commute and $\mathfrak{h}^\bullet$ is a comodule over the symmetric coalgebra Sym V^*. Its Spencer cohomology in form degree 0 is very simple:

Remark 6.6 (Stabilizer Filtration and Spencer Cohomology).
The only non–vanishing Spencer cohomology in form degree 0 *of the comodule* $\mathfrak{h}^\bullet$ *associated to a connection–curvature–torsion triple* (A, R, T) *in the algebraic variety* $\mathfrak{M}(\,\mathfrak{gl}\, V\,)$ *is:*

$$H^{0,0}(\,\mathfrak{h}\,) = \mathfrak{h}^0 := \text{End } V/_{\mathfrak{h}_0}$$

In fact $H^{r,0}(\,\mathfrak{h}\,) = \{0\}$ *for all positive* $r > 0$ *by the definition the stabilizer filtration (7).*

Whereas the Spencer cohomology in form degree 0 is thus not particularly interesting, the Spencer cohomology $H^{\bullet,1}(\,\mathfrak{h}\,)$ in form degree 1 has the following neat geometric interpretation in terms of deformations of a formal affine homogeneous space $\mathfrak{g} \supset \mathfrak{h}$.

Lemma 6.7 (Formal Tangent Space of the Moduli Space).
Given a point $[\, A,\, R,\, T\,]$ *in the moduli space* $\mathfrak{M}_\infty(\, V\,)$ *of formal affine homogeneous spaces of dimension* dim V *we may consider the subsets of formal affine homogeneous spaces*

$$\mathfrak{M}^d_\infty(\, A,\, R,\, T\,) := \{\, [\,\tilde{A},\, \tilde{R},\, \tilde{T}\,] \mid [\,\tilde{A},\, \tilde{R},\, \tilde{T}\,] \sim_d [\, A,\, R,\, T\,]\,\} \subset \mathfrak{M}_\infty(\,\mathfrak{gl}\, V\,)$$

in contact to $[\, A,\, R,\, T\,]$ *to order* $d \geq 0$. *The resulting decreasing filtration of* $\mathfrak{M}_\infty(\,\mathfrak{gl}\, V\,)$

$$\mathfrak{M}_\infty(\,\mathfrak{gl}\, V\,) \supseteq \mathfrak{M}^0_\infty(A, R, T) \supseteq \mathfrak{M}^1_\infty(A, R, T) \supseteq \ldots$$
$$\supseteq \mathfrak{M}^s_\infty(A, R, T) \supseteq \{\, [A, R, T]\,\}$$

provides an interpretation of the Spencer cohomology of $\mathfrak{h}$ *by means of formal tangent spaces:*

$$H^{\bullet,1}(\,\mathfrak{h}\,) = T_{[A,R,T]}\mathfrak{M}^\bullet_\infty(A, R, T)/_{T_{[A,R,T]}\mathfrak{M}^{\bullet-1}_\infty(A, R, T)}$$

Proof. Consider a curve of connection–curvature–torsion triples $\varepsilon \longmapsto (\, A_\varepsilon,\, R_\varepsilon,\, T_\varepsilon\,)$ representing a tangent vector to $\mathfrak{M}^d_\infty(A, R, T)$ in the point $[\, A,\, R,\, T\,] = [\, A_0,\, R_0,\, T_0\,]$:

$$\frac{d}{d\varepsilon}\bigg|_0 [\, A_\varepsilon,\, R_\varepsilon,\, T_\varepsilon\,] \in T_{[\, A, R, T\,]}\mathfrak{M}^d_\infty(A, R, T)$$

By assumption all triples $[A_\varepsilon, R_\varepsilon, T_\varepsilon]$ are in contact to order $d \geq 0$ so that we may find a smooth curve $\varepsilon \longmapsto F_\varepsilon$ in $\mathbf{GL}\,V$ satisfying $F^*_\varepsilon T_\varepsilon = T$ and $F^*_\varepsilon R_\varepsilon = R$ as well as $F^*_\varepsilon A_\varepsilon \equiv A$ modulo $V^* \otimes \mathfrak{h}_{d-1}$. The infinitesimal variation of the curve $\varepsilon \longmapsto [\, A_\varepsilon, R_\varepsilon, T_\varepsilon \,]$ defined by

$$\delta A \;:=\; \frac{d}{d\varepsilon}\Big|_0 F^*_\varepsilon A_\varepsilon \;\in\; V^* \otimes \mathfrak{h}_{d-1}$$

clearly satisfies $\delta A \in V^* \otimes \mathfrak{h}_d$, if the original curve $\varepsilon \longmapsto (\, A_\varepsilon, R_\varepsilon, T_\varepsilon \,)$ stays actually in contact to $[\, A, R, T \,]$ to order $d+1$. For this reason we are interested in the class represented by δA in $V^* \otimes \mathfrak{h}^d = V^* \otimes (\mathfrak{h}_{d-1}/\mathfrak{h}_d)$. This infinitesimal variation class is closed, because

