Growth on Meromorphic Solutions of Non-linear Delay Differential Equations*

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Abstract

By using Nevanlinna theory and linear algebra, we show that the number one is a lower bound of the hyper-order of any meromorphic solution of a non-linear delay differential equation under certain conditions.

1 Introduction

Nevanlinna theory is the value distribution theory established by R. Nevanlinna, it is a very useful tool for studying both the growth of meromorphic functions in the complex plane $\mathbb{C}$ and meromorphic solutions of differential equations. The well-known mathematician K. Yoshida [18] applied the Nevanlinna theory to extend Malmquist's celebrated work [14] in showing that a first order algebraic differential equation of the form $y' = R(z, y)$, where R is a rational function in y with polynomial coefficients in z, admits a meromorphic (i.e., global) solution, then it must reduce to a Riccati equation. N.Steinmitz [15], Bank and Kaufman [1] independently extended earlier works of Hermite and Painlevé on first order algebraic differential equations $(y')^m = P(y)$ when the corresponding algebraic curves have genus 0 or 1 by using Nevanlinna theory.

The classification of $y'' = R(z, y, y')$ that would yield Painlevé's (I-V) equations has yet to be completed. Recently, A. Eremenko and A. Gabrielov [6],

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Conte, Ng and Wong [4], etc have successfully derived meromorphic solutions out of a set of nonlinear PDE with wide range of physical applications by combining Nevanlinna theory and local series analysis. On the other hand, also recently, Halburd and Korhonen [8] showed, again using Nevanlinna theory, that if the difference equation $y(z+1)+y(z-1)=R(z,y)$ (e.g. R rational in both arguments) admits a finite order meromorphic solution, then the equations must reduce to one of the known discrete-Painlevé equations. In this paper, we study the growth of any meromorphic solution of a non-linear delay differential (or differential difference) equation under certain conditions.

2 Main Results

Take positive integers t and k. For $t+1$ complex numbers $c_0(=0), c_1, ..., c_t$, it is an interesting question to study properties of entire (or meromorphic) solutions of differential (or difference, or differential-difference) equations in the complex plane $\mathbb{C}$,

$$P(f)=\sum_{\mathbf{I}\in\mathcal{I}} a_{\mathbf{I}}\left(\mathbf{f}^{(\mathbf{k})}\right)^{\mathbf{I}}=\sum_{\mathbf{I}} a_{\mathbf{I}}\prod_{l=0}^{t}\left(f_{c_l}^{(\mathbf{k})}\right)^{I_l}=0, \tag{2.1}$$

where $\mathbf{k}=(0,1,...,k)$; $\mathbf{I}=(I_0,...,I_t)$, $I_l=(i_{l0},i_{l1},...,i_{lk})$ are multi-indices of non-negative integers $\mathbb{Z}_+$; $\mathcal{I}$ is a finite set of $\mathbb{Z}_+^{(t+1)(k+1)}$; $\mathbf{f}=(f_{c_0},...,f_{c_t})$ in which f_{c_l} is defined by $f_{c_l}(z)=f(z+c_l)$; $f_{c_l}^{(\mathbf{k})}=\left(f_{c_l},f'_{c_l},...,f_{c_l}^{(k)}\right)$; $a_{\mathbf{I}}$ are non-zero meromorphic functions in $\mathbb{C}$; and where

$$\left(f_{c_l}^{(\mathbf{k})}\right)^{I_l}=f_{c_l}^{i_{l0}}\left(f'_{c_l}\right)^{i_{l1}}\cdots\left(f_{c_l}^{(k)}\right)^{i_{lk}}.$$

Obviously, this kind of problems are closely related to those of delay differential equations. For example, some authors (cf. [7]) are concerned with an investigation of the asymptotic behavior, as $t\to\infty$ of positive nonconstant solutions of the autonomous delay differential equation

$$\frac{dx(t)}{dt}=x(t)\left\{a-\sum_{j=1}^{n} b_j x(t-\tau_j)\right\};\quad t\geq 0 \tag{2.2}$$

and several of its variants where a, b_j, τ_j $(j=1,...,n)$ are positive constants, or the stability and fundamental theory of delay (or functional) differential equations (see, e.g., [5], [10]).

Many complex analysts have investigated some special cases of the question (2.1) by using value distribution theory of Nevanlinna (see e.g. [2], [3]). In particular, fixed a polynomial $p(\not\equiv 0)$ and considered

$$f^n(z)+p(z)f(z+c)=\sum_{l=1}^{s}\beta_l e^{\alpha_l z}, \tag{2.3}$$

which is also called a difference equation of f by some complex analysts, under the following assumptions:

(A) Fix $c \in \mathbb{C}$. Take positive integers n, s with $n \geq s+2$. Let $\beta_1, \beta_2, ..., \beta_s$ be non-zero constants and let $\alpha_1, \alpha_2, ..., \alpha_s$ be distinct non-zero constants satisfying $\frac{\alpha_i}{\alpha_j} \neq n$ for all $i, j \in \{1, 2, ..., s\}$. When $s \geq 5$, one further assumes that $n\alpha_l$ $(5 \leq l \leq s)$ are not linear combinations of $\alpha_1, ..., \alpha_s$ with the weight n over $\{0, 1, ..., n-1\}$, that is,

$$n\alpha_l \neq \langle \hat{\mathbf{m}}, \alpha \rangle = \sum_{j=1}^{s} m_j \alpha_j,\ l = 5, \cdots, s,$$

where $\hat{\mathbf{m}} = (m_1, m_2, \cdots, m_s) \in \{0, 1, ..., n-1\}^s$ and $|\hat{\mathbf{m}}| = n$.

