Periodic orbits of the three dimensional logarithm galactic potential

Daniel Paşca Bogdan Mircea Tătaru

Abstract

We apply the averaging theory of first order to study analytically families of periodic orbits for a three dimensional logarithmic galactic potential $H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{v_0^2}{2}\ln(x^2 - \lambda x^3 + \alpha y^2 + bz^2 + c_b^2)$, that is relevant in the study of elliptic galactic dynamics. We first introduce a scale transformation in the coordinates and momenta with a parameter ε and we find, using averaging theory of first order in ε, the existence up to three periodic orbits if α, β are irrational, and one periodic orbit if either α is irrational and β is rational, or β is irrational and α is rational, for ε sufficiently small.

1 Introduction and statement of the main results

We consider the Hamiltonian

$$H = H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z),$$

where the potential $V = V(x, y, z)$ is given by

$$V = \frac{v_0^2}{2}\ln(x^2 - \lambda x^3 + \alpha y^2 + bz^2 + c_b^2),$$

where α, b are the flattening parameters, c_b is the scale length of the bulge component, while the parameter $\lambda \ll 1$ introduces a small asymmetry in the system (see [2]). The parameter v_0 stands for the consistency of the galactic units.

Received by the editors in November 2016 - In revised form in July 2017.

Communicated by P. De Maesschalck.

2010 *Mathematics Subject Classification* : Primary 34C10, 34C25.

Key words and phrases : three dimensional galactic potential, periodic solution, averaging theory.

The choice of this potential is justified by the fact that triaxilities are common in elliptical galaxies (see e.g. [4], [9], [1], [5], [11]). Due to the fact that $\lambda \ll 1$ we will replace in the following λ by $\varepsilon\lambda$ where ε is a small parameter. So the Hamiltonian will be

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{v_0^2}{2} \ln(x^2 - \varepsilon\lambda x^3 + \alpha y^2 + bz^2 + c_b^2). \quad (1)$$

We start with the Hamiltonian system associated to the logarithmic potentials (1), having the first integral given by the total energy H whose logarithmic Hamiltonian system is given by

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = p_x, \\
\dot{y} &= \frac{\partial H}{\partial p_y} = p_y, \\
\dot{z} &= \frac{\partial H}{\partial p_z} = p_z, \\
\dot{p_x} &= -\frac{\partial H}{\partial x} = -\frac{v_0^2(2x - 3\varepsilon\lambda x^2)}{2(c_b^2 + x^2 + \alpha y^2 + bz^2 - \varepsilon\lambda x^3)}, \\
\dot{p_y} &= -\frac{\partial H}{\partial y} = -\frac{av_0^2 y}{c_b^2 + x^2 + \alpha y^2 + bz^2 - \varepsilon\lambda x^3}, \\
\dot{p_z} &= -\frac{\partial H}{\partial z} = -\frac{bv_0^2 z}{c_b^2 + x^2 + \alpha y^2 + bz^2 - \varepsilon\lambda x^3}.
\end{aligned} \quad (2)$$

We denote the vector field associated to equation (2) by $\mathcal{F}$:

$$\mathcal{F} = \mathcal{F}(x, y, z, p_x, p_y, p_z) := \Big(p_x, p_y, p_z, -\frac{v_0^2(2x - 3\varepsilon\lambda x^2)}{2(c_b^2 + x^2 + \alpha y^2 + bz^2 - \varepsilon\lambda x^3)}, - \frac{av_0^2 y}{c_b^2 + x^2 + \alpha y^2 + bz^2 - \varepsilon\lambda x^3}, -\frac{bv_0^2 z}{c_b^2 + x^2 + \alpha y^2 + bz^2 - \varepsilon\lambda x^3}\Big).$$

After introducing a non-canonical scale transformation in the coordinates and momenta with a parameter ε of the form

$$\{x \to \sqrt{\varepsilon}x, y \to \sqrt{\varepsilon}y, z \to \sqrt{\varepsilon}z, p_x \to \sqrt{\varepsilon}p_x, p_y \to \sqrt{\varepsilon}p_y, p_z \to \sqrt{\varepsilon}p_z\}$$

the Hamiltonian system (2) can be reduced to study the differential system

$$\begin{aligned}
\dot{x} &= p_x, \\
\dot{y} &= p_y, \\
\dot{z} &= p_z, \\
\dot{p_x} &= -\frac{v_0^2 x}{c_b^2} + \varepsilon \frac{v_0^2 x^3 + \alpha v_0^2 x y^2 + bv_0^2 x z^2}{c_b^4} + \mathcal{O}(\varepsilon^2), \\
\dot{p_y} &= -\frac{\alpha v_0^2 y}{c_b^2} + \varepsilon \frac{\alpha v_0^2 x^2 y + \alpha^2 v_0^2 y^3 + \alpha b v_0^2 y z^2}{c_b^4} + \mathcal{O}(\varepsilon^2), \\
\dot{p_z} &= -\frac{bv_0^2 z}{c_b^2} + \varepsilon \frac{bv_0^2 x^2 z + \alpha b v_0^2 y^2 z + b^2 v_0^2 z^3}{c_b^4} + \mathcal{O}(\varepsilon^2),
\end{aligned} \quad (3)$$

having the first integral

$$K = \frac{1}{2}v_0^2 \ln c_b^2 + \varepsilon \frac{c_b^2(p_x^2 + p_y^2 + p_z^2) + v_0^2(x^2 + \alpha y^2 + bz^2)}{2c_b^2} - \varepsilon^2 \frac{v_0^2(x^2 + \alpha y^2 + bz^2)^2}{4c_b^4} + \mathcal{O}(\varepsilon^3).$$

As K is a first integral, also $H = (K - \frac{1}{2}v_0^2 \ln c_b^2)/\varepsilon$ is a first integral, and we are going to use this first integral

$$H = \frac{c_b^2(p_x^2 + p_y^2 + p_z^2) + v_0^2(x^2 + \alpha y^2 + bz^2)}{2c_b^2} - \varepsilon \frac{v_0^2(x^2 + \alpha y^2 + bz^2)^2}{4c_b^4} + \mathcal{O}(\varepsilon^2).$$

Thus, the conditions for finding families of periodic orbits using the averaging theory up to first order in ε, apply for this system.

We apply the *averaging theory* of first order in the small parameter ε to compute periodic orbits of a perturbed periodic differential system depending on ε. We recall in section 2 the basic theorem of this tool: the *Averaging Theorem* of first order. This theorem provides, under certain conditions, perturbed periodic orbits for ε sufficiently small that bifurcate from some unperturbed periodic orbits for $\varepsilon = 0$. The method goes back to [6] and [7], and a shorter proof is given by [3]. For a general introduction to the averaging theory see the books [8] and [10].

We find five families of periodic orbits parameterized by the energy when the parameters α and b are such that $\sqrt{\alpha}$ or/and $\sqrt{b}$ are irrational, all of them bifurcating from unperturbed periodic orbits around the center: one bifurcating from the two-dimensional plane $(x, 0, 0, p_x, 0, 0)$, another one from the two-dimensional plane $(0, y, 0, 0, p_y, 0)$ and another one from the two-dimensional plane $(0, 0, z, 0, 0, p_z)$. When $\sqrt{\alpha}$ and $\sqrt{b}$ are rational, the Averaging Theorem gives no information about periodic orbits.

Our main results on the periodic orbits of the tridimensional logarithmic Hamiltonian systems is summarized in the next three theorems, which is proved in section. 4. We denote the periodic solutions $\gamma_i(t) = (x(t), y(t), z(t), p_x(t), p_y(t), p_z(t))$ for $i = 1, \ldots, 5$.