$$\begin{aligned} B(\,\delta A\,)(\,y,z\,) \;&\equiv\; \frac{\partial}{\partial y}(\,\delta A_z\,) - \frac{\partial}{\partial z}(\,\delta A_y\,) \\ &\equiv\; [\,\delta A_z,\, A_y\,] - [\,\delta A_y,\, A_z\,] + A_{\delta A_y z \,-\, \delta A_z y} \\ &\equiv\; -\,\delta Q(\,A,\, T\,)_{y,z} \;-\; \delta A_{A_y z \,-\, A_z y} \end{aligned}$$

vanishes modulo $\mathfrak{h}_{d-1}$, where δQ is the infinitesimal variation of the approximate curvature:

$$\delta Q \;:=\; \frac{d}{d\varepsilon}\Big|_0 Q(\,F^*_\varepsilon A_\varepsilon,\; T\,) \;\in\; \Lambda^2 V^* \otimes \mathfrak{h}_{d-1} \tag{11}$$

In fact all connection–curvature–torsion triples $(\,F^*_\varepsilon A_\varepsilon, R, T\,) \in \mathfrak{M}(\,\mathfrak{gl}\,V\,)$ are in contact to order $d \geq 0$ by assumption and thus share the initial part $\mathfrak{h}_{-1} \supsetneq \ldots \supsetneq \mathfrak{h}_{d-1}$ of their stabilizer filtrations by Lemma 4.5. Among the equations defining $\mathfrak{M}(\,\mathfrak{gl}\,V\,)$ is the congruence

$$Q(\,F^*_\varepsilon A_\varepsilon,\; T\,) \;\equiv\; R \;\bmod\; \Lambda^2 V^* \otimes (\,\mathfrak{h}_\varepsilon\,)_\infty$$

which implies $Q(F^*_\varepsilon A_\varepsilon, T) \equiv R$ modulo $\Lambda^2 V^* \otimes \mathfrak{h}_{d-1}$ for all ε and thus proves the implicit claim in definition (11). Being closed for the Spencer coboundary operator B the infinitesimal variation represents a class $[\,\delta A\,] \in H^{d,1}(\,\mathfrak{h}\,)$ in the Spencer cohomology of the comodule $\mathfrak{h}^\bullet$ associated to $[\,A, R, T\,]$. In case this class vanishes and the infinitesimal variation is exact

$$\delta A \;=\; B\,\mathfrak{X} \;\equiv\; \mathfrak{X} \star A \;\bmod\; V^* \otimes \mathfrak{h}_d$$

for some $\mathfrak{X} \in \mathfrak{h}_d$, hence the two curves $\varepsilon \longmapsto (\,e^{-\varepsilon\mathfrak{X}} \star F^*_\varepsilon A_\varepsilon, R, T\,)$ and $\varepsilon \longmapsto (\,A_\varepsilon, R_\varepsilon, T_\varepsilon\,)$ represent the same tangent vector in $[\,A, R, T\,]$ tangent to $\mathfrak{M}^{d+1}_\infty(\,\mathfrak{gl}\,V\,)$. ■

In order to illustrate the relation between the Spencer cohomology of Definition 6.4 and the formal tangent spaces to the moduli space $\mathfrak{M}_\infty(\,\mathfrak{gl}\,V\,)$ we want to study the family of examples of Riemannian homogeneous spaces with large Singer invariant constructed by Meusers [M] in somewhat more detail. Calculations can be streamlined significantly using the standard identification of the Lie algebra $\mathfrak{so}\,V$ of skew symmetric endomorphisms on a euclidean vector space V with scalar product g with the second exterior power $\Lambda^2 V$

$$\Lambda^2 V \;\stackrel{\cong}{\longrightarrow}\; \mathfrak{so}\,V, \qquad X \wedge Y \longmapsto \Big(\, Z \longmapsto g(\,X, Z\,)\,Y - g(\,Y, Z\,)\,X \,\Big)$$

characterized by $g(\mathfrak{X}, Y \wedge Z) = g(\mathfrak{X}Y, Z)$ for all $Y, Z \in V$ and $\mathfrak{X} \in \Lambda^2 V$ on the left, but $\mathfrak{X} \in \mathfrak{so}\, V$ on the right hand side. The Lie bracket on $\Lambda^2 V = \mathfrak{so}\, V$ satisfies the formulas

$$[\, X \wedge Y,\, X \wedge Z \,] \;=\; g(X,X)\, Y \wedge Z \qquad\qquad [\, \mathfrak{X},\, Y \wedge Z \,] \;=\; \mathfrak{X} Y \wedge Z \,+\, Y \wedge \mathfrak{X} Z$$