Zhang and Huang [19] proved that any meromorphic solution f on $\mathbb{C}$ of the functional equation (2.3) must satisfy $\sigma_2(f) \geq 1$, where $\sigma_2(f)$ is the hyper-order of f defined by the Nevanlinna characteristic function $T(r, f)$

$$\sigma_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$

In this paper, we will extend the result of Zhang and Huang mentioned above to the following delay differential equation on f

$$f^n(z) f^{(k)}(z) + p(z) f(z+c) = \sum_{l=1}^{s} \beta_l e^{\alpha_l z}, \tag{2.4}$$

which also is called a differential-difference equation of f by some complex analysts, under the following assumptions:

(B) Fix $c \in \mathbb{C}$. Take positive integers n, s, k with $n \geq s+2$. Let $\beta_1, \beta_2, ..., \beta_s$ be non-zero constants and let $\alpha_1, \alpha_2, ..., \alpha_s$ be distinct non-zero constants satisfying $\frac{\alpha_i}{\alpha_j} \neq n+1$ for all $i, j \in \{1, 2, ..., s\}$. When $s \geq 5$, one further assumes that $(n+1)\alpha_l$ $(5 \leq l \leq s)$ are not linear combinations of $\alpha_1, ..., \alpha_s$ with the weight $n+1$ over $\{0, 1, ..., n\}$, that is,

$$(n+1)\alpha_l \neq \langle \mathbf{m}, \alpha \rangle = \sum_{j=1}^{s} m_j \alpha_j,\ l = 5, \cdots, s,$$

where $\mathbf{m} = (m_1, m_2, \cdots, m_s) \in \{0, 1, ..., n\}^s$ and $|\mathbf{m}| = n+1$.

In this paper, we prove the following theorem:

Theorem 2.1. *If $p(\not\equiv 0)$ is a polynomial, then any meromorphic solution f on $\mathbb{C}$ of the delay differential equation* (2.4) *under the assumptions* (B) *must satisfy $\sigma_2(f) \geq 1$.*

If $n < 1+s$, the following example shows Theorem 2.1 is not true.

Example 2.2. *The delay differential equation*

$$f^4(z) f'(z) - 2f\left(z + \frac{\pi}{2}\right) = ie^{5iz} + 3ie^{3iz} - 3ie^{-3iz} - ie^{-5iz}$$

has an entire solution

$$f(z) = e^{iz} + e^{-iz}$$

with $\sigma_2(f) = 0$. For this case, we have $4 = n < s+1 = 5$.

The following example 2.3 shows that the condition $\frac{\alpha_i}{\alpha_j} \neq n+1$ for all $i, j \in \{1, 2, ..., s\}$ is necessary.

Example 2.3. *For $k \geq 1$, the delay differential equation*

$$f^4(z)f^{(k)}(z) - f(z + 6\pi i) = 3^{-k}e^{\frac{5}{3}z} - e^{\frac{z}{3}}$$

has an entire function with $\sigma_2(f) = 0$,

$$f(z) = e^{\frac{1}{3}z}.$$

3 Preliminaries

We assume that the reader is familiar with the standard notations and fundamental results in Nevanlinna theory (see, e.g., [11], [17]). The hyper-exponent of convergence of poles of f is defined by

$$\lambda_2\left(\frac{1}{f}\right) = \limsup_{r\to\infty} \frac{\log\log N(r, f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log n(r, f)}{\log r}.$$

We denote by $S(r, f)$ any real function of growth $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. A meromorphic function α on $\mathbb{C}$ is said to be a small function of f if $T(r, \alpha) = S(r, f)$. The function $P(f)$ defined by left side of (2.1) is called a differential-difference polynomial of f if the coefficients $a_{\mathbf{I}}$ are small functions of f.

The first Lemma is referred to [13, Lemma2.2].

Lemma 3.1. *Let f be a non-constant meromorphic function, let c, h be two complex numbers such that $c \neq h$. If $\sigma_2(f) < 1$, then*

$$m\left(r, \frac{f_h}{f_c}\right) = S(r, f),$$

for all r outside a set of finite logarithmic measure, where $f_h(z) = f(z + h), f_c(z) = f(z + c)$.

Take complex numbers $d_0 (= 0), d_1, ..., d_t$. Let $R(f)$ be a differential-difference polynomial of f defined by

$$R(f) = \sum_{\mathbf{J}\in\mathcal{J}} b_{\mathbf{J}} \prod_{l=0}^{t} \left(f_{d_l}^{(\mathbf{k})}\right)^{J_l}, \tag{3.1}$$

where $\mathbf{k} = (0, 1, ..., k)$; $\mathbf{J} = (J_0, ..., J_t)$, $J_l = (j_{l0}, j_{l1}, ..., j_{lk})$ are multi-indices of non-negative integers $\mathbb{Z}_+$; $\mathcal{J}$ is a finite set of $\mathbb{Z}_+^{(t+1)(k+1)}$, and where $b_{\mathbf{J}}$ are non-zero small functions of f. For complex numbers $e_0 (= 0), e_1, ..., e_t$, we use $Q(f)$ to denote a difference polynomial of f as follows:

$$Q(f) = \sum_{\mathbf{K}\in\mathcal{K}} C_{\mathbf{K}} f_{e_0}^{K_0} \cdots f_{e_t}^{K_t}, \tag{3.2}$$

where $\mathbf{K} = (K_0, ..., K_t)$ are multi-indices of non-negative integers $\mathbb{Z}_+$; $\mathcal{K}$ is a finite set of $\mathbb{Z}_+^{t+1}$, and where $C_{\mathbf{K}}$ are non-zero small functions of f. Next we consider the following equation

$$R(f)Q(f) = P(f), \tag{3.3}$$

$P(f)$ is a differential-difference polynomial defined by the left side of (2.1).

The second lemma is a variant of the result due to Laine and Yang [12].

Lemma 3.2. *Let f be a transcendental meromorphic solution of hyper-order $\sigma_2(f) < 1$ of the equation (3.3) with $\deg P(f) \leq \deg Q(f)$. Assume that there is only unique monomial of degree $\deg Q(f)$ in $Q(f)$. Then,*

$$m(r, R(f)) = S(r, f)$$

holds possibly outside an exceptional set of finite logarithmic measure.

