Ricci solitons in almost f-cosymplectic manifolds

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Abstract

In this article we study an almost f-cosymplectic manifold admitting a Ricci soliton. We first prove that there do not exist Ricci solitons on an almost cosymplectic (κ, μ)-manifold. Further, we consider an almost f-cosymplectic manifold admitting a Ricci soliton whose potential vector field is the Reeb vector field and show that a three dimensional almost f-cosymplectic is a cosymplectic manifold. Finally we classify a three dimensional η-Einstein almost f-cosymplectic manifold admitting a Ricci soliton.

1 Introduction

A *Ricci soliton* is a Riemannian metric defined on manifold M such that

$$\frac{1}{2}\mathcal{L}_V g + Ric - \lambda g = 0, \tag{1.1}$$

where V and λ are the potential vector field and some constant on M, respectively. Moreover, the Ricci soliton is called *shrinking, steady* and *expanding* according as λ is positive, zero and negative respectively. The Ricci solitons are of interest to physicists as well and are known as quasi Einstein metrics in the physics literature [6]. Compact Ricci solitons are the fixed point of the Ricci flow: $\frac{\partial}{\partial t}g = -2Ric$,

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projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flows on compact manifolds. The study on the Ricci solitons has a long history and a lot of achievements were acquired, see [5, 13, 15]etc. On the other hand, the normal almost contact manifolds admitting Ricci solitons were also been studied by many researchers (see [3, 7, 8, 9]).

Recently, we note that the three dimensional almost Kenmotsu manifolds admitting Ricci solitons were considered ([18, 19]) and Cho ([2]) gave the classification of an almost cosymplectic manifold admitting a Ricci soliton whose potential vector field is the Reeb vector field. Here the *almost cosymplectic manifold*, defined by Goldberg and Yano [10], was an almost contact manifold whose 1-form η and fundamental 2-form ω are closed, and the *almost Kenmotsu manifold* is an almost contact manifold satisfying $d\eta = 0$ and $d\omega = 2\eta \wedge \omega$. Based on this Kim and Pak [11] introduced the concept of *almost α-cosymplectic manifold*, i.e., an almost contact manifold satisfying $d\eta = 0$ and $d\omega = 2\alpha\eta \wedge \omega$ for any real number α. In particular, if α is non-zero it is said to be an *almost α-Kenmotsu* manifold. Later Aktan et al.[1] defined an *almost f-cosymplectic manifold M* by generalizing the real number α to a smooth function f on M, i.e., an almost contact manifold satisfies $d\omega = 2f\eta \wedge \omega$ and $d\eta = 0$ for a smooth function f satisfying $df \wedge \eta = 0$. Clearly, an almost f-cosymplectic manifold is an almost cosymplectic manifold under the condition that $f = 0$ and an almost α-Kenmotsu manifold if f is constant($\neq 0$). In particular, if $f = 1$ then M is an almost Kenmotsu manifold.

On the other hand, we observe that a remarkable class of contact metric manifold is (κ, μ)-space whose curvature tensor satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{1.2}$$

for any vector fields X, Y, where κ and μ are constants and $h := \frac{1}{2}\mathcal{L}_\xi\phi$ is a self-dual operator. In fact Sasakian manifolds are special (κ, μ)-spaces with $\kappa = 1$ and $h = 0$. An *almost cosymplectic (κ, μ)-manifold* is an almost cosymplectic manifold with curvature tensor satisfying (1.2). Endo proved that if $\kappa \neq 0$ any almost cosymplectic (κ, μ)-manifolds are not cosymplectic ([4]). Furthermore, since $\kappa\phi^2 = h^2$, $\kappa \leq 0$ and the equality holds if and only if the almost cosymplectic (κ, μ)-manifolds are cosymplectic. Notice that Wang proved the non-existence of gradient Ricci solitons in almost cosymplectic (κ, μ)-manifolds(see [17]).

In this paper we first obtain an non-existence of a Ricci soliton in almost cosymplectic (κ, μ)-manifolds, namely

Theorem 1.1. *There do not exist Ricci solitons on almost cosymplectic (κ, μ)-manifolds with $\kappa < 0$.*

Next we consider a three-dimensional almost f-cosymplectic manifold admitting a Ricci soliton whose potential vector field is the Reeb vector field ξ, and prove the following theorem.