for all $\mathfrak{X} \in \mathfrak{so}\, V$ and all $Y, Z \in V$ satisfying $g(X,Y) = 0 = g(X,Z)$. Consider now a euclidean vector space $V_\circ$ endowed with a scalar product $g \in \mathrm{Sym}^2_+ V^*$ and an endomorphism $F : V_\circ \longrightarrow V_\circ$. The direct sum $V := \mathbb{R} \oplus V_\circ$ is then a euclidean Lie algebra under the extension $g(x \oplus X, y \oplus Y) = xy + g(X,Y)$ of the scalar product and the Lie bracket:

$$[\, x \oplus X,\, y \oplus Y \,] \;:=\; 0 \oplus (\, xFY \,-\, yFX \,)$$

Evidently V is a solvable Lie algebra with abelian nilpotent subalgebra $[\, V, V \,] = V_\circ$, which can be realized as the Lie algebra of left invariant vector fields on the corresponding simply connected solvable Lie group G. After a short calculation it turns out that the Levi–Civita connection of the left invariant metric g on G depends on the decomposition of $F = F_+ + F_-$ into its symmetric part F_+ and its skew symmetric part F_-, more precisely we obtain:

$$A : \quad V \longrightarrow \mathrm{End}\, V, \qquad x \oplus X \longrightarrow (\, F_+ X \,) \wedge \mathbf{1} \,+\, x F_-$$

In fact $A_{x \oplus X} \in \mathfrak{so}\, V$ is a skew symmetric endomorphism of V for all $x \oplus X \in V$ and

$$\begin{aligned} A_{x \oplus X}(\, y \oplus Y \,) \,-\, A_{y \oplus Y}(\, x \oplus X \,) \\ &= \Big(g(\, F_+ X, Y \,) - g(\, F_+ Y, X \,) \Big) \oplus \\ &\quad \Big((\, -y F_+ X + x F_- Y \,) - (\, -x F_+ Y + y F_- X \,) \Big) \\ &= 0 \oplus \Big(xFY \,-\, yFX \Big) \;=\; [\, x \oplus X,\, y \oplus Y \,] \end{aligned}$$

agrees with the Lie bracket. Using this piece of information we calculate the curvature to be

$$\begin{aligned} &R_{x \oplus X, y \oplus Y} \\ &= [\, (\, F_+ X \,) \wedge \mathbf{1} \,+\, x F_-,\, (\, F_+ Y \,) \wedge \mathbf{1} \,+\, y F_- \,] \,-\, (\, x F_+ F Y \,-\, y F_+ F X \,) \wedge \mathbf{1} \\ &= (\, F_+ X \,) \wedge (\, F_+ Y \,) \,-\, x\, (\, F_+^2 \,+\, [\, F_+, F_- \,] \,)\, Y \wedge \mathbf{1} \\ &\quad + y\, (\, F_+^2 \,+\, [\, F_+, F_- \,] \,)\, X \wedge \mathbf{1} \end{aligned}$$

In order to determine the stabilizer of R in $\mathfrak{so}\, V$ it seems prudent to study the Ricci curvature:

$$\mathrm{Ric}(\, y \oplus Y,\, z \oplus Z \,) \;:=\; \mathrm{tr}_V \Big(x \oplus X \longmapsto R_{x \oplus X, y \oplus Y}(\, z \oplus Z \,) \Big)$$

Calculating the traces of the following expressions appearing in $R_{x \oplus X, y \oplus Y}(z \oplus Z)$ over $x \oplus X$

$$\Big(\qquad F_+ X \,\wedge\, F_+ Y \qquad \Big)(z \oplus Z) =$$

$$0 \oplus \Big(g(\, F_+ Z, X \,)\, F_+ Y \,-\, g(\, F_+ Y, Z \,)\, F_+ X \Big)$$

$$\Big(x\,(F_+^2 + [F_+,F_-])\,Y \wedge \mathbf{1}\Big)\,(z \oplus Z) = \Big(g(\,(F_+^2 + [F_+,F_-])Y,\,Z\,)\,x\Big) \oplus \Big(\quad \ldots \quad\Big)$$

$$\Big(y\,(F_+^2 + [F_+,F_-])\,X \wedge \mathbf{1}\Big)\,(z \oplus Z) = \Big(\quad \ldots \quad\Big) \oplus \Big(-\,yz\,(F_+^2 + [F_+,F_-])X\Big)$$

we obtain $g(\,F_+^2Y \;-\; (\operatorname{tr}F_+)\,F_+Y,\,Z\,)$ and $g(\,F_+^2Y \;+\; [F_+,F_-]Y,\,Z\,)$ as well as $-yz\,(\operatorname{tr}F_+^2)$ so that the Ricci endomorphism $g(\,\mathrm{Ric}(\,y \oplus Y\,),\,z \oplus Z\,) := \mathrm{Ric}(\,y \oplus Y,\,z \oplus Z\,)$ reads:

$$\mathrm{Ric}(\,y \oplus Y\,) \;=\; -\,\Big(\operatorname{tr}(\,F_+^2\,)\,y \oplus (\,[\,F_+,\,F_-\,] \;+\; (\operatorname{tr}F_+)\,F_+\,)\,Y\Big) \tag{12}$$