Proof. Set $n = \deg Q(f)$ and put

$$|\mathbf{I}| = |I_0| + \cdots + |I_t|,\ |I_l| = i_{l0} + \cdots + i_{lk}.$$

Note that

$$\deg P(f) = \max_{\mathbf{I}\in\mathcal{I}} |\mathbf{I}| \leq \deg Q(f) = \max_{\mathbf{K}\in\mathcal{K}} |\mathbf{K}|. \tag{3.4}$$

Rewrite $Q(f)$ into the following form

$$Q(f) = \sum_{\eta=0}^{n} \widetilde{C}_\eta f^\eta, \tag{3.5}$$

where

$$\widetilde{C}_\eta = \sum_{|\mathbf{K}|=\eta} C_{\mathbf{K}} \left(\frac{f_{e_0}}{f}\right)^{K_0} \cdots \left(\frac{f_{e_t}}{f}\right)^{K_t}.$$

In particular, by the assumption, we have

$$\widetilde{C}_n = C_{\mathbf{K}} \left(\frac{f_{e_0}}{f}\right)^{K_0} \cdots \left(\frac{f_{e_t}}{f}\right)^{K_t}$$

with $|\mathbf{K}| = n$. By Lemma 3.1, we obtain

$$m\left(r, \widetilde{C}_\eta\right) = S(r, f),\ \eta = 0, ..., n \tag{3.6}$$

for $\varepsilon > 0$ small enough, as well as

$$m\left(r, \frac{1}{\widetilde{C}_n}\right) = S(r, f), \tag{3.7}$$

for all r outside a set of finite logarithmic measure.

Making use of the reasoning in [16], we first define

$$\widetilde{c}(z) := \max_{1\leq\eta\leq n} \left(1, 2\left|\frac{\widetilde{C}_{n-\eta}}{\widetilde{C}_\eta}\right|^{\frac{1}{\eta}}\right). \tag{3.8}$$

Although $\widetilde{c}$ is not meromorphic, however we may estimate $m(r,\widetilde{c})$,

$$m(r,\widetilde{c}) \le \sum_{\eta=0}^{n} m\left(r,\widetilde{C}_\eta\right) + m\left(r,\frac{1}{\widetilde{C}_n}\right) + O(1) = S(r,f).$$

Take $z \in \mathbb{C}$ and write $z = re^{i\theta}$. Set

$$E_1 := \left\{\theta \in [0,2\pi) : \left|f(re^{i\theta})\right| \le \widetilde{c}(re^{i\theta})\right\},\ E_2 := [0,2\pi)\backslash E_1. \tag{3.9}$$

In the set E_1, we have the following estimate

$$\begin{aligned} |R(f)| &\le \sum_{\mathbf{J}\in\mathcal{J}} |b_{\mathbf{J}}|\,|f|^{|\mathbf{J}|} \prod_{l=0}^{t} \left|\frac{f_{d_l}}{f}\right|^{j_{l0}} \cdots \left|\frac{f_{d_l}^{(k)}}{f}\right|^{j_{lk}} \\ &\le \sum_{\mathbf{J}\in\mathcal{J}} |b_{\mathbf{J}}|\,|\widetilde{c}|^{|\mathbf{J}|} \prod_{l=0}^{t} \left|\frac{f_{d_l}}{f}\right|^{j_{l0}} \cdots \left|\frac{f_{d_l}^{(k)}}{f}\right|^{j_{lk}} \\ &\le |\widetilde{c}|^{\varsigma} \sum_{\mathbf{J}\in\mathcal{J}} |b_{\mathbf{J}}| \prod_{l=0}^{t} \left|\frac{f_{d_l}}{f}\right|^{j_{l0}} \cdots \left|\frac{f_{d_l}^{(k)}}{f}\right|^{j_{lk}}, \end{aligned} \tag{3.10}$$

where

$$\varsigma = \deg R(f) = \max_{\mathbf{J}\in\mathcal{J}} |\mathbf{J}|.$$

In the set E_2, noting that

$$|f| > \widetilde{c} \ge 2\left|\frac{\widetilde{C}_{n-\eta}}{\widetilde{C}_n}\right|^{\frac{1}{\eta}},$$

and hence

$$\left|\frac{\widetilde{C}_{n-\eta}}{\widetilde{C}_n}\right| \le \frac{|f|^\eta}{2^\eta}$$

for $\eta = 1,...,n$, which means

$$|Q(f)| = \left|\sum_{\eta=0}^{n} \widetilde{C}_\eta f^\eta\right| \ge \left|\widetilde{C}_n f^n\right| \left(1 - \sum_{\eta=1}^{n} \frac{|\widetilde{C}_{n-\eta}|}{|\widetilde{C}_n f^\eta|}\right) \ge \frac{\left|\widetilde{C}_n\right| |f|^n}{2^n},$$

we also obtain an estimate

$$\begin{aligned} |R(f)| = \left|\frac{P(f)}{Q(f)}\right| &\le \frac{2^n}{\left|\widetilde{C}_n\right| |f|^n} \sum_{\mathbf{I}\in\mathcal{I}} |a_{\mathbf{I}}|\,|f|^{|\mathbf{I}|} \prod_{l=0}^{t} \left|\frac{f_{c_l}}{f}\right|^{i_{l0}} \cdots \left|\frac{f_{c_l}^{(k)}}{f}\right|^{i_{lk}} \\ &= \frac{2^n}{\left|\widetilde{C}_n\right|} \sum_{\mathbf{I}\in\mathcal{I}} |a_{\mathbf{I}}||f|^{|\mathbf{I}|-n} \prod_{l=0}^{t} \left|\frac{f_{c_l}}{f}\right|^{i_{l0}} \cdots \left|\frac{f_{c_l}^{(k)}}{f}\right|^{i_{lk}} \\ &\le \frac{2^n}{\left|\widetilde{C}_n\right|} \sum_{\mathbf{I}\in\mathcal{I}} |a_{\mathbf{I}}| \prod_{l=0}^{t} \left|\frac{f_{c_l}}{f}\right|^{i_{l0}} \cdots \left|\frac{f_{c_l}^{(k)}}{f}\right|^{i_{lk}}, \end{aligned} \tag{3.11}$$

since $|\mathbf{I}| \leq \deg(P(f)) \leq n$ and $|f| \geq 1$.