Theorem 1.2. *A three-dimensional almost f-cosymplectic manifold M admits a Ricci soliton whose potential vector field is ξ if and only if ξ is Killing and M is Ricci flat.*

As it is well known that a $(2n+1)$-dimensional almost contact manifold (M,ϕ,ξ,η,g) is said to be *η-Einstein* if its Ricci tensor satisfies

$$Ric = ag + b\eta \otimes \eta, \tag{1.3}$$

where a and b are smooth functions. For a three-dimensional η-Einstein almost f-cosymplectic manifold M with a Ricci soliton we prove the following result:

Theorem 1.3. *Let (M,ϕ,η,ξ,g) be a three-dimensional η-Einstein almost f-cosymplectic manifold admitting a Ricci soliton. Then either M is an α-cosymplectic manifold, or M is an Einstein manifold of constant sectional curvature $\frac{\lambda}{2}$ with $\lambda = -2\xi(f) - 2f^2$.*

Remark 1.1. Our theorem extends the Wang and Liu's result [18]. In fact, when $f = 1$ then $a = -2$ in view of Proposition 5.1 in Section 5. Thus it follows from (2.8) that trace$(h^2) = 0$, i.e., $h = 0$. So M is also a Kenmotsu manifold of sectional curvature -1.

In order to prove these conclusions, in Section 2 we recall some basic concepts and formulas. The proofs of theorems will be given in Section 3, Section 4 and Section 5, respectively.

2 Some basic concepts and related results

In this section we will recall some basic concepts and equations. Let M^{2n+1} be a $(2n+1)$-dimensional Riemannian manifold. An *almost contact structure* on M is a triple (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field, ξ a unit vector field, η a one-form dual to ξ satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \eta \circ \phi = 0,\ \phi \circ \xi = 0. \tag{2.1}$$

A smooth manifold with such a structure is called an *almost contact manifold*. It is well-known that there exists a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any $X, Y \in \mathfrak{X}(M)$. It is easy to get from (2.1) and (2.2) that

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X,\xi) = \eta(X). \tag{2.3}$$

An almost contact structure (ϕ,ξ,η) is said to be *normal* if the corresponding complex structure J on $M \times \mathbb{R}$ is integrable.

Denote by ω the fundamental 2-form on M defined by $\omega(X,Y) := g(\phi X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. An *almost α-cosymplectic manifold* ([11, 14]) is an almost contact metric manifold (M,ϕ,ξ,η,g) such that the fundamental form ω and 1-form η satisfy $d\eta = 0$ and $d\omega = 2\alpha\eta \wedge \omega$, where α is a real number. In particular, M is an *almost cosymplectic manifold* if $\alpha = 0$ and an *almost Kenmotsu manifold* if $\alpha = 1$. In [1], a class of more general almost contact manifolds was defined by generalizing the real number α to a smooth function f. More precisely, an almost contact metric manifold is called an *almost f-cosymplectic manifold* if $d\eta = 0$ and $d\omega = 2f\eta \wedge \omega$ are satisfied, where f is a smooth function with $df \wedge \eta = 0$. In addition, a normal almost f-cosymplectic manifold is said to be an *f-cosymplectic manifold*.

Let M be an almost f-cosymplectic manifold, we recall that there is an operator $h = \frac{1}{2}\mathcal{L}_\xi\phi$ which is a self-dual operator. The Levi-Civita connection is given by (see [1])

$$2g((\nabla_X\phi)Y, Z) = 2fg(g(\phi X, Y)\xi - \eta(Y)\phi X, Z) + g(N(Y, Z), \phi X) \tag{2.4}$$

for arbitrary vector fields X, Y, where N is the Nijenhuis torsion of M. Then by a simple calculation, we have

$$\mathrm{trace}(h) = 0, \quad h\xi = 0, \quad \phi h = -h\phi, \quad g(hX, Y) = g(X, hY), \quad \forall X, Y \in \mathfrak{X}(M).$$

Write $AX := \nabla_X\xi$ for any vector field X. Thus A is a $(1, 1)$-tensor of M. Using (2.4), a straightforward calculation gives

$$AX = -f\phi^2X - \phi hX \tag{2.5}$$

and $\nabla_\xi\phi = 0$. By (2.3), it is obvious that $A\xi = 0$ and A is symmetric with respect to metric g, i.e., $g(AX, Y) = g(X, AY)$ for all $X, Y \in \mathfrak{X}(M)$. We denote by R and Ric the Riemannian curvature tensor and Ricci tensor, respectively. For an almost f-cosymplectic manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ the following equations were proved([1]):

$$R(X, \xi)\xi - \phi R(\phi X, \xi)\xi = 2[(\xi(f) + f^2)\phi^2X - h^2X], \tag{2.6}$$

$$(\nabla_\xi h)X = -\phi R(X, \xi)\xi - [\xi(f) + f^2]\phi X - 2fhX - \phi h^2X, \tag{2.7}$$

$$Ric(\xi, \xi) = -2n(\xi(f) + f^2) - \mathrm{trace}(h^2), \tag{2.8}$$

$$\mathrm{trace}(\phi h) = 0, \tag{2.9}$$

$$R(X, \xi)\xi = [\xi(f) + f^2]\phi^2X + 2f\phi hX - h^2X + \phi(\nabla_\xi h)X, \tag{2.10}$$

for any vector fields X, Y on M.