In this way we obtain the following upper and lower bound on the stabilizer of R in $\mathfrak{so}\,V$

$$\mathfrak{stab}\;\mathrm{Ric} \;\supset\; \mathfrak{stab}\;R \;\supset\; \mathfrak{stab}\;F_+ \,\cap\, \mathfrak{stab}\;[\,F_+,\,F_-\,] \,\cap\, \mathfrak{so}\,V_\circ \tag{13}$$

because the explicit formula for R tells us that every $\mathfrak{X} \in \mathfrak{so}\,V_\circ$ stabilizing F_+ and $[\,F_+,\,F_-\,]$ stabilizes R. In order to proceed we need to be somewhat more specific about the special form of the endomorphism F in the family of examples found by Meusers:

Definition 6.8 (Meusers' Family of Examples)**.**
An endomorphism $F:\,V_\circ \longrightarrow V_\circ$ on a euclidean vector space $V_\circ$ of dimension $m-1 \geq 3$ is called special provided its diagonalizable symmetric part F_+ has only two different eigenvalues of multiplicities 1 and $m-2$ respectively and every eigenvector $e \neq 0$ in the 1–dimensional eigenspace is cyclic for the skew symmetric part F_- of F in the sense:

$$V_\circ \;=\; \mathrm{span}\,\{\;\; e,\;\; F_-\,e,\;\; F_-^2\,e,\;\; F_-^3\,e,\;\; F_-^4\,e,\;\ldots \qquad\}$$

Using the cyclicity of the eigenvector $e \neq 0$ of the symmetric part F_+ of a special endomorphism F under its skew symmetric part F_- we may construct a complete flag on $V_\circ$ via:

$$\{\,0\,\} \;\subsetneq\; \mathrm{span}\,\{\,e\,\} \;\subsetneq\; \mathrm{span}\,\{\,e,\;F_-e\,\} \;\subsetneq\; \ldots \;\subsetneq\; \mathrm{span}\,\{\,e,\;F_-e,\;\ldots,\;F_-^{m-3}e\,\} \;\subsetneq\; V_\circ$$

Up to the choice of signs there exists a unique orthonormal basis $e_2,\ldots,e_m$ of $V_\circ$ adapted to this flag, in this basis the matrix of the special endomorphism F is tridiagonal of the form

$$F \;\widehat{=}\; \begin{pmatrix} f_1 & -f_3 & & & \\ +f_3 & f_2 & -f_4 & & \\ & +f_4 & \ddots & \ddots & \\ & & \ddots & f_2 & -f_m \\ & & & +f_m & f_2 \end{pmatrix} \tag{14}$$

where the parameters $f_1, \ldots, f_m \in \mathbb{R}$ are arbitrary except for $f_1 \neq f_2$ and $f_3, \ldots, f_m \neq 0$ to ensure the cyclicity of the basis vector e_2. The special endomorphisms form in this way an m–parameter family of orbits in End $V_\circ$ under the action of the orthogonal group $\mathbf{O}(V_\circ)$. Extending this orthonormal basis to the orthonormal basis $\mathbf{1}, e_2, \ldots, e_m$ of V we obtain the following explicit matrix for the Ricci endomorphism calculated in equation (12):

$$\mathrm{Ric} \;\widehat{=}\; \begin{pmatrix} -\operatorname{tr} F_+^2 & & & & & \\ & -(\operatorname{tr} F_+)f_1 & (f_1 - f_2)f_3 & & & \\ & (f_1 - f_2)f_3 & -(\operatorname{tr} F_+)f_2 & & & \\ & & & -(\operatorname{tr} F_+)f_2 & & \\ & & & & \ddots & \\ & & & & & -(\operatorname{tr} F_+)f_2 \end{pmatrix}$$

This matrix certainly has 4 different eigenvalues of multiplicities $m-3$ and 1, 1, 1 for a generic special endomorphism F, say for $f_1 = 1$ and $f_2 = 0$ these eigenvalues are 0 and $-1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{4f_3^2 + 1})$ respectively. For such a generic special endomorphism F

$$\mathfrak{stab}\ \mathrm{Ric} \;=\; \mathfrak{stab}\ F_+ \cap \mathfrak{stab}\ [\,F_+, F_-\,] \cap \mathfrak{so}\, V_\circ \;=\; \mathfrak{so}\{\,\mathbf{1}, e_2, e_3\,\}^\perp$$

and equation (13) tells us that this agrees with the stabilizer of R:

$$\mathfrak{h}_0 \;:=\; \mathfrak{stab}\ R \;=\; \mathfrak{so}\{\,e_4, e_5, \ldots, e_m\,\}$$