Theorem 1. *The following statements hold for the perturbed differential system* (3)*:*

(a) For $\varepsilon > 0$ sufficiently small and $\sqrt{\alpha}$ and $\sqrt{b}$ irrational, at every energy level $H = h > 0$, it has at least one periodic solution $\gamma_1(t)$ such that $\gamma_1(0) \to (\frac{c_b\sqrt{2h}}{v_0}, 0, 0, 0, 0, 0)$; at least one periodic solution $\gamma_2(t)$ satisfying $\gamma_2(0) \to (0, \frac{c_b\sqrt{2h}}{\sqrt{\alpha}v_0}, 0, 0, 0, 0)$ and at least one periodic solution $\gamma_3(t)$ such that $\gamma_3(0) \to (0, 0, \frac{c_b\sqrt{2h}}{\sqrt{b}v_0}, 0, 0, 0)$;

(b) For $\varepsilon > 0$ sufficiently small, $\sqrt{\alpha}$ irrational and $\sqrt{b}$ rational, at every energy level $H = h > 0$ it has at least one periodic solution $\gamma_4(t)$ such that $\gamma_4(0) \to (0, \frac{c_b\sqrt{2h}}{\sqrt{\alpha}v_0}, 0, 0, 0, 0)$;

(c) *For* $\varepsilon > 0$ *sufficiently small,* $\sqrt{\alpha}$ *rational and* $\sqrt{b}$ *irrational, at every energy level* $H = h > 0$ *it has at least one periodic* $\gamma_5(t)$ *satisfying* $\gamma_5(0) \to (0, 0, \frac{c_b\sqrt{2h}}{\sqrt{b}v_0}, 0, 0, 0)$.

2 The averaging theory of first order

Now we shall provide the basic results from averaging theory that we need for proving the results of this paper.

We consider the problem of the bifurcation of T-periodic solutions from the differential system

$$\dot{\mathbf{x}}(t) = \mathbf{F}_0(t,\mathbf{x}) + \varepsilon \mathbf{F}_1(t,\mathbf{x}) + \varepsilon^2 \mathbf{F}_2(t,\mathbf{x},\varepsilon), \tag{4}$$

with $\varepsilon \neq 0$ sufficiently small. Here the functions $\mathbf{F}_0, \mathbf{F}_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $\mathbf{F}_2 : \mathbb{R} \times \Omega \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\dot{\mathbf{x}}(t) = \mathbf{F}_0(t,\mathbf{x}) \tag{5}$$

has a k-dimensional submanifold of T-periodic solutions. We assume that the coordinates have been taken in such a way that the k-dimensional submanifold of periodic orbits is contained in $\{x_1, \dots, x_k, 0, \dots, 0\} \in \Omega$. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z})$ be the solution of the unperturbed system (5) such that $\mathbf{x}(0,\mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t,\mathbf{z})$ as

$$\dot{\mathbf{y}}(t) = D_{\mathbf{x}}\mathbf{F}_0(t,\mathbf{x}(t,\mathbf{z}))\mathbf{y}. \tag{6}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (6), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e., $\xi(x_1, \dots, x_n) = (x_1, \dots, x_k)$.

Theorem 2 (Averaging Theorem of first order)**.** *Let* $V \subset \mathbb{R}^k$ *be open and bounded, and let* $\beta_0 : Cl(V) \to \mathbb{R}^{n-k}$ *be a* $\mathcal{C}^2$ *function. We assume that*

(i) $\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)), \alpha \in Cl(V) \subset \Omega\}$ *and that for each* $\mathbf{z}_\alpha \in \mathcal{Z}$ *the solution* $\mathbf{x}(t,\mathbf{z}_\alpha)$ *of (5) is T-periodic;*

(ii) *for each* $\mathbf{z}_\alpha \in \mathcal{Z}$ *there is a fundamental matrix* $M_{\mathbf{z}_\alpha}(t)$ *of (6) such that the matrix* $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ *has in the upper right corner the* $k \times (n-k)$ *zero matrix, and in the lower right corner a* $(n-k) \times (n-k)$ *matrix* Δ_α *with* $\det \Delta_\alpha \neq 0$.

We consider the function $\mathcal{F} : Cl(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t,\mathbf{x}(t,\mathbf{z}_\alpha))dt \right). \tag{7}$$

If there exists $a \in V$ *with* $\mathcal{F}(a) = 0$ *and*

$$\det\left((d\mathcal{F}/d\alpha)(a)\right) \neq 0, \tag{8}$$

then there exists a T-periodic solution $\varphi(t,\varepsilon)$ *of system (4) such that* $\varphi(0,\varepsilon) \to \mathbf{z}_\alpha$ *as* $\varepsilon \to 0$.

3 The scale transformation and the first order differential system

Of course, the Hamiltonian system (2) is not into the normal form (4) for applying the averaging theory. So we first introduce a non-canonical rescaling transformation with a factor $\sqrt{\varepsilon}$ in order to have a small parameter $\varepsilon > 0$ in the differential system

$$\{x \to \sqrt{\varepsilon}x, y \to \sqrt{\varepsilon}y, z \to \sqrt{\varepsilon}z, p_x \to \sqrt{\varepsilon}p_x, p_y \to \sqrt{\varepsilon}p_y, p_z \to \sqrt{\varepsilon}p_z\}.$$

The differential system (2) of the logarithm potential in the rescaled variables is given by

$$\begin{aligned}
\dot{x} &= p_x,\\
\dot{y} &= p_y,\\
\dot{z} &= p_z,\\
\dot{p_x} &= -\frac{v_0^2(2x\sqrt{\varepsilon} - 3x^2\varepsilon^2\lambda)}{2\sqrt{\varepsilon}(c_b^2 + \varepsilon x^2 + \varepsilon\alpha y^2 + \varepsilon b z^2 - \varepsilon^{5/2}\lambda x^3)},\\
\dot{p_y} &= -\frac{\alpha v_0^2 y}{c_b^2 + \varepsilon x^2 + \varepsilon\alpha y^2 + \varepsilon b z^2 - \varepsilon^{5/2}\lambda x^3},\\
\dot{p_z} &= -\frac{b v_0^2 z}{c_b^2 + \varepsilon x^2 + \varepsilon\alpha y^2 + \varepsilon b z^2 - \varepsilon^{5/2}\lambda x^3},
\end{aligned} \tag{9}$$

with the first integral

$$H = \frac{\varepsilon}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{v_0^2}{2}\ln(\varepsilon x^2 - \varepsilon^{5/2}\lambda x^3 + \varepsilon\alpha y^2 + \varepsilon b z^2 + c_b^2).$$

As the change to the new variables is only a rescaling transformation, the differential system (9) for all $\varepsilon > 0$ is topologically equivalent to the Hamiltonian system (2). Therefore studying the differential system (9) for small values of $\varepsilon \neq 0$, we are also studying the original Hamiltonian system (2) with $\varepsilon = 1$. Now we expand equation (9) in powers of the small parameter ε and the first integral H up to first order in ε, thus we have

$$\begin{aligned}
\dot{x} &= p_x,\\
\dot{y} &= p_y,\\
\dot{z} &= p_z,\\
\dot{p_x} &= -\frac{v_0^2 x}{c_b^2} + \varepsilon\frac{v_0^2 x^3 + \alpha v_0^2 x y^2 + b v_0^2 x z^2}{c_b^4} + \mathcal{O}(\varepsilon^2),\\
\dot{p_y} &= -\frac{\alpha v_0^2 y}{c_b^2} + \varepsilon\frac{\alpha v_0^2 x^2 y + \alpha^2 v_0^2 y^3 + \alpha b v_0^2 y z^2}{c_b^4} + \mathcal{O}(\varepsilon^2),\\
\dot{p_z} &= -\frac{b v_0^2 z}{c_b^2} + \varepsilon\frac{b v_0^2 x^2 z + \alpha b v_0^2 y^2 z + b^2 v_0^2 z^3}{c_b^4} + \mathcal{O}(\varepsilon^2),
\end{aligned} \tag{10}$$

and the first integral becomes

$$K = \frac{1}{2}v_0^2 \ln c_b^2 + \varepsilon \frac{c_b^2(p_x^2 + p_y^2 + p_z^2) + v_0^2(x^2 + \alpha y^2 + bz^2)}{2c_b^2} - \varepsilon^2 \frac{v_0^2(x^2 + \alpha y^2 + bz^2)^2}{4c_b^4} + \mathcal{O}(\varepsilon^3).$$