3 Proof of Theorem 1.1

In this section we suppose that (M, ϕ, ξ, η, g) is an almost cosymplectic (κ, μ)-manifold, i.e., the curvature tensor satisfies (1.2). In the following we always suppose $\kappa < 0$. The following relations are provided(see [12, Eq.(3.22) and Eq.(3.23)]):

$$Q = 2n\kappa\eta \otimes \xi + \mu h, \tag{3.1}$$

$$h^2 = \kappa\phi^2, \tag{3.2}$$

where Q is the Ricci operator defined by $\mathrm{Ric}(X, Y) = g(QX, Y)$ for any vectors X, Y. In particular, $Q\xi = 2n\kappa\xi$ because of $h\xi = 0$.

In view of (3.1) and the Ricci soliton equation (1.1), we obtain

$$(\mathcal{L}_V g)(Y, Z) = 2\lambda g(Y, Z) - 2\mu g(hY, Z) - 4n\kappa\eta(Y)\eta(Z) \tag{3.3}$$

for any vectors Y, Z. Since κ, μ are two real numbers and $\nabla_X \xi = AX$, differentiating (3.3) along any vector field X provides

$$\begin{aligned}(\nabla_X \mathcal{L}_V g)(Y,Z) =& \nabla_X((\mathcal{L}_V g)(Y,Z)) - \mathcal{L}_V g(\nabla_X Y, Z) - \mathcal{L}_V g(Y, \nabla_X Z) \\ =& -2\mu g((\nabla_X h)Y, Z) - 4n\kappa \nabla_X(\eta(Y))\eta(Z) - 4n\kappa\eta(Y)\nabla_X(\eta(Z)) \\ &+ 4n\kappa\eta(\nabla_X Y)\eta(Z) + 4n\kappa\eta(Y)\eta(\nabla_X Z) \\ =& -2\mu g((\nabla_X h)Y, Z) - 4n\kappa g(Y, AX)\eta(Z) - 4n\kappa\eta(Y)g(Z, AX).\end{aligned} \tag{3.4}$$

Moreover, making use of the commutation formula (see [20]):

$$\begin{aligned}&(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y,Z) = \\ &- g((\mathcal{L}_V \nabla)(X,Y), Z) - g((\mathcal{L}_V \nabla)(X,Z), Y),\end{aligned}$$

we derive

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V \nabla)(X,Y), Z) + g((\mathcal{L}_V \nabla)(X,Z), Y). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}g((\mathcal{L}_V \nabla)(Y,Z), X) =& \frac{1}{2}\Big\{(\nabla_Z \mathcal{L}_V g)(Y,X) + (\nabla_Y \mathcal{L}_V g)(Z,X) - (\nabla_X \mathcal{L}_V g)(Y,Z)\Big\} \\ =& -\mu\Big\{g((\nabla_Z h)Y, X) + g((\nabla_Y h)Z, X) - g((\nabla_X h)Y, Z) \\ &- 4n\kappa g(Y, AZ)\eta(X)\Big\}.\end{aligned} \tag{3.6}$$

Hence for any vector Y,

$$(\mathcal{L}_V \nabla)(Y, \xi) = -\mu\Big\{(\nabla_\xi h)Y + 2\kappa\phi Y\Big\} \tag{3.7}$$

by using (3.2) and (2.5). Lie differentiating (3.7) along V and making use of the identity([20]):

$$(\mathcal{L}_V R)(X,Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z), \tag{3.8}$$

we obtain

$$\begin{aligned}(\mathcal{L}_V R)(X,\xi)\xi =& (\nabla_X \mathcal{L}_V \nabla)(\xi,\xi) - (\nabla_\xi \mathcal{L}_V \nabla)(X,\xi) \\ =& -2(\mathcal{L}_V \nabla)(\nabla_X \xi, \xi) - (\nabla_\xi \mathcal{L}_V \nabla)(X,\xi) \\ =& -2\mu\Big\{(\nabla_\xi h)\nabla_X \xi + 2\kappa\phi\nabla_X \xi\Big\} + \mu\{(\nabla_\xi \nabla_\xi h)X)\} \\ =& -2\mu\Big\{\phi(\nabla_\xi h)hX + 2\kappa hX\Big\} + \mu\{(\nabla_\xi \nabla_\xi h)X)\}.\end{aligned} \tag{3.9}$$