Because every $\mathfrak{X} \in \mathfrak{h}_0$ commutes with F_+, the comodule directional derivatives are given by

$$\frac{\partial \mathfrak{X}}{\partial \mathbf{1}} \;=\; [\,\mathfrak{X}, F_-\,] \qquad\qquad \frac{\partial \mathfrak{X}}{\partial X} \;=\; [\,\mathfrak{X}, F_+X \wedge \mathbf{1}\,] - F_+(\,\mathfrak{X}X\,) \wedge \mathbf{1} \;=\; 0$$

for all $X \in V_\circ$, in particular for $\mathfrak{X} := e_\mu \wedge e_\nu \in \mathfrak{h}_0$ with $4 \leq \mu < \nu \leq m$ we obtain this way

$$\frac{\partial}{\partial \mathbf{1}}(\,e_\mu \wedge e_\nu\,) \;=\; -\Big(\,F_-e_\mu \wedge e_\nu + e_\mu \wedge F_-e_\nu\,\Big) \;\equiv\; f_\mu\, e_{\mu-1} \wedge e_\nu$$

modulo $\mathfrak{so}\,\{e_\mu, \ldots, e_m\}$. Consequently the subalgebras in the stabilizer filtration (7) read

$$\mathfrak{h}_r \;=\; \mathfrak{so}\{\,e_{r+4}, \ldots, e_m\,\} \tag{15}$$

for all $r = 0, \ldots, m-4$ due to our cyclicity assumption $f_3, \ldots, f_m \neq 0$. In consequence the affine homogeneous spaces of dimension m associated to generic special endomorphisms $F : V_\circ \longrightarrow V_\circ$ on euclidean vector spaces $V_\circ$ of dimension $m-1$ have Singer invariant $m-4$, because $\mathfrak{h}_{m-5} \neq \{0\}$, but $\mathfrak{h}_{m-4} = \mathfrak{h}_\infty = \{0\}$.

Before closing this section we want to calculate the Spencer cohomologies in this family of examples of affine spaces with large Singer invariant. With all directional derivatives $\frac{\partial \mathfrak{X}}{\partial X} = 0$ in directions $X \in V_\circ$ vanishing we may use the isomorphism (10) to reduce this calculation to the calculation of the Spencer cohomology of the comodule $\mathfrak{h}^\bullet$ considered as a comodule over Sym $V/V_\circ$ = Sym $\mathbb{R}$. By definition the comodule $\mathfrak{h}^\bullet$ is the direct sum of the successive quotients $\mathfrak{h}_{r-1}/\mathfrak{h}_r$, which are spanned for $r > 0$ by the bivectors $e_{r+3} \wedge e_\nu + \mathfrak{h}_r$ with $r + 3 < \nu \leq m$. The only non–trivial directional derivatives are injective for $r > 0$

$$\frac{\partial}{\partial \mathbf{1}} : \quad \mathfrak{h}_r/\mathfrak{h}_{r+1} \longrightarrow \mathfrak{h}_{r-1}/\mathfrak{h}_r, \qquad e_{r+4} \wedge e_\nu + \mathfrak{h}_{r+1} \longmapsto f_{r+3}\, e_{r+3} \wedge e_\nu + \mathfrak{h}_r$$

with cokernel spanned by $e_{r+3} \wedge e_{r+4} + \mathfrak{h}_r$. In summary we have proved the following lemma:

Lemma 6.9 (Singer Invariant and Spencer Cohomology).
Consider a special endomorphism $F : V_\circ \longrightarrow V_\circ$ on a euclidean vector space $V_\circ$ of dimension $m - 1 \geq 3$ in the sense of Definition 6.8 and let $e_2, \ldots, e_m$ be the essentially unique orthonormal basis of $V_\circ$, in which F takes the trilinear form (14). The euclidean Lie algebra $\mathfrak{g} = \mathbb{R} \oplus V_\circ = V$ associated to the special endomorphism F has Singer invariant

$$\mathrm{Singer}(\,\mathfrak{g}\,) \;=\; m - 4$$

provided the Ricci endomorphism of $\mathfrak{g}$ has 4 different eigenvalues. For such a generic special endomorphism F the Spencer cohomology $H^{r,\circ}(\,\mathfrak{h}\,)$ of the associated comodule $\mathfrak{h}^\bullet$ over the coalgebra Sym V^* *is a free $\Lambda^\circ V_\circ^*$–module of rank 1 for all $r = 1, \ldots, m - 4$ with isomorphism*

$$\Lambda^{\circ-1}V_\circ^* \stackrel{\cong}{\longrightarrow} H^{r,\circ}(\,\mathfrak{h}\,), \qquad \eta \longmapsto \Big[\, \mathbf{1}^\sharp \wedge \eta \otimes (\, e_{r+3} \wedge e_{r+4} + \mathfrak{h}_r\,)\,\Big]$$

where $\mathbf{1}^\sharp \in V^*$ is the linear form $\mathbf{1}^\sharp(x \oplus X) := x$. In particular the interesting Spencer cohomologies $H^{r,1}(\mathfrak{h})$ for $r = 1, \ldots, m - 4$ are all one–dimensional.