As K is a first integral, also $H = (K - \frac{1}{2}v_0^2 \ln c_b^2)/\varepsilon$ is a first integral, and we are going to use this first integral

$$H = \frac{c_b^2(p_x^2 + p_y^2 + p_z^2) + v_0^2(x^2 + \alpha y^2 + bz^2)}{2c_b^2} - \varepsilon \frac{v_0^2(x^2 + \alpha y^2 + bz^2)^2}{4c_b^4} + \mathcal{O}(\varepsilon^2). \tag{11}$$

The unperturbed equations with $\varepsilon = 0$ represent a tridimensional harmonic oscillator that can be easily solved with arbitrary initial conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$, $p_x(0) = p_{x_0}$, $p_y(0) = p_{y_0}$, $p_z(0) = p_{z_0}$

$$\begin{aligned}
x(t) &= x_0 \cos\left(\frac{tv_0}{c_b}\right) + \frac{c_b p_{x_0}}{v_0} \sin\left(\frac{tv_0}{c_b}\right),\\
p_x(t) &= p_{x_0} \cos\left(\frac{tv_0}{c_b}\right) - \frac{v_0 x_0}{c_b} \sin\left(\frac{tv_0}{c_b}\right),\\
y(t) &= y_0 \cos\left(\frac{t\sqrt{\alpha}v_0}{c_b}\right) + \frac{c_b p_{y_0}}{\sqrt{\alpha}v_0} \sin\left(\frac{t\sqrt{\alpha}v_0}{c_b}\right),\\
p_y(t) &= p_{y_0} \cos\left(\frac{t\sqrt{\alpha}v_0}{c_b}v\right) - \frac{\sqrt{\alpha}v_0 y_0}{c_b} \sin\left(\frac{t\sqrt{\alpha}v_0}{c_b}\right),\\
z(t) &= z_0 \cos\left(\frac{t\sqrt{b}v_0}{c_b}\right) + \frac{c_b p_{z_0}}{\sqrt{b}v_0} \sin\left(\frac{t\sqrt{b}v_0}{c_b}\right),\\
p_z(t) &= p_{z_0} \cos\left(\frac{t\sqrt{b}v_0}{c_b}\right) - \frac{\sqrt{b}v_0 z_0}{c_b} \sin\left(\frac{t\sqrt{b}v_0}{c_b}\right).
\end{aligned}$$

We have the following situations:

(A) If $\sqrt{\alpha}$ and $\sqrt{b}$ are rational then the dimension of the space generated by the periodic solutions is six.

(B) If $\sqrt{\alpha}$ is rational and $\sqrt{b}$ is irrational then we have the periodic solutions $(x(t), y(t), 0, p_x(t), p_y(t), 0)$. Then the dimension of the space generated by the periodic solutions is four. Moreover in this case we also have a plane generated by periodic solutions, namely $(0, 0, z(t), 0, 0, p_z(t))$.

(C) If $\sqrt{\alpha}$ is irrational and $\sqrt{b}$ is rational then we have the periodic solutions $(x(t), 0, z(t), p_x(t), 0, p_z(t))$. Then the dimension of the space generated by the periodic solutions is four. Moreover in this case we also have a plane generated by periodic solutions, namely $(0, y(t), 0, 0, p_y(t), 0)$.

(D) If $\sqrt{\alpha}$ and $\sqrt{b}$ are irrational then we have three planes generated by periodic solutions, namely the plane $(x(t), 0, 0, p_x(t), 0, 0)$, the plane $(0, y(t), 0, 0, p_y(t), 0)$ and the plane $(0, 0, z(t), 0, 0, p_z(t))$.

However, using the averaging method of section 2 we was able to find periodic orbits just in the planar cases.

4 Proof of Theorem 1

We start to consider the case when both $\sqrt{\alpha}$ and $\sqrt{b}$ are irrational and the unperturbed periodic orbit at the plane $(x(t), 0, 0, p_x(t), 0, 0)$

$$\begin{aligned} x(t) &= x_0 \cos\left(\frac{tv_0}{c_b}\right) + \frac{c_b p_{x_0}}{v_0} \sin\left(\frac{tv_0}{c_b}\right), \\ p_x(t) &= p_{x_0} \cos\left(\frac{tv_0}{c_b}\right) - \frac{v_0 x_0}{c_b} \sin\left(\frac{tv_0}{c_b}\right), \\ y(t) &= 0, \\ p_y(t) &= 0, \\ z(t) &= 0, \\ p_z(t) &= 0, \end{aligned}$$

with the first integral (11) when $\varepsilon = 0$ taking the energy value

$$h = \frac{c_b^2 p_{x_0}^2 + v_0^2 x_0^2}{2c_b^2}. \tag{12}$$

Generically, the periodic orbits of a Hamiltonian system with more than one degree of freedom are on cylinders filled of periodic orbits. Therefore we cannot apply directly the Averaging Theorem to the Hamiltonian system, since the determinant (8) would be always zero. Then we must apply Averaging Theorem to every Hamiltonian fixed level where the periodic orbits generically are isolated. This allows to eliminate one of the coordinates, say p_x, and to reduce the study to dimension five.

We thus compute p_x at the energy level $H = h$ with H given by (11) and h given by (12) and we take the expansion to first order in ε. We introduce the notation

$$R_{b,\alpha,v_0,x_0,c_b} = \sqrt{c_b^2\Big(c_b^2\Big(-p_y^2 - p_z^2 + \frac{c_b^2 p_{x_0}^2 + v_0^2 x_0^2}{c_b^2}\Big) - v_0^2(x^2 + \alpha y^2 + bz^2)\Big)}$$

and we obtain

$$p_x = \pm\frac{R_{b,\alpha,v_0,x_0,c_b}}{c_b^2} \pm \varepsilon \frac{v_0^2(x^2 + \alpha y^2 + bz^2)^2}{4c_b^2 R_{b,\alpha,v_0,x_0,c_b}} + \mathcal{O}(\varepsilon^2). \tag{13}$$

We will consider first the positive solution for p_x. The equations of motion (10) on the energy level $H = h$ are given by

$$\begin{aligned}
\dot{x} &= \frac{R_{b,\alpha,v_0,x_0,c_b}}{c_b^2} + \varepsilon\frac{v_0^2(x^2+\alpha y^2+bz^2)^2}{4c_b^2R_{b,\alpha,v_0,x_0,c_b}} + \mathcal{O}(\varepsilon^2),\\
\dot{p_y} &= -\frac{\alpha v_0^2 y}{c_b^2} + \varepsilon\frac{\alpha v_0^2x^2y+\alpha^2v_0^2y^3+\alpha bv_0^2yz^2}{c_b^4} + \mathcal{O}(\varepsilon^2),\\
\dot{y} &= p_y,\\
\dot{p_z} &= -\frac{bv_0^2 z}{c_b^2} + \varepsilon\frac{bv_0^2x^2z+\alpha bv_0^2y^2z+b^2v_0^2z^3}{c_b^4} + \mathcal{O}(\varepsilon^2),\\
\dot{z} &= p_z.
\end{aligned} \tag{14}$$