Since $\mathrm{trace}(h) = 0$, contracting (3.9) over X gives

$$(\mathcal{L}_V Ric)(\xi,\xi) = \mu\ \mathrm{trace}(\nabla_\xi \nabla_\xi h - 2\phi(\nabla_\xi h)h).$$

Next we compute $\nabla_\xi \nabla_\xi h - 2\phi(\nabla_\xi h)h$: By (2.7), (3.2) and (2.1), we get $\nabla_\xi h = -\mu\phi h$, thus

$$\nabla_\xi \nabla_\xi h - 2\phi(\nabla_\xi h)h = \mu h - 2\kappa\mu\phi^2.$$

This means that

$$(\mathcal{L}_V Ric)(\xi, \xi) = 4n\kappa\mu^2. \tag{3.10}$$

On the other hand, by Lie differentiating the formula $Ric(\xi, \xi) = 2n\kappa$ along V, we also obtain

$$(\mathcal{L}_V Ric)(\xi, \xi) = -4n\kappa g(\mathcal{L}_V \xi, \xi).$$

Thus it follows from (3.10) that $g(\mathcal{L}_V \xi, \xi) = -\mu^2$.

Furthermore, notice that the Ricci tensor equation (3.1) implies the scalar curvature $r = 2n\kappa$ and recall the following integrability formula (see [16, Eq.(5)]):

$$\mathcal{L}_V r = -\Delta r - 2\lambda r + 2||Q||^2$$

for a Ricci soliton. By (3.1), (3.2) and the foregoing formula we thus obtain $\lambda = 2n\kappa - \mu^2$. Also, it follows from (3.3) that $g(\mathcal{L}_V \xi, \xi) = 2n\kappa - \lambda$. Therefore $g(\mathcal{L}_V \xi, \xi) = \mu^2$. Hence we find $\mu = 0$. However, from (3.4), (3.5) and (3.6), we have

$$g(Y, AX)\eta(Z) + g(Z, AX)\eta(Y) = 0$$

since $\kappa < 0$. Now putting $Z = \xi$ gives $g(Y, AX) = 0$ for any vector fields X, Y because $A\xi = 0$. That means that $AX = 0$ for any vector field X. From (2.5) with $f = 0$, we get $h = 0$. Clearly, it is impossible. Therefore we complete the proof.

4 Proof of Theorem 1.2

In this section we assume that M is a three dimensional almost f-cosymplectic manifold and the potential vector field V is the Reeb vector field. Before giving the proof, we need to prove the following lemma.

Lemma 4.1. *For any almost f-cosymplectic manifold the following formula holds:*

$$\begin{aligned}(\mathcal{L}_\xi R)(X, \xi)\xi =& 2\xi(f)\phi h X - 2[f(\nabla_\xi h)\phi X + (\nabla_\xi h)hX] \\ &+ [\xi(\xi(f)) + 2f\xi(f)]\phi^2 X + \phi(\nabla_\xi \nabla_\xi h)X.\end{aligned}$$

Proof. Obviously, $\mathcal{L}_\xi \eta = 0$ because $A\xi = 0$. Notice that for any vector fields X, Y, Z the 1-form η satisfies the following relation([20]):

$$-\eta((\mathcal{L}_X \nabla)(Y, Z)) = (\mathcal{L}_X(\nabla_Y \eta) - \nabla_Y(\mathcal{L}_X \eta) - \nabla_{[X,Y]}\eta)(Z). \tag{4.1}$$

Putting $X = \xi$ and using (2.5) yields

$$\begin{aligned}-\eta((\mathcal{L}_\xi \nabla)(Y, Z)) =& (\mathcal{L}_\xi(\nabla_Y \eta) - \nabla_Y(\mathcal{L}_\xi \eta) - \nabla_{[\xi,Y]}\eta)(Z) \\ =& \nabla_\xi((\nabla_Y \eta)(Z)) - (\nabla_Y \eta)(\mathcal{L}_\xi Z) - g(A([\xi, Y]), Z) \\ =& \nabla_\xi g(AY, Z) - g(AY, [\xi, Z]) - g(A([\xi, Y]), Z) \\ =& g((\nabla_\xi A)Y, Z) + 2g(AY, AZ).\end{aligned} \tag{4.2}$$

In view of (3.5), we obtain from (4.2) that

$$\begin{aligned}
g((\mathcal{L}_\xi\nabla)(X,\xi),Y) =&(\nabla_X\mathcal{L}_\xi g)(Y,\xi) - g((\mathcal{L}_\xi\nabla)(X,Y),\xi)\\
=&-2g(AX,AY) + g((\nabla_\xi A)X,Y) + 2g(AX,AY)\\
=&\xi(f)g(\phi X,\phi Y) + g((\nabla_\xi h)X,\phi Y).
\end{aligned}$$