Combing (3.10) and (3.11), we obtain a complete estimate

$$\begin{aligned}|R(f)| \leq & |\tilde{c}|^{\varsigma} \sum_{\mathbf{J}\in\mathcal{J}} |b_{\mathbf{J}}| \prod_{l=0}^{t} \left|\frac{f_{d_l}}{f}\right|^{j_{l0}} \cdots \left|\frac{f_{d_l}^{(k)}}{f}\right|^{j_{lk}} \\ & + \frac{2^n}{\left|\tilde{C}_n\right|} \sum_{\mathbf{I}\in\mathcal{I}} |a_{\mathbf{I}}| \prod_{l=0}^{t} \left|\frac{f_{c_l}}{f}\right|^{i_{l0}} \cdots \left|\frac{f_{c_l}^{(k)}}{f}\right|^{i_{lk}},\end{aligned}$$

which yields immediately

$$\begin{aligned}m(r, R(f)) \leq & \varsigma m(r, \tilde{c}) + m\left(r, \frac{1}{\tilde{C}_n}\right) + \sum_{\mathbf{I}\in\mathcal{I}} m(r, a_{\mathbf{I}}) + \sum_{\mathbf{J}\in\mathcal{J}} m(r, b_{\mathbf{J}}) \\ & + \sum_{\mathbf{I}\in\mathcal{I}} \sum_{l=0}^{t} \left[i_{l0} m\left(r, \frac{f_{c_l}}{f}\right) + \cdots + i_{lk} m\left(r, \frac{f_{c_l}^{(k)}}{f}\right)\right] \\ & + \sum_{\mathbf{J}\in\mathcal{J}} \sum_{l=0}^{t} \left[j_{l0} m\left(r, \frac{f_{d_l}}{f}\right) + \cdots + j_{lk} m\left(r, \frac{f_{d_l}^{(k)}}{f}\right)\right] + O(1).\end{aligned}$$

Note that

$$m\left(r, \frac{f_{\delta}^{(v)}}{f}\right) \leq m\left(r, \frac{f_{\delta}^{(v)}}{f^{(v)}}\right) + m\left(r, \frac{f^{(v)}}{f}\right), \ \delta \in \mathbb{C}, v = 1, 2, \cdots, k.$$

Applying (3.7), (3.9), Lemma 2.1 and logarithmic derivative lemma to the inequality on $m(r, R(f))$, it follows that

$$m(r, R(f)) = S(r, f)$$

since $a_{\mathbf{I}}, b_{\mathbf{J}}$ are small functions of f. Hence Lemma 3.2 is proved. ∎

To state next lemma, we introduce some notations. The determinant

$$V_{n0} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_n \\ \cdots & \cdots & \cdots & \cdots \\ d_1^{n-1} & d_2^{n-1} & \cdots & d_n^{n-1} \end{vmatrix}$$

is called the principal Vandermondian, which is determined by

$$V_{n0} = \prod_{1\leq j<i\leq n} (d_i - d_j).$$

For every $k = 1, 2, ..., n-1$, the determinant

$$V_{nk} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_n \\ \cdots & \cdots & \cdots & \cdots \\ d_1^{n-k-1} & d_2^{n-k-1} & \cdots & d_n^{n-k-1} \\ d_1^{n-k+1} & d_2^{n-k+1} & \cdots & d_n^{n-k+1} \\ \cdots & \cdots & \cdots & \cdots \\ d_1^{n} & d_2^{n} & \cdots & d_n^{n} \end{vmatrix}$$

is called the secondary Vandermondian. The coefficients G_k defined by

$$(x-d_1)\cdots(x-d_n)=x^n+\sum_{k=1}^{n}(-1)^k G_k x^{n-k}$$

are called elementary (or fundamental) symmetric functions (or polynomials) of $d_1,...,d_n$. The following lemma establishes a relationship between V_{n0} and V_{nk}, which is referred to [9].

Lemma 3.3. *The k-th elementary symmetric function G_k of the n variables $d_1,d_2,...,d_n$ is equal to the quotient of the secondary Vandermondian V_{nk} by the principal Vandermondian V_{n0}.*

In the final lemma, the elementary row transformations consist of the following three cases: (i) switch two rows; (ii) multiply a row by a non-zero number; (iii) add to a row by a multiple of another row.

Lemma 3.4. *Take positive integers n,k and s with $n\geq 2$. Let $c_l(1\leq l\leq s)$ be constants and let $b_j(1\leq j\leq 4)$ be rational functions. If there exist distinct nonzero constants $\alpha_l(1\leq l\leq s)$ satisfying $(n+1)\alpha_j\neq\alpha_l(1\leq j\leq 4, 1\leq l\leq s)$ such that*

$$\left(\sum_{j=1}^{4}b_j(z)e^{\alpha_j z}\right)^n\sum_{j=1}^{4}\alpha_j^k b_j(z)e^{\alpha_j z}=\sum_{l=1}^{s}c_l e^{\alpha_l z} \tag{3.12}$$

holds, then we have $b_j=0(1\leq j\leq 4)$.

Proof. It follows from (3.12) that

$$\sum_{l=1}^{s}c_l e^{\alpha_l z}=\sum_{j=1}^{4}\alpha_j^k b_j^{n+1}(z)e^{(n+1)\alpha_j z}+\sum_{|\widetilde{\mathbf{m}}|=n+1}c_{\widetilde{\mathbf{m}}}(z)e^{\langle\widetilde{\mathbf{m}},\widetilde{\alpha}\rangle z}, \tag{3.13}$$

where $\widetilde{\mathbf{m}}=(m_1,m_2,m_3,m_4)\in\{0,1,...,n\}^4$, $c_{\widetilde{\mathbf{m}}}$ are rational functions, and

$$\langle\widetilde{\mathbf{m}},\widetilde{\alpha}\rangle=\sum_{j=1}^{4}m_j\alpha_j.$$

Now we claim that there is some $i\in\{1,2,3,4\}$ such that $(n+1)\alpha_i$ is not a linear combination of $\alpha_1,...,\alpha_4$ with the weight $n+1$ over $\{0,1,...,n\}$. Otherwise, for each $i\in\{1,2,3,4\}$ there exist non-negative integers $d_{ij}\in\{0,1,...,n\}(j=1,2,3,4)$ with $d_{i1}+d_{i2}+d_{i3}+d_{i4}=n+1$, such that

$$\begin{cases}(n+1)\alpha_1=d_{11}\alpha_1+d_{12}\alpha_2+d_{13}\alpha_3+d_{14}\alpha_4,\\(n+1)\alpha_2=d_{21}\alpha_1+d_{22}\alpha_2+d_{23}\alpha_3+d_{24}\alpha_4,\\(n+1)\alpha_3=d_{31}\alpha_1+d_{32}\alpha_2+d_{33}\alpha_3+d_{34}\alpha_4,\\(n+1)\alpha_4=d_{41}\alpha_1+d_{42}\alpha_2+d_{43}\alpha_3+d_{44}\alpha_4.\end{cases} \tag{3.14}$$

Next we will deduce a contradiction from the system (3.14).