Now the differential system (14) has the form of (4), where

$$F_0(x,p_y,y,p_z,z) = \Big(\frac{R_{b,\alpha,v_0,x_0,c_b}}{c_b^2}, -\frac{\alpha v_0^2 y}{c_b^2}, p_y, -\frac{bv_0^2z}{c_b^2}, p_z\Big)$$

and

$$\begin{aligned}
F_1(x,p_y,y,p_z,z) = \Big(&\frac{v_0^2(x^2+\alpha y^2+bz^2)^2\varepsilon}{4c_b^2R_{b,\alpha,v_0,x_0,c_b}}, \frac{\alpha v_0^2x^2y+\alpha^2v_0^2y^3+\alpha bv_0^2yz^2}{c_b^4},\\
&0, \frac{bv_0^2x^2z+\alpha bv_0^2y^2z+b^2v_0^2z^3}{c_b^4}, 0\Big),
\end{aligned}$$

with the unperturbed solution $(x_0\cos(\frac{tv_0}{c_b}) + \frac{c_bp_{x_0}}{v_0}\sin(\frac{tv_0}{c_b}), 0, 0, 0, 0)$.

Set

$$S_{b,\alpha,v_0,x_0,c_b} = \sqrt{c_b^2\Big(c_b^2p_{x_0}^2 + v_0^2x_0^2 - v_0^2\big(x_0\cos(\frac{tv_0}{c_b}) + \frac{c_bp_{x_0}}{v_0}\sin(\frac{tv_0}{c_b})\big)^2\Big)}.$$

Now we compute the linearization of the unperturbed system along the periodic solution, $D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z}_{x_0}))$

$$\begin{pmatrix}
-\frac{v_0^2\left(x_0\cos(\frac{tv_0}{c_b})+\frac{c_bp_{x_0}}{v_0}\sin(\frac{tv_0}{c_b})\right)}{S_{b,\alpha,v_0,x_0,c_b}} & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{\alpha v_0^2}{c_b^2} & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\frac{bv_0^2}{c_b^2}\\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

and the fundamental matrix $M_{\mathbf{z}_{x_0}}(t)$ is obtained solving (6), that is, it is equal to

$$\begin{pmatrix} \cos(\frac{tv_0}{c_b})-\frac{v_0x_0}{c_bp_{x_0}}\sin(\frac{tv_0}{c_b}) & 0 & 0 & 0 & 0 \\ 0 & \cos(\frac{\sqrt{\alpha}tv_0}{c_b}) & -\frac{\sqrt{\alpha}v_0}{c_b}\sin(\frac{\sqrt{\alpha}tv_0}{c_b}) & 0 & 0 \\ 0 & \frac{c_b}{\sqrt{\alpha}v_0}\sin(\frac{\sqrt{\alpha}tv_0}{c_b}) & \cos(\frac{\sqrt{\alpha}tv_0}{c_b}) & 0 & 0 \\ 0 & 0 & 0 & \cos(\frac{\sqrt{b}tv_0}{c_b}) & -\frac{\sqrt{b}v_0}{c_b}\sin(\frac{\sqrt{b}tv_0}{c_b}) \\ 0 & 0 & 0 & \frac{c_b}{\sqrt{b}v_0}\sin(\frac{\sqrt{b}tv_0}{c_b}) & \cos(\frac{\sqrt{b}tv_0}{c_b}) \end{pmatrix},$$

which satisfies $M_{\mathbf{z}_{x_0}}(0) = I$, and the inverse, $M^{-1}_{\mathbf{z}_{x_0}}(t)$ is given by

$$\begin{pmatrix} \frac{c_bp_{x_0}}{c_bp_{x_0}\cos(\frac{tv_0}{c_b})-v_0x_0\sin(\frac{tv_0}{c_b})} & 0 & 0 & 0 & 0 \\ 0 & \cos(\frac{\sqrt{\alpha}tv_0}{c_b}) & \frac{\sqrt{\alpha}v_0}{c_b}\sin(\frac{\sqrt{\alpha}tv_0}{c_b}) & 0 & 0 \\ 0 & -\frac{c_b}{\sqrt{\alpha}v_0}\sin(\frac{\sqrt{\alpha}tv_0}{c_b}) & \cos(\frac{\sqrt{\alpha}tv_0}{c_b}) & 0 & 0 \\ 0 & 0 & 0 & \cos(\frac{\sqrt{b}tv_0}{c_b}) & \frac{\sqrt{b}v_0}{c_b}\sin(\frac{\sqrt{b}tv_0}{c_b}) \\ 0 & 0 & 0 & -\frac{c_b}{\sqrt{b}v_0}\sin(\frac{\sqrt{b}tv_0}{c_b}) & \cos(\frac{\sqrt{b}tv_0}{c_b}) \end{pmatrix}.$$

In order to apply the Averaging Theorem, we verify the condition $\det \Delta_{x_0} \neq 0$, thus we compute

$$M^{-1}_{\mathbf{z}_{x_0}}(0)-M^{-1}_{\mathbf{z}_{x_0}}(\tfrac{2\pi c_b}{v_0})=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\cos(2\sqrt{\alpha}\pi) & -\frac{\sqrt{\alpha}v_0}{c_b}\sin(2\sqrt{\alpha}\pi) & 0 & 0 \\ 0 & \frac{c_b}{\sqrt{\alpha}v_0}\sin(2\sqrt{\alpha}\pi) & 1-\cos(2\sqrt{\alpha}\pi) & 0 & 0 \\ 0 & 0 & 0 & 1-\cos(2\sqrt{b}\pi) & -\frac{\sqrt{b}v_0}{c_b}\sin(2\sqrt{b}\pi) \\ 0 & 0 & 0 & \frac{c_b}{\sqrt{b}v_0}\sin(2\sqrt{b}\pi) & 1-\cos(2\sqrt{b}\pi) \end{pmatrix}.$$

In the upper right corner, the 1×4 matrix is zero, and for each $\mathbf{z}_{x_0}$ in the lower right corner the matrix Δ_{x_0} has determinant non-zero, $\Delta_{x_0} = 16\sin^2(\sqrt{\alpha}\pi)\sin^2(\sqrt{b}\pi)$ since $\sqrt{\alpha}$ and $\sqrt{b}$ are both irrational.

The function F_1 along the periodic orbit is given by

$$F_1(t, \mathbf{x}(t, \mathbf{z}_{x_0})) = \Big(\frac{v_0^2(x_0\cos(\frac{tv_0}{c_b}) + \frac{c_bp_{x_0}}{v_0}\sin(\frac{tv_0}{c_b}))^4}{4c_b^2 S_{b,\alpha,v_0,x_0,c_b}}, 0, 0, 0, 0\Big),$$

and we must apply to it the inverse of the fundamental matrix

$$M^{-1}_{\mathbf{z}_{x_0}}(t)F_1(t, \mathbf{x}(t, \mathbf{z}_{x_0})) = \Big(\frac{p_{x_0}v_0^2(x_0\cos(\frac{tv_0}{c_b}) + \frac{c_bp_{x_0}}{v_0}\sin(\frac{tv_0}{c_b}))^4}{4c_b(c_bp_{x_0}\cos(\frac{tv_0}{c_b}) - v_0x_0\sin(\frac{tv_0}{c_b}))S_{b,\alpha,v_0,x_0,c_b}}, 0, 0, 0, 0\Big). \tag{15}$$