That is,

$$(\mathcal{L}_\xi\nabla)(X,\xi) = -[\xi(f)]\phi^2 X - \phi(\nabla_\xi h)X.$$

Obviously, for any vector field X, we know $(\mathcal{L}_\xi\nabla)(X,\xi) = (\mathcal{L}_\xi\nabla)(\xi,X)$ from (3.6). Therefore we compute the Lie derivative of $R(X,\xi)\xi$ along ξ as follows:

$$\begin{aligned}
(\mathcal{L}_\xi R)(X,\xi)\xi =&(\nabla_X\mathcal{L}_\xi\nabla)(\xi,\xi) - (\nabla_\xi\mathcal{L}_\xi\nabla)(X,\xi)\\
=&-(\mathcal{L}_\xi\nabla)(\nabla_X\xi,\xi) - (\mathcal{L}_\xi\nabla)(\xi,\nabla_X\xi) - (\nabla_\xi\mathcal{L}_\xi\nabla)(X,\xi)\\
=&2[\xi(f)]\phi^2 AX + 2\phi(\nabla_\xi h)AX - (\nabla_\xi\mathcal{L}_\xi\nabla)(X,\xi)\\
=&2\xi(f)(f\phi^2 X + \phi hX) + 2[-f(\nabla_\xi h)\phi X - (\nabla_\xi h)hX]\\
&+[\xi(\xi(f))]\phi^2 X + \phi(\nabla_\xi\nabla_\xi h)X.
\end{aligned}$$

Here we used $\nabla_\xi\phi = 0$ and $\phi\nabla_\xi h = -(\nabla_\xi h)\phi$ followed from $h\phi + \phi h = 0$. ■

Proof of Theorem 1.2. Now we suppose that the potential vector $V = \xi$ in the Ricci equation (1.1). Then for any $X \in \mathfrak{X}(M)$,

$$-f\phi^2 X - \phi hX + QX = \lambda X. \tag{4.3}$$

Putting $X = \xi$ in (4.3), we have

$$Q\xi = \lambda\xi. \tag{4.4}$$

Moreover, the above formula together (2.8) with $n = 1$ leads to

$$\mathrm{trace}(h^2) = -\lambda - 2f^2 - 2\xi(f). \tag{4.5}$$

Since the curvature tensor of a 3-dimension Riemannian manifold is given by

$$\begin{aligned}
R(X,Y)Z =&g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y\\
&-\frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\},
\end{aligned}\tag{4.6}$$

where r denotes the scalar curvature. Putting $Y = Z = \xi$ in (4.6) and applying (4.3) and (4.4), we obtain

$$R(X,\xi)\xi = \Big(\frac{r}{2} + f - 2\lambda\Big)\phi^2 X + \phi hX. \tag{4.7}$$

Contracting the above formula over X leads to $Ric(\xi,\xi) = -r - 2f + 4\lambda$, which follows from (4.4) that

$$r + 2f = 3\lambda. \tag{4.8}$$

Taking the Lie derivative of (4.7) along ξ and using (4.8), we obtain

$$(\mathcal{L}_\xi R)(X,\xi)\xi = 2h^2 X + \phi(\mathcal{L}_\xi h)X \tag{4.9}$$

since $\mathcal{L}_\xi\phi^2 = 2\phi h + 2h\phi = 0$ and $h = \frac{1}{2}\mathcal{L}_\xi\phi$. By Lemma 4.1 and (4.9), we get

$$\begin{aligned} &2\xi(f)\phi hX - 2[f(\nabla_\xi h)\phi X + (\nabla_\xi h)hX] \\ &\quad + [\xi(\xi(f)) + 2f\xi(f)]\phi^2 X + \phi(\nabla_\xi\nabla_\xi h)X \\ =&2h^2X + \phi(\mathcal{L}_\xi h)X. \end{aligned} \tag{4.10}$$

By virtue of (2.7) and (4.7), we have

$$(\nabla_\xi h)X = \Big(-\frac{\lambda}{2} - \xi(f) - f^2\Big)\phi X + (1-2f)hX - \phi h^2 X. \tag{4.11}$$

Making use of the above equation we further compute

$$\begin{aligned} \phi(\nabla_\xi\nabla_\xi h)X =&\Big(-\xi\xi(f) - 2f\xi(f)\Big)\phi^2 X - 2\xi(f)\phi hX + (1-2f)\phi(\nabla_\xi h)X \\ &+ (\nabla_\xi h)hX + h(\nabla_\xi h)X. \end{aligned} \tag{4.12}$$