If there is some $i \in \{1,2,3,4\}$ such that only one of three integers d_{ij} $(j \neq i)$ is greater than zero, say $d_{21} > 0$, so that $d_{2j} = 0$ $(j \neq 2)$, then from the second equation of system (3.14), we see $(n+1-d_{22})\alpha_2 = d_{21}\alpha_1$. Since $d_{21}+d_{22} = n+1$ and $d_{21} \neq 0$, we obtain $\alpha_1 = \alpha_2$. This is a contradiction.

Hence for each $i \in \{1,2,3,4\}$, at least two of three integers d_{ij} $(j \neq i)$ are greater than zero, so that when $j \neq i$, we have $d_{ij} < n+1-d_{ii}$. We rewrite the system (3.14) as follows:

$$\begin{cases} (d_{11}-n-1)\alpha_1 + d_{12}\alpha_2 + d_{13}\alpha_3 + d_{14}\alpha_4 = 0, \\ d_{21}\alpha_1 + (d_{22}-n-1)\alpha_2 + d_{23}\alpha_3 + d_{24}\alpha_4 = 0, \\ d_{31}\alpha_1 + d_{32}\alpha_2 + (d_{33}-n-1)\alpha_3 + d_{34}\alpha_4 = 0, \\ d_{41}\alpha_1 + d_{42}\alpha_2 + d_{43}\alpha_3 + (d_{44}-n-1)\alpha_4 = 0, \end{cases} \tag{3.15}$$

and denote the matrix of coefficients of system (3.15) by

$$B = \begin{pmatrix} d_{11}-n-1 & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22}-n-1 & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33}-n-1 & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44}-n-1 \end{pmatrix}.$$

Then we claim that the rank of B is 3.

Adding columns 2, 3 and 4 to column 1, and noting that $d_{i1}+d_{i2}+d_{i3}+d_{i4} = n+1(i = 1,2,3,4)$, we find $\det(B) = 0$. Next we discuss minor determinants of order 3 in B by distinguishing two cases.

Case 1. $d_{13} = d_{23} = d_{43} = 0$.

This case implies $d_{12} > 0, d_{14} > 0, d_{21} > 0, d_{24} > 0, d_{41} > 0, d_{42} > 0$. For the 3×3 submatrix

$$B_1 = \begin{pmatrix} d_{11}-n-1 & d_{12} & d_{14} \\ d_{31} & d_{32} & d_{34} \\ d_{41} & d_{42} & d_{44}-n-1 \end{pmatrix}$$

of the matrix B, we have

$$\begin{aligned} \det(B_1) =& (d_{11}-n-1)d_{32}(d_{44}-n-1) + d_{12}d_{34}d_{41} + d_{14}d_{31}d_{42} \\ &- d_{14}d_{32}d_{41} - d_{12}d_{31}(d_{44}-n-1) - (d_{11}-n-1)d_{34}d_{42}. \end{aligned}$$

Hence when $d_{32} > 0$, it follows that

$$\det(B_1) > d_{12}d_{34}d_{41} + d_{14}d_{31}d_{42} + d_{12}d_{31}d_{41} + d_{14}d_{34}d_{42} \geq 0$$

since $d_{14} < n+1-d_{11}$ and $d_{41} < n+1-d_{44}$.

If $d_{32} = 0$, we have $d_{31} > 0$, $d_{34} > 0$, and hence

$$\det(B_1) > d_{12}d_{34}d_{41} + d_{14}d_{31}d_{42} + d_{12}d_{31}d_{41} + d_{14}d_{34}d_{42} > 0.$$

Thus, we proved $\det(B_1) > 0$ in this case.

Case 2. At least one of d_{13}, d_{23}, d_{43} is greater than zero.

Now we consider the 3×3 submatrix

$$B_2 = \begin{pmatrix} d_{11}-n-1 & d_{12} & d_{13} \\ d_{21} & d_{22}-n-1 & d_{23} \\ d_{41} & d_{42} & d_{43} \end{pmatrix}$$

of the matrix B with

$$\begin{aligned} \det(B_2) =&(d_{11}-n-1)(d_{22}-n-1)d_{43} + d_{12}d_{23}d_{41} + d_{13}d_{21}d_{42} \\ &- d_{13}(d_{22}-n-1)d_{41} - d_{12}d_{21}d_{43} - (d_{11}-n-1)d_{23}d_{42}. \end{aligned}$$

Hence when $d_{43} > 0$, it follows that

$$\det(B_2) > d_{12}d_{23}d_{41} + d_{13}d_{21}d_{42} + d_{13}d_{21}d_{41} + d_{12}d_{23}d_{42} \geq 0$$

since $d_{12} < n+1-d_{11}$ and $d_{21} < n+1-d_{22}$. However, if $d_{43} = 0$, we have $d_{41} > 0, d_{42} > 0$, and hence

$$\det(B_2) > d_{12}d_{23}d_{41} + d_{13}d_{21}d_{42} + d_{13}d_{21}d_{41} + d_{12}d_{23}d_{42} > 0$$

because at least one of d_{13}, d_{23} is greater than zero.

Therefore, we proved $\text{rank}(B) = 3$. By using elementary row transformations, we can deduce the matrix B into the form

$$D = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

Then (3.15) and (3.16) yield $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. This is a contradiction. Hence (3.14) does not hold.

Without loss of generality, we may assume that $(n+1)\alpha_4$ is not a linear combination of $\alpha_1, ..., \alpha_4$ with the weight $n+1$ over $\{0, 1, ..., n\}$, that is,

$$(n+1)\alpha_4 \neq m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 + m_4\alpha_4 \tag{3.17}$$

for all $m_1, m_2, m_3, m_4 \in \{0, 1, ..., n\}$ such that $m_1 + m_2 + m_3 + m_4 = n+1$. Noting that $(n+1)\alpha_4 \neq \alpha_l (l = 1, 2, ..., s), \alpha_4 \neq \alpha_l \ (l = 1, 2, 3)$ and (3.17), then multiplying (3.13) by $e^{-(n+1)\alpha_4 z}$, we see that $\alpha_4^k b_4^{n+1}$ is a linear combination of exponential functions, thus $b_4 = 0$ by comparing its growth.