The function $\mathcal{F}(x_0)$ defined in (7) is the projection ξ in the first component of the integral of (15) in one period

$$\begin{aligned}\mathcal{F}(x_0) &= \int_0^{\frac{2\pi c_b}{v_0}} \frac{p_{x_0} v_0^2 (x_0 \cos(\frac{tv_0}{c_b}) + \frac{c_b p_{x_0}}{v_0} \sin(\frac{tv_0}{c_b}))^4}{4c_b (c_b p_{x_0} \cos(\frac{tv_0}{c_b}) - v_0 x_0 \sin(\frac{tv_0}{c_b})) S_{b,\alpha,v_0,x_0,c_b}} dt \\ &= -\frac{3\pi p_{x_0}(c_b^2 p_{x_0}^2 + v_0^2 x_0^2)}{4c_b v_0^3},\end{aligned}$$

where $p_{x_0} = \pm \frac{\sqrt{2c_b^2 h - v_0^2 x_0^2}}{c_b}$ at the energy level (12), thus

$$\mathcal{F}(x_0) = \pm \frac{3\pi h \sqrt{2c_b^2 h - v_0^2 x_0^2}}{2v_0^3}.$$

Now we look for the zeros of $\mathcal{F}(x_0) = 0 : x_0 = \pm \frac{c_b\sqrt{2h}}{v_0}$ which implies $p_{x_0} = 0$. Every simple zero of $\mathcal{F}(x_0)$ provides a periodic orbit for the perturbed differential system in the energy level $H = h > 0$. Finally, the negative solution of (13) provides the same solutions as the positive one because $p_{x_0} = 0$. Note that both initial conditions $(\pm\frac{c_b\sqrt{2h}}{v_0}, 0, 0, 0, 0, 0)$ provides the same periodic orbits. This conclude the first statement in Theorem 1(a).

Now we consider the case when both $\sqrt{\alpha}$ and $\sqrt{b}$ are irrational or the case when $\sqrt{\alpha}$ is irrational and $\sqrt{b}$ is rational, and the unperturbed periodic orbit at the plane $(0, y(t), 0, 0, p_y(t), 0)$

$$\begin{aligned}x(t) &= 0,\\ p_x(t) &= 0,\\ y(t) &= y_0 \cos\Big(\frac{t\sqrt{\alpha}v_0}{c_b}\Big) + \frac{c_b p_{y_0}}{\sqrt{\alpha}v_0} \sin\Big(\frac{t\sqrt{\alpha}v_0}{c_b}\Big),\\ p_y(t) &= p_{y_0} \cos\Big(\frac{t\sqrt{\alpha}v_0}{c_b}\Big) - \frac{v_0 y_0 \sqrt{\alpha}}{c_b} \sin\Big(\frac{t\sqrt{\alpha}v_0}{c_b}\Big),\\ z(t) &= 0,\\ p_z(t) &= 0,\end{aligned}$$

with the first integral (11) when $\varepsilon = 0$ taking the energy value

$$h = \frac{c_b^2 p_{y_0}^2 + v_0^2 \alpha y_0^2}{2c_b^2}. \tag{16}$$

Like in the previous case we eliminate now p_y, and we reduce the study to dimension five.

Let

$$U_{b,\alpha,v_0,y_0,c_b} = \sqrt{c_b^2\Big(c_b^2\big(-p_x^2 - p_z^2 + \frac{c_b^2 p_{y_0}^2 + \alpha v_0^2 y_0^2}{c_b^2}\big) - v_0^2(x^2 + \alpha y^2 + bz^2)\Big)}.$$

We compute p_y at the energy level $H = h$ with H given by (11) and h given by (16) and we take the expansion to first order in ε

$$p_y = \pm\frac{U_{b,\alpha,v_0,y_0,c_b}}{c_b^2} \pm \varepsilon\frac{v_0^2(x^2+\alpha y^2+bz^2)^2}{4c_b^2 U_{b,\alpha,v_0,y_0,c_b}} + \mathcal{O}(\varepsilon^2).$$

We will consider first the positive solution for p_y. The equations of motion (10) on the energy level $H = h$ are given by

$$\begin{aligned}
\dot{y} &= \frac{U_{b,\alpha,v_0,y_0,c_b}}{c_b^2} + \varepsilon\frac{v_0^2(x^2+\alpha y^2+bz^2)^2}{4c_b^2 U_{b,\alpha,v_0,y_0,c_b}} + \mathcal{O}(\varepsilon^2),\\
\dot{p_x} &= -\frac{v_0^2 x}{c_b^2} + \varepsilon\frac{v_0^2x^3+\alpha v_0^2 xy^2+bv_0^2xz^2}{c_b^4} + \mathcal{O}(\varepsilon^2),\\
\dot{x} &= p_x,\\
\dot{p_z} &= -\frac{bv_0^2 z}{c_b^2} + \varepsilon\frac{bv_0^2x^2z+\alpha bv_0^2y^2z+b^2v_0^2z^3}{c_b^4} + \mathcal{O}(\varepsilon^2),\\
\dot{z} &= p_z.
\end{aligned} \tag{17}$$

Now the differential system (17) has the form of (4), where

$$F_0(y,p_x,x,p_z,z) = \Big(\frac{U_{b,\alpha,v_0,y_0,c_b}}{c_b^2}, -\frac{v_0^2x}{c_b^2}, -\frac{bv_0^2z}{c_b^2}, p_z\Big)$$

and

$$\begin{aligned}
F_1(y,p_x,x,p_z,z) = \Big(&\frac{v_0^2(x^2+\alpha y^2+bz^2)^2}{4c_b^2U_{b,\alpha,v_0,y_0,c_b}}, \frac{v_0^2x^3+\alpha^2v_0^2xy^2+bv_0^2xz^2}{c_b^4},\\
&0, \frac{bv_0^2x^2z+\alpha bv_0^2y^2z+b^2v_0^2z^3}{c_b^4}, 0\Big),
\end{aligned}$$

with the unperturbed solution $(y_0\cos(\frac{t\sqrt{\alpha}v_0}{c_b}) + \frac{c_bp_{y_0}}{\sqrt{\alpha}v_0}\sin(\frac{t\sqrt{\alpha}v_0}{c_b}),0,0,0,0)$.

Set now

$$V_{b,\alpha,v_0,y_0,c_b} = \sqrt{c_b^2\Big(c_b^2p_{y_0}^2+\alpha v_0^2y_0^2-\alpha v_0^2\big(y_0\cos(\frac{t\sqrt{\alpha}v_0}{c_b})+\frac{c_bp_{y_0}}{\sqrt{\alpha}v_0}\sin(\frac{t\sqrt{\alpha}v_0}{c_b})\big)^2\Big)}.$$