The remarkable conclusion of our calculation of the Spencer cohomology of Meusers' examples of Riemannian homogeneous spaces with large Singer invariant is that this specific family of examples is in a sense maximal in the moduli space $\mathfrak{M}(\,\mathfrak{so}\,V\,)_\infty$: Every vector formally tangent to the moduli space $\mathfrak{M}(\,\mathfrak{so}V\,)_\infty$ in a point corresponding to a Riemannian homogeneous space in Meusers' family is integrable to a real deformation by changing one of the parameters $f_5, \ldots, f_m - 1 \neq 0$ of this family.

A Locally Homogeneous Spaces

Intuitively a locally or infinitesimally homogeneous space of contact order $d \in \mathbb{N}_0$ is a smooth manifold endowed with a connection on its tangent bundle such that the d–th order jets of the curvature and torsion tensors look the same in all of its points. Despite the fact that no Lie group is acting transitively on a generic locally homogeneous space, its curvature and torsion tensors still share some of the algebraic properties of the curvature–torsion of formal affine homogeneous spaces encoded in the moduli spaces constructed in this article. Sketching the construction of $\mathfrak{M}(\,\mathfrak{gl}\,V\,)$ from the point of view of locally homogeneous spaces we derive a rather simple differential equation characterizing locally homogeneous spaces which can be used to study old and construct new examples of such affine manifolds.

Recall to begin with that the curvature and the torsion of a connection ∇ on the tangent bundle of a manifold M are tensors as are all their covariant derivatives. All these tensors can be pulled back under linear isomorphisms $F : V \longrightarrow T_pM$, for the curvature say via

$$F^*[\, R^M_p \,]_{x,y}z \;:=\; F^{-1}(\, R^M_p\,)_{Fx,\,Fy}Fz$$

with a very similar formula for the torsion. An affine manifold $(\, M,\, \nabla\,)$ is called locally homogeneous of contact order $d \in \mathbb{N}_0$ provided for every two points $p,\, q \in M$ there exists a linear isomorphism $F_{q,\,p} : T_qM \longrightarrow T_pM$ between the tangent spaces such that

$$F^*[\, (\nabla^r R^M)_p \,] \;=\; (\nabla^r R^M)_q \qquad\qquad F^*[\, (\nabla^r T^M)_p \,] \;=\; (\nabla^r T^M)_q \tag{16}$$

for all $r = 0, \ldots, d$. Choosing a model vector space V of dimension $m = \dim M$ as before we translate the condition of M being locally homogeneous of order d into the existence of a reduction of the full frame bundle $\mathrm{Frame}(\, V,\, TM\,)$ to a subbundle $\mathrm{Frame}^d(\, V,\, TM\,)$

$$\{\; (\,p, F\,) \;|\; F^*[\,(\nabla^r T^M)_p\,] \;=\; \nabla^r T \text{ and } F^*[\,(\,\nabla^r R^M\,)_p\,] \;=\; \nabla^r R \text{ for } r = 0, \ldots, d \;\} \tag{17}$$

with $R,\, \nabla R,\, \ldots,\, \nabla^d R$ and $T,\, \nabla T,\, \ldots,\, \nabla^d T$ defined for $r = 0, \ldots, d$ as the pull backs

$$\begin{aligned} \nabla^r T \;&:=\; F^*_\circ[\,(\,\nabla^r T^M\,)_{p_\circ}\,] \;\in\; \bigotimes\nolimits^r V^* \otimes (\,\Lambda^2 V^* \otimes V\,) \\ \nabla^r R \;&:=\; F^*_\circ[\,(\,\nabla^r R^M\,)_{p_\circ}\,] \;\in\; \bigotimes\nolimits^r V^* \otimes (\,\Lambda^2 V^* \otimes \mathrm{End}\; V\,) \end{aligned} \tag{18}$$

for some arbitrarily chosen point $p_\circ \in M$ and frame $F_\circ : V \longrightarrow T_{p_\circ}M$; the common stabilizer

$$H_d \;:=\; \mathbf{Stab}\,(\, T \oplus \nabla T \oplus \ldots \oplus \nabla^d T\,) \cap \mathbf{Stab}\,(\, R \oplus \nabla R \oplus \ldots \oplus \nabla^d R\,) \;\subset\; \mathbf{GL}\, V$$