Thus, equation (3.12) becomes

$$\left(\sum_{j=1}^{3} b_j(z)e^{\alpha_j z}\right)^n \sum_{j=1}^{3} \alpha_j^k b_j(z)e^{\alpha_j z} = \sum_{l=1}^{s} c_l e^{\alpha_l z}.$$

Repeating above arguments, it is same to show one of $\{b_1, b_2, b_3\}$, say b_3, is zero, so that the equation (3.12) further becomes

$$\left(\sum_{j=1}^{2} b_j(z)e^{\alpha_j z}\right)^n \sum_{j=1}^{2} \alpha_j^k b_j(z)e^{\alpha_j z} = \sum_{l=1}^{s} c_l e^{\alpha_l z}.$$

We can deduce $b_2 = b_1 = 0$ similarly. Hence Lemma 3.4 follows. ∎

4 Proof of Theorem 2.1

Suppose that the equation (2.4) has a meromorphic solution f with $\sigma_2(f) < 1$. We will deduce contradictions by distinguishing two cases.

Case 1. f has at least one pole.

Let z_0 be a pole of f with multiplicity q (≥ 1). For the case $c = 0$, we get a contradiction by comparing the multiplicities of the pole z_0 at both sides of (2.4). If $c \neq 0$, it follows from (2.4) that $z_0 + c$ is also a pole of f with multiplicity $\geq (n+1)q + k$. Substituting $z + c$ into (2.4), we get

$$f^n(z+c)f^{(k)}(z+c) + p(z+c)f(z+2c) = \sum_{l=1}^{s} \beta_l e^{\alpha_l(z+c)}. \tag{4.1}$$

It follows from (4.1) that $z_0 + 2c$ also is a pole of f with a multiplicity $\geq (n+1)^2 q + k(n+1) + k$ since $z_0 + c$ is a pole of $f^n f^{(k)}$ with a multiplicity $\geq (n+1)^2 q + k(n+1) + k$. By using induction, we know that for each integer $j \geq 1$, the point $z_0 + jc$ is a pole of f with a multiplicity $\geq (n+1)^j q + k\left[(n+1)^{j-1} + (n+1)^{j-2} + \cdots + 1\right]$. Hence for each integer $m \geq 1$, we get an estimate on the number $n(r, f)$ of poles of f in the disc $|z| \leq r$ as follows:

$$n(r_m, f) \geq q + \sum_{j=1}^{m} (n+1)^j q + k\left[(n+1)^{j-1} + (n+1)^{j-2} + \cdots + 1\right],$$

where $r_m = m|c| + |z_0| + 1$. Thus, we have

$$\begin{aligned}\sigma_2(f) \geq \lambda_2\left(\frac{1}{f}\right) &= \limsup_{r\to\infty} \frac{\log\log n(r,f)}{\log r} \\ &\geq \limsup_{m\to\infty} \frac{\log\log n(r_m,f)}{\log r_m} \\ &\geq \limsup_{m\to\infty} \frac{\log\log (n+1)^m}{\log m} = 1\end{aligned}$$

since $n \geq 2 + s \geq 3$. It contradicts with $\sigma_2(f) < 1$.

Case 2. f is an entire function.

If f is a polynomial, by comparing the growth at both sides of the equation (2.4), we find the order of the function at left side of (2.4) is 0, but the order of the function at right side of (2.4) is 1. It is a contradiction. Hence f is transcendental. Further, we divide our discussion into two subcases:

Subcase 2.1. $s = 1$. Now the equation (2.4) becomes

$$f(z)^n f^{(k)}(z) + p(z)f_c(z) = \beta_1 e^{\alpha_1 z}. \tag{4.2}$$

Differentiating both sides of (4.2), we get

$$nf(z)^{n-1}f'(z)f^{(k)}(z) + f(z)^n f^{(k+1)}(z) + (p(z)f_c(z))' = \alpha_1\beta_1 e^{\alpha_1 z}.$$

Combining this equation with (4.2), we get

$$f^{n-1}F = \alpha_1 p f_c - (pf_c)', \tag{4.3}$$

where $F = nf'f^{(k)} - \alpha_1 ff^{(k)} + ff^{(k+1)}$.

If $F \neq 0$, it follow from (4.3) and Lemma 3.2 that

$$T(r,F) = m(r,F) = m\left(r, \frac{\alpha_1 pf_c - (pf_c)'}{f^{n-1}}\right) = S(r,f), \tag{4.4}$$

$$T(r,fF) = m(r,fF) = m\left(r, \frac{\alpha_1 pf_c - (pf_c)'}{f^{n-2}}\right) = S(r,f) \tag{4.5}$$

since $n \geq 2+s = 3$. Combining (4.4) with (4.5), we get

$$T(r,f) \leq T(r,fF) + T\left(r, \frac{1}{F}\right) = T(r,fF) + T(r,F) + O(1) = S(r,f).$$

This is a contradiction.

When $F = 0$, or equivalently $n\frac{f'}{f} + \frac{f^{(k+1)}}{f^{(k)}} = \alpha_1$, then by integrating, it follows that $f^n(z)f^{(k)}(z) = \tau_1 e^{\alpha_1 z}$, where τ_1 is a nonzero constant. Combining this equation with equation (4.2) , we get

$$f_c(z) = \frac{\beta_1 - \tau_1}{p(z)} e^{\alpha_1 z}, \tag{4.6}$$

which means that $\beta_1 \neq \tau_1$ and $p(z)$ is a nonzero constant, say τ_2, since f is a transcendental entire function. Thus we obtain

$$f(z) = \frac{\beta_1 - \tau_1}{\tau_2} e^{\alpha_1(z-c)} = \tau e^{\alpha_1 z},$$

where $\tau = \frac{\beta_1-\tau_1}{\tau_2} e^{-\alpha_1 c}$ is a nonzero constant. It contradicts with $f^n(z)f^{(k)}(z) = \tau_1 e^{\alpha_1 z}$ since $k \geq 1, n \geq s+2 \geq 3$.