Now we compute the linearization of the unperturbed system along the periodic solution, $D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z}_{x_0}))$

$$\begin{pmatrix}
-\frac{\alpha v_0^2\left(y_0\cos(\frac{t\sqrt{\alpha}v_0}{c_b})+\frac{c_bp_{y_0}}{\sqrt{\alpha}v_0}\sin(\frac{t\sqrt{\alpha}v_0}{c_b})\right)}{V_{b,\alpha,v_0,y_0,c_b}} & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{v_0^2}{c_b^2} & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\frac{bv_0^2}{c_b^2}\\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

and the fundamental matrix $M_{\mathbf{z}_{x_0}}(t)$ is obtained solving (6) and is equal to

$$\begin{pmatrix} \cos(\frac{t\sqrt{\alpha}v_0}{c_b})-\frac{v_0y_0\sqrt{\alpha}}{c_bp_{y_0}}\sin(\frac{t\sqrt{\alpha}v_0}{c_b}) & 0 & 0 & 0 & 0 \\ 0 & \cos(\frac{tv_0}{c_b}) & -\frac{v_0}{c_b}\sin(\frac{tv_0}{c_b}) & 0 & 0 \\ 0 & \frac{c_b}{v_0}\sin(\frac{tv_0}{c_b}) & \cos(\frac{tv_0}{c_b}) & 0 & 0 \\ 0 & 0 & 0 & \cos(\frac{\sqrt{b}tv_0}{c_b}) & -\frac{\sqrt{b}v_0}{c_b}\sin(\frac{\sqrt{b}tv_0}{c_b}) \\ 0 & 0 & 0 & \frac{c_b}{\sqrt{b}v_0}\sin(\frac{\sqrt{b}tv_0}{c_b}) & \cos(\frac{\sqrt{b}tv_0}{c_b}) \end{pmatrix},$$

which satisfies $M_{\mathbf{z}_{x_0}}(0) = I$, and the inverse $M^{-1}_{\mathbf{z}_{x_0}}(t)$ is given by

$$\begin{pmatrix} \frac{c_bp_{y_0}}{c_bp_{y_0}\cos(\frac{t\sqrt{\alpha}v_0}{c_b})-\sqrt{\alpha}v_0y_0\sin(\frac{t\sqrt{\alpha}v_0}{c_b})} & 0 & 0 & 0 & 0 \\ 0 & \cos(\frac{tv_0}{c_b}) & \frac{v_0}{c_b}\sin(\frac{tv_0}{c_b}) & 0 & 0 \\ 0 & -\frac{c_b}{v_0}\sin(\frac{tv_0}{c_b}) & \cos(\frac{tv_0}{c_b}) & 0 & 0 \\ 0 & 0 & 0 & \cos(\frac{\sqrt{b}tv_0}{c_b}) & \frac{\sqrt{b}v_0}{c_b}\sin(\frac{\sqrt{b}tv_0}{c_b}) \\ 0 & 0 & 0 & -\frac{c_b}{\sqrt{b}v_0}\sin(\frac{\sqrt{b}tv_0}{c_b}) & \cos(\frac{\sqrt{b}tv_0}{c_b}) \end{pmatrix}.$$

In order to apply the Averaging Theorem, we verify the condition $\det \Delta_{x_0} \neq 0$, thus we compute

$$M^{-1}_{\mathbf{z}_{x_0}}(0)-M^{-1}_{\mathbf{z}_{x_0}}(\frac{2\pi c_b}{v_0\sqrt{\alpha}})=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\cos(\frac{2\pi}{\sqrt{\alpha}}) & -\frac{v_0}{c_b}\sin(\frac{2\pi}{\sqrt{\alpha}}) & 0 & 0 \\ 0 & \frac{c_b}{v_0}\sin(\frac{2\pi}{\sqrt{\alpha}}) & 1-\cos(\frac{2\pi}{\sqrt{\alpha}}) & 0 & 0 \\ 0 & 0 & 0 & 1-\cos(\frac{2\sqrt{b}\pi}{\sqrt{\alpha}}) & -\frac{\sqrt{b}v_0}{c_b}\sin(\frac{2\sqrt{b}\pi}{\sqrt{\alpha}}) \\ 0 & 0 & 0 & \frac{c_b}{\sqrt{b}v_0}\sin(\frac{2\sqrt{b}\pi}{\sqrt{\alpha}}) & 1-\cos(\frac{2\sqrt{b}\pi}{\sqrt{\alpha}}) \end{pmatrix}.$$

In the upper right corner, the 1×4 matrix is zero, and for each $\mathbf{z}_{x_0}$ in the lower right corner the matrix Δ_{x_0} has determinant non-zero, $\Delta_{x_0} = 16\sin^2(\frac{\pi}{\sqrt{\alpha}})\sin^2(\frac{\sqrt{b}\pi}{\sqrt{\alpha}})$ since $\sqrt{\alpha}$ and $\sqrt{b}$ are both irrational, or $\sqrt{\alpha}$ is irrational and $\sqrt{b}$ is rational.

The function F_1 along the periodic orbit is given by

$$F_1(t,\mathbf{x}(t,\mathbf{z}_{x_0})) = \Big(\frac{\alpha^2v_0^2(y_0\cos(\frac{t\sqrt{\alpha}v_0}{c_b}) + \frac{c_bp_{y_0}}{\sqrt{\alpha}v_0}\sin(\frac{t\sqrt{\alpha}v_0}{c_b}))^4}{4c_b^2V_{b,\alpha,v_0,y_0,c_b}},0,0,0,0\Big),$$

and we must apply to it the inverse of the fundamental matrix

$$\begin{aligned} &M^{-1}_{\mathbf{z}_{x_0}}(t)F_1(t,\mathbf{x}(t,\mathbf{z}_{x_0})) \\ &= \Big(\frac{\alpha^2p_{y_0}v_0^2(y_0\cos(\frac{t\sqrt{\alpha}v_0}{c_b}) + \frac{c_bp_{y_0}}{\sqrt{\alpha}v_0}\sin(\frac{t\sqrt{\alpha}v_0}{c_b}))^4}{4c_b(c_bp_{y_0}\cos(\frac{t\sqrt{\alpha}v_0}{c_b}) - \sqrt{\alpha}v_0y_0\sin(\frac{t\sqrt{\alpha}v_0}{c_b}))V_{b,\alpha,v_0,y_0,c_b}},0,0,0,0\Big). \end{aligned}$$

The function $\mathcal{F}(x_0)$ defined in (7) is the projection ξ in the first component of the integral of (15) in one period

$$\begin{aligned}\mathcal{F}(x_0) &= \int_0^{\frac{2\pi c_b}{v_0}} \frac{\alpha^2 p_{y_0} v_0^2 (y_0 \cos(\frac{t\sqrt{\alpha}v_0}{c_b}) + \frac{c_b p_{y_0}}{\sqrt{\alpha}v_0} \sin(\frac{t\sqrt{\alpha}v_0}{c_b}))^4}{4c_b(c_b p_{y_0} \cos(\frac{t\sqrt{\alpha}v_0}{c_b}) - \sqrt{\alpha}v_0 y_0 \sin(\frac{t\sqrt{\alpha}v_0}{c_b}))V_{b,\alpha,v_0,y_0,c_b}} dt \\ &= -\frac{3\pi p_{y_0}(c_b^2 p_{y_0}^2 + \alpha v_0^2 y_0^2)}{4\sqrt{\alpha} c_b v_0^3},\end{aligned}$$

where $p_{y_0} = \pm \frac{\sqrt{2c_b^2 h - \alpha v_0^2 y_0^2}}{c_b}$ at the energy level (12), thus

$$\mathcal{F}(x_0) = \pm \frac{3\pi h \sqrt{2c_b^2 h - \alpha v_0^2 y_0^2}}{2\sqrt{\alpha} v_0^3}.$$

Now we look for the zeros of $\mathcal{F}(x_0) = 0$. They satisfy $y_0 = \pm \frac{c_b\sqrt{2h}}{\sqrt{\alpha}v_0}$ which implies $p_{y_0} = 0$. Every simple zero of $\mathcal{F}(x_0)$ provides a periodic orbit for the perturbed differential system in the energy level $H = h > 0$. Finally, the negative solution of (13) provides the same solutions as the positive one because $p_{y_0} = 0$. Note that both initial conditions $(0, \pm\frac{c_b\sqrt{2h}}{\sqrt{\alpha}v_0}, 0, 0, 0, 0)$ provides the same periodic orbits. This conclude the second case of Theorem 1(a) and also Theorem 1(b).