As well as via (2.5) we get

$$\begin{aligned} \phi(\mathcal{L}_\xi h)X =&\phi\mathcal{L}_\xi(hX) - \phi h([\xi, X]) \\ =&\phi(\nabla_\xi h)X + \phi(hA - Ah)X \\ =&\phi(\nabla_\xi h)X - 2h^2X. \end{aligned} \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.10), we derive

$$-(\nabla_\xi h)hX + h(\nabla_\xi h)X = 0. \tag{4.14}$$

Further, applying (4.11) in the formula (4.14), we get

$$\Big(-\lambda - 2f^2 - 2\xi(f)\Big)hX - h^3X = 0.$$

Write $\beta := -\lambda - 2f^2 - 2\xi(f)$, then the above equation is rewritten as $h^3X = \beta hX$ for every vector X. Denote by e_i and λ_i the eigenvectors and the corresponding eigenvalues for $i = 1, 2, 3$, respectively. If $h \neq 0$ then there is a nonzero eigenvalue $\lambda_1 \neq 0$ satisfying $\lambda_1^3 = \beta\lambda_1$, i.e., $\lambda_1^2 = \beta$. Since $\mathrm{trace}(h) = 0$ and $\mathrm{trace}(h^2) = \beta$ by (4.4), we have $\sum_{i=1}^3 \lambda_i = 0$ and $\sum_{i=1}^3 \lambda_i^2 = \beta$. This shows $\lambda_2 = \lambda_3 = 0$, which further leads to $\lambda_1 = 0$. It is a contradiction. Thus we have

$$\beta = -(\lambda + 2f^2 + 2\xi(f)) = 0 \quad \text{and} \quad h = 0. \tag{4.15}$$

On the other hand, taking the covariant differentiation of $Q\xi = \lambda\xi$ (see (4.4)) along arbitrary vector field X and using (2.5), one can easily deduce

$$(\nabla_X Q)\xi + Q(-f\phi^2X - \phi hX) = -\lambda(f\phi^2X + \phi hX).$$

Contracting this equation over X and using (4.3), we derive

$$\frac{1}{2}\xi(r) - 2f^2 - \mathrm{trace}(h^2) = 0.$$

By virtue of (4.4) and (4.8), the foregoing equation yields

$$\xi(f) + \lambda = 0. \tag{4.16}$$

This shows that $\xi(f)$ is constant, then it infers from (4.15) that $f = 0$ and $\lambda = 0$, that means that M is cosymplectic. Therefore it follows from (2.5) that $(\mathcal{L}_\xi g)(X,Y) = 2g(AX,Y) = 0$ for any vectors X, Y. By (4.3), it is obvious that $Q = 0$. Thus we complete the proof of Theorem 1.2.

By the proof of Theorem 1.2, the following result is clear.

Corollary 4.1. *A three-dimensional almost α-Kenmotsu manifold (M,ϕ,η,ξ,g) does not admit a Ricci soliton with potential vector field being ξ.*

5 Proof of Theorem 1.3

In this section we assume that M is a three dimensional η-Einstein almost f-cosymplectic manifold, i.e., the Ricci tensor $Ric = ag + b\eta \otimes \eta$. We first prove the following proposition.

Proposition 5.1. *Let M be a three dimensional η-Einstein almost f-cosymplectic manifold. Then the following relation is satisfied:*

$$2\xi(f) + 2f^2 + (a+b) = 0.$$

Proof. Since M is η-Einstein, we know that $Q\xi = (a+b)\xi$ and the scalar curvature $r = 3a + b$. Hence it follows from (4.6) that

$$R(X,\xi)\xi = -\frac{a+b}{2}\phi^2 X. \tag{5.1}$$

By Lie differentiating (5.1), we derive

$$(\mathcal{L}_\xi R)(X,\xi)\xi = -\frac{\xi(a+b)}{2}\phi^2 X. \tag{5.2}$$

Also, it follows from (5.1) and (2.7) that

$$(\nabla_\xi h)X = -\left[\xi(f) + f^2 + \frac{1}{2}(a+b)\right]\phi X - 2fhX - \phi h^2 X. \tag{5.3}$$

Moreover, we get

$$\begin{aligned}\phi(\nabla_\xi \nabla_\xi h)X = &-\Big[\xi(\xi(f)) + 2f\xi(f) + \frac{\xi(a+b)}{2}\Big]\phi^2 X \\ &- 2\xi(f)\phi hX - 2f\phi(\nabla_\xi h)X + (\nabla_\xi h)hX + h(\nabla_\xi h)X.\end{aligned} \tag{5.4}$$

Therefore by using (5.3) and (5.4), the formula of Lemma 4.1 becomes

$$\begin{aligned}(\mathcal{L}_\xi R)(X,\xi)\xi &= -\frac{\xi(a+b)}{2}\phi^2 X - (\nabla_\xi h)hX + h(\nabla_\xi h)X \\ &= -\frac{\xi(a+b)}{2}\phi^2 X + 2\Big[\xi(f) + f^2 + \frac{1}{2}(a+b)\Big]\phi hX + 2\phi h^3 X.\end{aligned}$$