Subcase 2.2. $s > 1$. Differentiating both sides of the equation (2.4), we obtain

$$G'(z) = \sum_{l=1}^{s} \alpha_l \beta_l e^{\alpha_l z}, \cdots, G^{(s-1)}(z) = \sum_{l=1}^{s} \alpha_l^{s-1} \beta_l e^{\alpha_l z},$$

where $G = f^n f^{(k)} + pf_c$. Combining these equations with (2.4) and using Cramer's Rule, we find

$$\beta_1 e^{\alpha_1 z} = \frac{E_1(z)}{E},$$

where

$$E_1(z) = \begin{vmatrix} G(z) & 1 & \cdots & 1 \\ G'(z) & \alpha_2 & \cdots & \alpha_s \\ G''(z) & \alpha_2^2 & \cdots & \alpha_s^2 \\ \cdots & \cdots & \cdots & \cdots \\ G^{(s-1)}(z) & \alpha_2^{s-1} & \cdots & \alpha_s^{s-1} \end{vmatrix}, \tag{4.7}$$

$$E = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_s^2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{s-1} & \alpha_2^{s-1} & \cdots & \alpha_s^{s-1} \end{vmatrix} = \prod_{1 \leq j < i \leq s} (\alpha_i - \alpha_j) \neq 0. \tag{4.8}$$

By expanding determinant (4.7) along the first column, we get a relation

$$\beta_1 e^{\alpha_1 z} = \frac{1}{E}\sum_{j=0}^{s-1}(-1)^{s-j+1} M_{s-j,1} G^{(s-j-1)}(z), \tag{4.9}$$

where $M_{s-j,1}(j = 0, 1, ..., s-1)$ is the determinant formed by throwing away the first column and $(s-j)$-th row from the determinant (4.7), that is,

$$M_{s,1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_2 & \alpha_3 & \cdots & \alpha_s \\ \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_s^2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_2^{s-2} & \alpha_3^{s-2} & \cdots & \alpha_s^{s-2} \end{vmatrix}, \tag{4.10}$$

$$M_{s-j,1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_2 & \alpha_3 & \cdots & \alpha_s \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_2^{s-j-2} & \alpha_3^{s-j-2} & \cdots & \alpha_s^{s-j-2} \\ \alpha_2^{s-j} & \alpha_3^{s-j} & \cdots & \alpha_s^{s-j} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_2^{s-1} & \alpha_3^{s-1} & \cdots & \alpha_s^{s-1} \end{vmatrix}, \ j = 1, 2, ..., s-1. \tag{4.11}$$

From (4.10) and (4.11), we see that $M_{s,1}$ is the principal Vandermondian with variables $\alpha_2, \alpha_3, ..., \alpha_s$. Let μ_j be the elementary symmetric function of $\alpha_2, \alpha_3, ..., \alpha_s$ defined by

$$(x-\alpha_2)\cdots(x-\alpha_s) = x^{s-1} + \sum_{j=1}^{s-1}(-1)^j \mu_j x^{s-1-j}. \tag{4.12}$$

By (4.10), (4.11) and Lemma 3.3, we get

$$\mu_j \equiv \frac{M_{s-j,1}}{M_{s,1}}, \ j = 1, 2, ..., s-1. \tag{4.13}$$

Differentiating both sides of (4.9), we get

$$\alpha_1\beta_1 e^{\alpha_1 z} = \frac{1}{E}\sum_{j=0}^{s-1}(-1)^{s-j+1} M_{s-j,1} G^{(s-j)}(z). \tag{4.14}$$

Eliminating $e^{\alpha_1 z}$ from (4.9) and (4.14), we have

$$M_{s,1}G^{(s)} + \sum_{j=1}^{s-1}(-1)^j \left(M_{s-j,1} + \alpha_1 M_{s-j+1,1}\right) G^{(s-j)} = (-1)^{s+1}\alpha_1 M_{1,1} G. \tag{4.15}$$

Let $L(w)$ be a linear differential operator defined by

$$L(w) = w^{(s)} + \sum_{j=1}^{s-1}\frac{(-1)^j \left(M_{s-j,1} + \alpha_1 M_{s-j+1,1}\right)}{M_{s,1}} w^{(s-j)} + \frac{(-1)^s \alpha_1 M_{1,1}}{M_{s,1}} w. \tag{4.16}$$

It follows from (4.15) and (4.16) that

$$L(f^n f^{(k)}) = -L(pf_c). \tag{4.17}$$

Let ν_j be the elementary symmetric function of $\alpha_1, \alpha_2, ..., \alpha_s$ defined by

$$(x-\alpha_1)\cdots(x-\alpha_s) = x^s + \sum_{j=1}^{s}(-1)^j \nu_j x^{s-j}. \tag{4.18}$$

It follow from (4.12), (4.13) and (4.18) that

$$-\frac{M_{s-1,1}+\alpha_1 M_{s,1}}{M_{s,1}} = -(\mu_1+\alpha_1) = -(\alpha_1+\alpha_2+\cdots+\alpha_s) = -\nu_1,$$

$$\frac{(-1)^j(M_{s-j,1}+\alpha_1 M_{s-j+1,1})}{M_{s,1}} = (-1)^j\left(\mu_j+\alpha_1\mu_{j-1}\right) = (-1)^j\nu_j$$

for $j = 2, 3, ..., s-1$, and

$$\frac{(-1)^s\alpha_1 M_{1,1}}{M_{s,1}} = (-1)^s\alpha_1\mu_{s-1} = (-1)^s(\alpha_1\alpha_2\cdots\alpha_s) = (-1)^s\nu_s.$$

Hence $L(w)$ becomes

$$L(w) = w^{(s)} - \nu_1 w^{(s-1)} + \cdots + (-1)^j\nu_j w^{(s-j)} + \cdots + (-1)^s\nu_s w. \tag{4.19}$$