Finally we consider the case when both $\sqrt{\alpha}$ and $\sqrt{b}$ are irrational or the case when $\sqrt{\alpha}$ is rational and $\sqrt{b}$ is irrational, and the unperturbed periodic orbit at the plane $(0, 0, z(t), 0, 0, p_z(t))$

$$\begin{aligned} x(t) &= 0, \\ p_x(t) &= 0, \\ y(t) &= 0, \\ p_y(t) &= 0, \\ z(t) &= z_0 \cos\left(\frac{t\sqrt{b}v_0}{c_b}\right) + \frac{c_b p_{z_0}}{\sqrt{b}v_0} \sin\left(\frac{t\sqrt{b}v_0}{c_b}\right), \\ p_z(t) &= p_{z_0} \cos\left(\frac{t\sqrt{b}v_0}{c_b}\right) - \frac{v_0 z_0 \sqrt{b}}{c_b} \sin\left(\frac{t\sqrt{b}v_0}{c_b}\right), \end{aligned}$$

with the first integral (11) when $\varepsilon = 0$ taking the energy value

$$h = \frac{c_b^2 p_{z_0}^2 + v_0^2 b z_0^2}{2c_b^2}. \tag{18}$$

Now we eliminate p_z to reduce the study to dimension five.

Let

$$W_{b,\alpha,z_0,y_0,c_b} = \sqrt{c_b^2\Big(c_b^2\big(- p_x^2 - p_y^2 + \frac{c_b^2 p_{z_0}^2 + b v_0^2 z_0^2}{c_b^2}\big) - v_0^2(x^2 + \alpha y^2 + bz^2)\Big)}.$$

We compute p_z at the energy level $H = h$ with H given by (11) and h given by (18) and we take the expansion to first order in ε

$$p_z = \pm\frac{W_{b,\alpha,z_0,y_0,c_b}}{c_b^2} \pm \varepsilon\frac{v_0^2(x^2+\alpha y^2+bz^2)^2}{4c_b^2 W_{b,\alpha,z_0,y_0,c_b}} + \mathcal{O}(\varepsilon^2).$$

We will consider first the positive solution for p_z. The equations of motion (10) on the energy level $H = h$ are given by

$$\begin{aligned}
\dot{z} &= p_z,\\
\dot{p_y} &= -\frac{\alpha v_0^2 y}{c_b^2} + \varepsilon\frac{\alpha v_0^2 x^2 y + \alpha^2 v_0^2 y^3 + \alpha b v_0^2 y z^2}{c_b^4} + \mathcal{O}(\varepsilon^2),\\
\dot{y} &= p_y,\\
\dot{p_x} &= -\frac{v_0^2 x}{c_b^2} + \varepsilon\frac{v_0^2 x^3 + \alpha v_0^2 x y^2 + b v_0^2 x z^2}{c_b^4} + \mathcal{O}(\varepsilon^2),\\
\dot{x} &= p_x.
\end{aligned} \tag{19}$$

Now the differential system (19) has the form of (4), where

$$F_0(z,p_y,y,p_x,x) = \Big(\frac{W_{b,\alpha,z_0,y_0,c_b}}{c_b^2}, -\frac{\alpha v_0^2 y}{c_b^2}, p_y, -\frac{v_0^2 x}{c_b^2}, p_x\Big)$$

and

$$\begin{aligned}
F_1(z,p_y,y,p_x,x) = \Big(&\frac{v_0^2(x^2+\alpha y^2+bz^2)^2}{4c_b^2 W_{b,\alpha,z_0,y_0,c_b}}, \frac{\alpha v_0^2 x^2 y + \alpha^2 v_0^2 y^3 + \alpha b v_0^2 y z^2}{c_b^4},\\
&0, \frac{v_0^2 x^3 + \alpha v_0^2 x y^2 + b v_0^2 x z^2}{c_b^4}, 0\Big),
\end{aligned}$$

with the unperturbed solution $(z_0\cos(\frac{t\sqrt{b}v_0}{c_b}) + \frac{c_b p_{z_0}}{\sqrt{b}v_0}\sin(\frac{t\sqrt{b}v_0}{c_b}), 0, 0, 0, 0)$.

Set now

$$X_{b,\alpha,v_0,z_0,c_b} = \sqrt{c_b^2\Big(c_b^2 p_{z_0}^2 + b v_0^2 z_0^2 - b v_0^2\big(z_0\cos(\frac{t\sqrt{b}v_0}{c_b}) + \frac{c_b p_{z_0}}{\sqrt{b}v_0}\sin(\frac{t\sqrt{b}v_0}{c_b})\big)^2\Big)}.$$

Now we compute the linearization of the unperturbed system along the periodic solution, $D_{\mathbf{x}}F_0(t, \mathbf{x}(t, \mathbf{z}_{x_0}))$

$$\begin{pmatrix}
-\frac{b v_0^2\big(z_0\cos(\frac{t\sqrt{b}v_0}{c_b}) + \frac{c_b p_{z_0}}{\sqrt{b}v_0}\sin(\frac{t\sqrt{b}v_0}{c_b})\big)}{X_{b,\alpha,v_0,z_0,c_b}} & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{\alpha v_0^2}{c_b^2} & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\frac{v_0^2}{c_b^2}\\
0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

and the fundamental matrix $M_{\mathbf{z}_{x_0}}(t)$ is obtained solving (6), is given by

$$\begin{pmatrix} \cos(\frac{t\sqrt{b}v_0}{c_b})-\frac{v_0 z_0\sqrt{b}}{c_b p z_0}\sin(\frac{t\sqrt{b}v_0}{c_b}) & 0 & 0 & 0 & 0 \\ 0 & \cos(\frac{\sqrt{\alpha}tv_0}{c_b}) & -\frac{\sqrt{\alpha}v_0}{c_b}\sin(\frac{\sqrt{\alpha}tv_0}{c_b}) & 0 & 0 \\ 0 & \frac{c_b}{\sqrt{\alpha}v_0}\sin(\frac{\sqrt{\alpha}tv_0}{c_b}) & \cos(\frac{\sqrt{\alpha}tv_0}{c_b}) & 0 & 0 \\ 0 & 0 & 0 & \cos(\frac{tv_0}{c_b}) & -\frac{v_0}{c_b}\sin(\frac{tv_0}{c_b}) \\ 0 & 0 & 0 & \frac{c_b}{v_0}\sin(\frac{tv_0}{c_b}) & \cos(\frac{tv_0}{c_b}) \end{pmatrix},$$

which satisfies $M_{\mathbf{z}_{x_0}}(0) = I$, and the inverse $M^{-1}_{\mathbf{z}_{x_0}}(t)$ is given by

$$\begin{pmatrix} \frac{c_b p z_0}{c_b p z_0\cos(\frac{t\sqrt{b}v_0}{c_b})-\sqrt{b}v_0 z_0\sin(\frac{t\sqrt{b}v_0}{c_b})} & 0 & 0 & 0 & 0 \\ 0 & \cos(\frac{\sqrt{\alpha}tv_0}{c_b}) & \frac{\sqrt{\alpha}v_0}{c_b}\sin(\frac{\sqrt{\alpha}tv_0}{c_b}) & 0 & 0 \\ 0 & -\frac{c_b}{\sqrt{\alpha}v_0}\sin(\frac{\sqrt{\alpha}tv_0}{c_b}) & \cos(\frac{\sqrt{\alpha}tv_0}{c_b}) & 0 & 0 \\ 0 & 0 & 0 & \cos(\frac{tv_0}{c_b}) & \frac{v_0}{c_b}\sin(\frac{tv_0}{c_b}) \\ 0 & 0 & 0 & -\frac{c_b}{v_0}\sin(\frac{tv_0}{c_b}) & \cos(\frac{tv_0}{c_b}) \end{pmatrix}.$$