of the constants (18) is of course the structure group of the reduction $\mathrm{Frame}^d(\, V,\, TM\,)$. Every smooth local section $F \in \Gamma_{\mathrm{loc}}(\,\mathrm{Frame}^d(V, TM)\,)$ of the reduction (17) of the frame bundle corresponds to a local trivialization of the tangent bundle TM of the manifold

$$F : \; C^\infty_{\mathrm{loc}}(\, M,\, V\,) \stackrel{\cong}{\longrightarrow} \Gamma_{\mathrm{loc}}(\, TM\,), \qquad x \longmapsto \Big(\; Fx : \; M \longrightarrow TM, \;\; p \longmapsto F_p x_p \;\Big)$$

whose inverse $F^{-1} \in \Gamma_{\mathrm{loc}}(\, T^*M \otimes V\,)$ is a locally defined V–valued form on M. In turn the given connection ∇ on the tangent bundle TM of M can be written locally in the form

$$F^{-1}(\, \nabla_{Fx}(\, Fy\,)\,) \;=:\; D_{Fx} y \;+\; A_x y \tag{19}$$

for a smooth map $A \in C^\infty_{\mathrm{loc}}(M, V^* \otimes \mathrm{End}\, V)$ and all $x, y \in C^\infty_{\mathrm{loc}}(M, V)$, where D denotes and will denote the trivial connection on $C^\infty(M, V)$ and similarly on $C^\infty(M, \Sigma)$ for an arbitrary vector space Σ. The connection form A is in essence the pull back $A_{F^{-1}} = F^*\omega^\nabla$ of the principal connection ω^∇ on $\mathrm{Frame}(V, TM)$ describing the connection ∇ by the section F, for simplicity however we will not use principal connections in our argument.

Calculating the torsion T^M and the curvature R^M of the connection ∇ in terms of its local description (19) via the local frame F and the associated connection form A allows us to bring the decisive property of F to be a local section of the reduction (17) to bear. For the torsion for example we obtain for all constant vectors $x, y \in V \subset C^\infty_{\mathrm{loc}}(M, V)$ the identity

$$T(x, y) \overset{!}{=} F^{-1}(T^M(Fx, Fy)) = A_x y - A_y x - F^{-1}([Fx, Fy])$$

due to $D_{Fx} y = 0 = D_{Fy} x$. The analogous calculation for the curvature results in

$$R_{x,y} \overset{!}{=} F^{-1} \circ R^M_{Fx,Fy} \circ F = (D_{Fx}A)_y - (D_{Fy}A)_x + [A_x, A_y] - A_{F^{-1}([Fx,Fy])}$$

for all constant vectors $x, y \in V$ due to $D_{Fx}(A_y) = (D_{Fx}A)_y$ etc. In light of the definition of the approximate curvature tensor in 4.6 the preceding two equations combine into

$$\begin{aligned} F^{-1}([Fx, Fy]) &= A_x y - A_y x - T(x, y) \\ (D_{Fx}A)_y - (D_{Fy}A)_x &= -(Q(A, T) - R)_{x,y} \end{aligned} \tag{20}$$

a coupled system of first order partial differential equations for the pair (F, A). The most important observation to be made to link locally homogeneous spaces to the moduli spaces $\mathfrak{M}(\mathfrak{gl}\, V)$ constructed in the main part of this article is that the iterated covariant derivatives of T^M and R^M can still be calculated via Definition 4.3 using the binary operation $\circledast$

$$\begin{aligned} F^*[\nabla^r T^M] &\overset{!}{=} A \circledast (A \circledast (\ldots (A \circledast T) \ldots)) \in C^\infty_{\mathrm{loc}}(M, \textstyle\bigotimes^r V^* \otimes (\Lambda^2 V^* \otimes V)) \\ F^*[\nabla^r R^M] &\overset{!}{=} A \circledast (A \circledast (\ldots (A \circledast R) \ldots)) \in C^\infty_{\mathrm{loc}}(M, \textstyle\bigotimes^r V^* \otimes (\Lambda^2 V^* \otimes \mathrm{End}\, V)) \end{aligned}$$

for all $r = 0, \ldots, d+1$. This somewhat surprising observation is really a consequence of the construction (17) of the reduction $\mathrm{Frame}^d(V, TM)$ of the frame bundle, according to which $F^*[\nabla^r T^M]$ and $F^*[\nabla^r R^M]$ are actually constant maps equal to the constants defined in (18) for all $r = 0, \ldots, d$. In turn they are parallel with respect to the trivial connection D and so their covariant derivatives are determined by the connection form $A \in C^\infty_{\mathrm{loc}}(M, V^* \otimes \mathrm{End}\, V)$ defined in (19) and can be calculated using the binary operation $\circledast$.