Combining this with (5.2) yields

$$0 = [\xi(f) + f^2 + \frac{1}{2}(a+b)]hX + h^3X. \tag{5.5}$$

As in the proof of (4.15), the formula (5.5) yields the assertion. ∎

Proof of Theorem 1.3. In view of Proposition 5.1, we know $a + b = -2\xi(f) - 2f^2$. Because $(\mathcal{L}_V g)(Y, Z) = 2\lambda g(Y, Z) - 2g(QY, Z)$ for any vector fields Y, Z, we compute

$$\begin{aligned}(\nabla_X \mathcal{L}_V g)(Y, Z) &= -2g((\nabla_X Q)Y, Z)\\ &= -2g(X(a)Y + X(b)\eta(Y)\xi + bg(AX, Y)\xi + b\eta(Y)AX, Z).\end{aligned}$$

Hence

$$\begin{aligned}g((\mathcal{L}_V\nabla)(Y, Z), X) =& \frac{1}{2}\Big\{(\nabla_Z \mathcal{L}_V g)(Y, X) + (\nabla_Y \mathcal{L}_V g)(Z, X) - (\nabla_X \mathcal{L}_V g)(Y, Z)\Big\}\\ =& -\Big\{g(Z(a)Y + Z(b)\eta(Y)\xi + bg(AZ, Y)\xi + b\eta(Y)AZ, X)\\ &+ g(Y(a)Z + Y(b)\eta(Z)\xi + bg(AY, Z)\xi + b\eta(Z)AY, X)\\ &- g(X(a)Y + X(b)\eta(Y)\xi + bg(AX, Y)\xi + b\eta(Y)AX, Z)\Big\}\\ =& -\Big\{Z(a)g(Y, X) + Z(b)\eta(Y)\eta(X) + Y(a)g(X, Z)+\\ &Y(b)\eta(Z)\eta(X) + 2b\eta(X)g(AY, Z) - X(a)g(Y, Z)-\\ &X(b)\eta(Y)\eta(Z)\Big\}.\end{aligned}$$

That means that

$$\begin{aligned}(\mathcal{L}_V\nabla)(Y, Z) = & -Z(a)Y - Z(b)\eta(Y)\xi - Y(a)Z - Y(b)\eta(Z)\xi\\ &- 2bg(AY, Z)\xi + g(Z, Y)\nabla a + \eta(Y)\eta(Z)\nabla b.\end{aligned} \tag{5.6}$$

Taking the covariant differentiation of $(\mathcal{L}_V\nabla)(Y, Z)$ along any vector field X, we may obtain

$$\begin{aligned}&(\nabla_X \mathcal{L}_V \nabla)(Y, Z)\\ &= -g(Z, \nabla_X\nabla a)Y - g(Z, \nabla_X\nabla b)\eta(Y)\xi - Z(b)g(AX, Y)\xi\\ &- Z(b)\eta(Y)AX - g(Y, \nabla_X\nabla a)Z - g(Y, \nabla_X\nabla b)\eta(Z)\xi - Y(b)g(AX, Z)\xi\\ &- Y(b)\eta(Z)AX - 2X(b)g(AY, Z)\xi - 2bg((\nabla_X A)Y, Z)\xi - 2bg(AY, Z)AX\\ &+ g(Z, Y)\nabla_X\nabla a + g(AX, Y)\eta(Z)\nabla b + \eta(Y)g(AX, Z)\nabla b + \eta(Y)\eta(Z)\nabla_X\nabla b.\end{aligned}$$

Thus by virtue of (3.8) we have

$$
\begin{aligned}
&(\mathcal{L}_V R)(X,Y)Z \\
=&(\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z) \\
=&-g(Z,\nabla_X \nabla a)Y - g(Z,\nabla_X\nabla b)\eta(Y)\xi - Z(b)\eta(Y)AX - Y(b)g(AX,Z)\xi \\
&- Y(b)\eta(Z)AX - 2X(b)g(AY,Z)\xi - 2bg((\nabla_X A)Y,Z)\xi - 2bg(AY,Z)AX \\
&+ g(Z,Y)\nabla_X\nabla a + \eta(Y)g(AX,Z)\nabla b + \eta(Y)\eta(Z)\nabla_X\nabla b \\
&- \Big[-g(Z,\nabla_Y\nabla a)X - g(Z,\nabla_Y\nabla b)\eta(X)\xi - Z(b)\eta(X)AY - X(b)g(AY,Z)\xi \\
&- X(b)\eta(Z)AY - 2Y(b)g(AX,Z)\xi - 2bg((\nabla_Y A)X,Z)\xi - 2bg(AX,Z)AY \\
&+ g(Z,X)\nabla_Y\nabla a + \eta(X)g(AY,Z)\nabla b + \eta(X)\eta(Z)\nabla_Y\nabla b \Big]
\end{aligned}
$$

since $g(X,\nabla_Y\nabla\zeta) = g(Y,\nabla_X\nabla\zeta)$ for any function ζ and vector fields X,Y followed from Poincaré lemma.