In order to apply the Averaging Theorem, we verify the condition $\det\Delta_{x_0} \neq 0$, thus we compute

$$M^{-1}_{\mathbf{z}_{x_0}}(0)-M^{-1}_{\mathbf{z}_{x_0}}(\frac{2\pi c_b}{v_0\sqrt{b}})=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\cos(\frac{2\sqrt{\alpha}\pi}{\sqrt{b}}) & -\frac{\sqrt{\alpha}v_0}{c_b}\sin(\frac{2\sqrt{\alpha}\pi}{\sqrt{b}}) & 0 & 0 \\ 0 & \frac{c_b}{\sqrt{\alpha}v_0}\sin(\frac{2\sqrt{\alpha}\pi}{\sqrt{b}}) & 1-\cos(\frac{2\sqrt{\alpha}\pi}{\sqrt{b}}) & 0 & 0 \\ 0 & 0 & 0 & 1-\cos(\frac{2\pi}{\sqrt{b}}) & -\frac{v_0}{c_b}\sin(\frac{2\pi}{\sqrt{b}}) \\ 0 & 0 & 0 & \frac{c_b}{v_0}\sin(\frac{2\pi}{\sqrt{b}}) & 1-\cos(\frac{2\pi}{\sqrt{b}}) \end{pmatrix}.$$

In the upper right corner, the 1×4 matrix is zero, and for each $\mathbf{z}_{x_0}$ in the lower right corner the matrix Δ_{x_0} has determinant non-zero, $\Delta_{x_0} = 16\sin^2(\frac{\pi}{\sqrt{b}})\sin^2(\frac{\sqrt{\alpha}\pi}{\sqrt{b}})$ since $\sqrt{\alpha}$ and $\sqrt{b}$ are both irrational, or $\sqrt{\alpha}$ is rational and $\sqrt{b}$ is irrational.

The function F_1 along the periodic orbit is given by

$$F_1(t,\mathbf{x}(t,\mathbf{z}_{x_0})) = \Big(\frac{b^2 v_0^2(z_0\cos(\frac{t\sqrt{b}v_0}{c_b}) + \frac{c_b p_{z_0}}{\sqrt{b}v_0}\sin(\frac{t\sqrt{b}v_0}{c_b}))^4}{4c_b^2 X_{b,\alpha,v_0,z_0,c_b}}, 0, 0, 0, 0\Big),$$

and we must apply to it the inverse of the fundamental matrix

$$\begin{aligned} &M^{-1}_{\mathbf{z}_{x_0}}(t)F_1(t,\mathbf{x}(t,\mathbf{z}_{x_0})) \\ &= \Big(\frac{b^2 v_0^2 p_{z_0}(z_0\cos(\frac{t\sqrt{b}v_0}{c_b}) + \frac{c_b p_{z_0}}{\sqrt{b}v_0}\sin(\frac{t\sqrt{b}v_0}{c_b}))^4}{4c_b(c_b p_{z_0}\cos(\frac{t\sqrt{b}v_0}{c_b}) - \sqrt{b}v_0 z_0\sin(\frac{t\sqrt{b}v_0}{c_b}))X_{b,\alpha,v_0,z_0,c_b}}, 0, 0, 0, 0\Big). \end{aligned} \tag{20}$$

The function $\mathcal{F}(x_0)$ defined in (7) is the projection ξ in the first component of the integral of (20) in one period

$$\mathcal{F}(x_0) = \int_0^{\frac{2\pi c_b}{\sqrt{b}v_0}} \frac{b^2 v_0^2 p_{z_0}(z_0 \cos(\frac{t\sqrt{b}v_0}{c_b}) + \frac{c_b p_{z_0}}{\sqrt{b}v_0}\sin(\frac{t\sqrt{b}v_0}{c_b}))^4}{4c_b\left(c_b p_{z_0}\cos(\frac{t\sqrt{b}v_0}{c_b}) - \sqrt{b}v_0 z_0 \sin(\frac{t\sqrt{b}v_0}{c_b})\right) X_{b,\alpha,v_0,z_0,c_b}} dt$$
$$= -\frac{3\pi p_{z_0}(c_b^2 p_{z_0}^2 + b v_0^2 z_0^2)}{4\sqrt{b}c_b v_0^3},$$

where $p_{z_0} = \pm\frac{\sqrt{2c_b^2 h - b v_0^2 z_0^2}}{c_b}$ at the energy level (12), thus

$$\mathcal{F}(x_0) = \pm\frac{3\pi h\sqrt{2c_b^2 h - b v_0^2 z_0^2}}{2\sqrt{b}v_0^3}.$$

Now we look for the zeros of $\mathcal{F}(x_0) = 0$. They satisfy $z_0 = \pm\frac{c_b\sqrt{2h}}{\sqrt{b}v_0}$ which implies $p_{z_0} = 0$. Every simple zero of $\mathcal{F}(x_0)$ provides a periodic orbit for the perturbed differential system in the energy level $H = h > 0$. Finally, the negative solution of (13) provides the same solutions as the positive one because $p_{z_0} = 0$. Note that both initial conditions $(0, 0, \pm\frac{c_b\sqrt{2h}}{\sqrt{b}v_0}, 0, 0, 0)$ provides the same periodic orbits. This conclude the third statement of Theorem 1(a) and also Theorem 1(c).

References

[1] J. Bak, T. Statler - *The intrinsic shape distribution of a sample of elliptical galaxies*, AJ, 120:110-122, 2000.

[2] Binney,J. & Tremaine, Sc. - *Galactic Dynamics*, Princeton Series in Astrophysics, 1987.

[3] Buică, A., Francoise, J., and Llibre, J. - *Periodic solutions of nonlinear periodic differential systems with a small parameter*, Commun. Pure Appl. Anal. 6, (2007), 103–111.

[4] R.L. Davies, M. Birkinshaw - *The orientations of the rotation axes of radio galaxies. II - Stellar rotation curves and velocity dispersion profiles*, ApJS, 68:409-447, 1988.

[5] Hibbard et al. in ASP Conf. Ser. 240 - *Gas and Galaxy Evolution*, ed. J. E. Hibbard, M. P. Rupen & J. H. van Gorkom (san Francisco:ASP), 2001.

[6] Malkin, I. G. - *Some Problems of the Theory of Nonlinear Oscillations* (Gosudarstv. Izdat. Tehn.- Teor. Lit.,Moscow, 1956) (in Russian).

[7] Roseau, M. - *Vibrations non linéaires et théorie de la stabilité*, Springer Tracts in Natural Philosophy Vol. 8 (Springer, Berlin, 1966) (in French).

[8] Sanders, J. A., Verhultst, F. and Murdock, J. *Averaging Methods in Nonlinear Dynamical Systems*, revised edition, Applied Mathematical Sci. Vol. **59** (Springer Verlag, New York, 2007).

[9] J. Van Gorkom, D. Schiminovisch. ASP Conf. Ser. 116 - *The Nature of Elliptical Galaxies*, ed. M. Arnaboldi, G. S. Da Costa & P. Saha (San Francisco:ASP), 1997.

[10] F. Verhulst - *Nonlinear Differential Equations and Dynamical Systems*, Universitext Springer Verlag, 1996.

[11] L.M. Young - *Molecular disks in the elliptical galaxies NGC 83 and NGC 2320* ApJ, 634:258-271, 2005.

Department of Mathematics and Informatics
University of Oradea, University Street 1,
410087 Oradea, Romania
email :dpasca@uoradea.ro

Department of Mechanical Engineering and Automotive
University of Oradea, University Street 1,
410087 Oradea, Romania
email :btataru@uoradea.ro