A direct consequence of the preceding observation is that the connection form A itself is constant on M modulo the vector space $V^* \otimes \mathfrak{h}_d$ of forms on V with values in the Lie algebra $\mathfrak{h}_d$ of the common stabilizer $H_d \subset \mathbf{GL}\, V$ of the constants (18), because the argument of Lemma 4.5 would otherwise tell us that $F^*[\,\nabla^r T^M\,]$ and $F^*[\,\nabla^r R^M\,]$ are not constant maps on M contrary to the basic assumption that M is a locally homogeneous space. The left hand side of the second equation of the couple (20) is thus a smooth map on M with values in $V^* \otimes \mathfrak{h}_d$ so that the right hand side difference $Q(A,T) - R \in C^\infty_{\mathrm{loc}}(\,M,\, \Lambda^2 V^* \otimes \mathrm{End}\, V\,)$ satisfies the algebraic equations characterizing the algebraic variety $\mathfrak{M}(\,\mathfrak{gl}\, V\,)$ up to order d:

Definition A.1 (Truncated Moduli Spaces).
The truncated moduli spaces $\mathfrak{M}^{\leq d}(\,\mathfrak{gl}\, V\,)$ are approximations to the variety $\mathfrak{M}(\,\mathfrak{gl}\, V\,)$ of formal affine homogeneous spaces defined for all $d \in \mathbb{N}_0$ as the set of connection–curvature–torsion triples $(\,A,\, R,\, T\,)$ solving the formal first $d^{(A,T)}T = R \wedge \mathrm{id}$ and second Bianchi identity $d^{(A,T)}R = 0$ and the other equations characterizing the algebraic variety $\mathfrak{M}(\,\mathfrak{gl}\, V\,)$

$$\Big(\, Q(A,T) \;-\; R \,\Big) \circledast \Big(\, \underbrace{A \circledast (\, A \circledast (\, \ldots (\, A}_{r \text{ times}} \circledast T\,) \ldots))\,\Big) \;=\; 0$$

$$\Big(\, Q(A,T) \;-\; R \,\Big) \circledast \Big(\, \underbrace{A \circledast (\, A \circledast (\, \ldots (\, A}_{r \text{ times}} \circledast R\,) \ldots))\,\Big) \;=\; 0$$

for all $r = 0, \ldots, d$, but not to infinite order. The stabilizer filtration of Definition 4.4 is still well–defined up to order d for every point the truncated moduli space $\mathfrak{M}^{\leq d}(\,\mathfrak{gl}\, V\,)$:

$$\mathrm{End}\; V \;\supseteq\; \mathfrak{h}_0 \;:=\; \mathfrak{stab}\, R \cap \mathfrak{stab}\, T \;\supseteq\; \mathfrak{h}_1 \;\supseteq\; \ldots \;\supseteq\; \mathfrak{h}_d$$

Summarizing these arguments we recall that the choice of a base frame $F_\circ : V \longrightarrow T_{p_*}M$ in a point $p_\circ$ in a locally homogeneous space M of order $d \in \mathbb{N}_0$ provides us with a curvature–torsion tuple $(\,R,\, T\,)$, which defines the reduction (17) of the frame bundle of M. Choosing a local section F of this reduction of the frame bundle provides us with the connection form $A \in C^\infty_{\mathrm{loc}}(\,M,\, V^* \otimes \mathrm{End}\, V\,)$ which complements the curvature–torsion tuple to a map

$$(\,A,\, R,\, T\,) : \quad M \;\longrightarrow\; \mathfrak{M}^{\leq d}(\,\mathfrak{gl}\, V\,), \qquad p \;\longmapsto\; (\,A_p,\, R,\, T\,)$$

from M to the truncated moduli space, which is actually constant modulo $V^* \otimes \mathfrak{h}_d$. Conversely we may construct new examples of locally homogeneous spaces of order $d \in \mathbb{N}_0$ by choosing first of all a point $(\,A_\circ,\, R,\, T\,) \in \mathfrak{M}^{\leq d}(\,\mathfrak{gl}\, V\,)$ in the truncated moduli space of order d to set up the system (20) of partial differential equations. Every solution $(\,F,\, A\,)$ to this system with a local section $F \in \Gamma_{\mathrm{loc}}(\,\mathrm{Frame}(V, TM)\,)$ of the frame bundle of a manifold M of dimension $\dim V$ and $A \in C^\infty_{\mathrm{loc}}(\,M,\, V^* \otimes \mathrm{End}\, V\,)$ constant equal to $A_\circ$ modulo $V^* \otimes \mathfrak{h}_d$ defines a connection ∇ on the tangent bundle TM by equation (19) under which M becomes a locally homogeneous space of order d with curvature and torsion tensor R and T.

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