We deduce inductively

$$\begin{aligned}\left(f^n f^{(k)}\right)^{(m)} &= \sum_{i=0}^{m}\binom{m}{i}(f^n)^{(i)}\left(f^{(k)}\right)^{(m-i)} = \sum_{i=1}^{m}\binom{m}{i}\left(f^{(k)}\right)^{(m-i)}\\ &\quad\cdot\left[nf^{n-1}f^{(i)} + \sum_{j=2}^{i-1}\sum_{\lambda}\gamma_{j\lambda}f^{n-j}\left(f'\right)^{\lambda_{j1}}\left(f''\right)^{\lambda_{j2}}\cdots\left(f^{(i-1)}\right)^{\lambda_{j,i-1}}\right.\\ &\quad\left.+\,n(n-1)\cdots(n-(i-1))f^{n-i}\left(f'\right)^{i}\right] + f^n f^{(k+m)}\end{aligned} \tag{4.20}$$

for $m = 1, 2, ..., s$, where $\gamma_{j\lambda}$ are positive integers, $\lambda_{j1}, \lambda_{j2}, ..., \lambda_{j,i-1}$ are non-negative integers and sum $\sum_\lambda$ is carried out such that $\lambda_{j1}+\lambda_{j2}+\cdots+\lambda_{j,i-1} = j$ and $\lambda_{j1}+2\lambda_{j2}+\cdots+(i-1)\lambda_{j,i-1} = i$. By (4.19) and (4.20), we get

$$L(f^n f^{(k)}) = f^{n-s}\psi, \tag{4.21}$$

where ψ is a differential polynomial in f of degree $s+1$ with constant coefficients. From (4.17), (4.19) and (4.21), we obtain

$$f^{n-s}\psi = -L(pf_c), \tag{4.22}$$

where $L(pf_c)$ is a differential-difference polynomial in f of degree 1 with polynomial coefficients.

If $\psi \neq 0$, then by (4.22) and Lemma 3.2, we get

$$\begin{aligned} T(r,\psi) &= m(r,\psi) = S(r,f), \\ T(r,f\psi) &= m(r,f\psi) = S(r,f). \end{aligned} \tag{4.23}$$

The above two equalities give

$$T(r,f) \leq T(r,f\psi) + T\left(r,\frac{1}{\psi}\right) = S(r,f).$$

This is a contradiction.

If $\psi = 0$, we have $L(f^n f^{(k)}) = 0$ and $L(pf_c) = 0$. Using (4.19), we get

$$L(pf_c) = (pf_c)^{(s)} + \sum_{l=1}^{s}(-1)^l \nu_l (pf_c)^{(s-l)} = 0.$$

The characteristic equation of this equation is

$$\lambda^s - \nu_1\lambda^{s-1} + \cdots + (-1)^j \nu_j \lambda^{s-j} + \cdots + (-1)^s \nu_s = 0. \tag{4.24}$$

Since (4.24) has s distinct roots $\alpha_1, \alpha_2, ..., \alpha_s$, we get that pf_c has the form

$$p(z)f(z+c) = \widetilde{b}_1 e^{\alpha_1 z} + \widetilde{b}_2 e^{\alpha_2 z} + \cdots + \widetilde{b}_s e^{\alpha_s z},$$

where $\widetilde{b}_j$ $(j = 1,2,...,s)$ are constants. So

$$f(z) = \widehat{b}_1(z)e^{\alpha_1 z} + \widehat{b}_2(z)e^{\alpha_2 z} + \cdots + \widehat{b}_s(z)e^{\alpha_s z}, \tag{4.25}$$

where $\widehat{b}_j(z) = \frac{\widetilde{b}_j e^{-\alpha_j c}}{p(z-c)} (j = 1,2,...,s)$ are rational functions.

Similarly, we deduce from $L(f^n f^{(k)}) = 0$ that

$$f(z)^n f^{(k)}(z) = \widetilde{c}_1 e^{\alpha_1 z} + \widetilde{c}_2 e^{\alpha_2 z} + \cdots + \widetilde{c}_s e^{\alpha_s z}, \tag{4.26}$$

where $\widetilde{c}_j$ $(j = 1,2,...,s)$ are constants. From (4.25) and (4.26), we obtain

$$\sum_{l=1}^{s} \widetilde{c}_l e^{\alpha_l z} = \sum_{l=1}^{s} \alpha_l^k \widehat{b}_l^{n+1}(z) e^{(n+1)\alpha_l z} + \sum_{|\mathbf{m}|=n+1} c_{\mathbf{m}}(z) e^{\langle \mathbf{m},\alpha\rangle z}, \tag{4.27}$$

where $\mathbf{m} = (m_1, m_2, \cdots, m_s) \in \{0,1,...,n\}^s$, $c_{\mathbf{m}}$ are rational functions, and

$$\langle \mathbf{m}, \alpha\rangle = \sum_{j=1}^{s} m_j \alpha_j.$$

Since $(n+1)\alpha_s \neq \langle \mathbf{m},\alpha\rangle$, $(n+1)\alpha_s \neq \alpha_l (l = 1,2,..., s)$ and $\alpha_s \neq \alpha_l$ $(l = 1,2,...,s-1)$, then multiplying (4.27) by $e^{-(n+1)\alpha_s z}$, we see that $\alpha_s^k \widehat{b}_s^{n+1}$ is a linear combination of exponential functions, and hence $\widehat{b}_s = 0$ by comparing its growth. Repeating above arguments, it is same to show that

$$\widehat{b}_5 = \widehat{b}_6 = \cdots = \widehat{b}_{s-1} = 0.$$

Then f becomes

$$f(z) = \widehat{b}_1(z)e^{\alpha_1 z} + \widehat{b}_2(z)e^{\alpha_2 z} + \widehat{b}_3(z)e^{\alpha_3 z} + \widehat{b}_4(z)e^{\alpha_4 z}. \tag{4.28}$$

From (4.26), (4.28) and Lemma 3.4, we obtain that $\widehat{b}_1 = \widehat{b}_2 = \widehat{b}_3 = \widehat{b}_4 = 0$, which implies $f = 0$. This is a contradiction. Therefore, the equation (2.4) does not have any entire solution of hyper-order less than one when $s > 1$.

According to the arguments above, we see that any meromorphic solution f of the equation (2.4) must satisfy $\sigma_2(f) \geq 1$. Thus we complete the proof.

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