By contracting over X in the previous formula, we have

$$
\begin{aligned}
&(\mathcal{L}_V Ric)(Y,Z) \\
=&g(Z,\nabla_Y\nabla a) - g(Z,\nabla_\xi\nabla b)\eta(Y) - 2fZ(b)\eta(Y) \\
&- 2fY(b)\eta(Z) - 2\xi(b)g(AY,Z) - 2bg((\nabla_\xi A)Y,Z) - 4fbg(AY,Z) \\
&+ g(Z,Y)\Delta a + \eta(Y)g(A\nabla b,Z) + \eta(Y)\eta(Z)\Delta b \\
&- \Big[-g(Z,\nabla_Y\nabla b) - AY(b)\eta(Z) + \eta(Z)g(\nabla_Y\nabla b,\xi)\Big].
\end{aligned}
\tag{5.7}
$$

On the other hand, since $QX = aX + b\eta(X)\xi$ and $r = 3a+b$, the equation (4.6) is expressed as

$$
\begin{aligned}
R(X,Y)Z =&bg(Y,Z)\eta(X)\xi - bg(X,Z)\eta(Y)\xi + b\eta(Y)\eta(Z)X - b\eta(X)\eta(Z)Y \\
&+ \frac{a-b}{2}[g(Y,Z)X - g(X,Z)Y].
\end{aligned}
\tag{5.8}
$$

Thus we get the Lie derivative of $R(X,Y)Z$ along V from (5.8)

$$
\begin{aligned}
&(\mathcal{L}_V R)(X,Y)Z \\
=&V(b)g(Y,Z)\eta(X)\xi + b(\mathcal{L}_V g)(Y,Z)\eta(X)\xi \\
&+ bg(Y,Z)(\mathcal{L}_V\eta)(X)\xi + bg(Y,Z)\eta(X)\mathcal{L}_V\xi \\
&- [V(b)g(X,Z)\eta(Y)\xi + b(\mathcal{L}_V g)(X,Z)\eta(Y)\xi \\
&+ bg(X,Z)(\mathcal{L}_V\eta)(Y)\xi + bg(X,Z)\eta(Y)\mathcal{L}_V\xi] \\
&+ V(b)\eta(Y)\eta(Z)X + b(\mathcal{L}_V\eta)(Y)\eta(Z)X + b\eta(Y)(\mathcal{L}_V\eta)(Z)X \\
&- [V(b)\eta(X)\eta(Z)Y + b(\mathcal{L}_V\eta)(X)\eta(Z)Y + b\eta(X)(\mathcal{L}_V\eta)(Z)Y] \\
&+ \frac{V(a-b)}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{a-b}{2}[(\mathcal{L}_V g)(Y,Z)X - (\mathcal{L}_V g)(X,Z)Y].
\end{aligned}
\tag{5.9}
$$

Contracting over X in (5.9) gives

$$\begin{aligned}
&(\mathcal{L}_V Ric)(Y,Z) \\
=&V(b)g(Y,Z)+\ b(\mathcal{L}_V g)(Y,Z)+\ bg(Y,Z)(\mathcal{L}_V\eta)(\xi)\\
&+\ bg(Y,Z)g(\mathcal{L}_V\xi,\xi)-[V(\ b)\eta(Z)\eta(Y)+\ b(\mathcal{L}_V g)(\xi,Z)\eta(Y)\\
&+b\eta(Z)(\mathcal{L}_V\eta)(Y)+b\eta(Y)g(\mathcal{L}_V\xi,Z)]\\
&+3V(b)\eta(Y)\eta(Z)+3b(\mathcal{L}_V\eta)(Y)\eta(Z)+3b\eta(Y)(\mathcal{L}_V\eta)(Z)\\
&-[V(b)\eta(Y)\eta(Z)+b(\mathcal{L}_V\eta)(Y)\eta(Z)+b\eta(Y)(\mathcal{L}_V\eta)(Z)]\\
&+\frac{V(a-b)}{2}[2g(Y,Z)]+\frac{a-b}{2}[2(\mathcal{L}_V g)(Y,Z)]\\
=&bg(Y,Z)g(\mathcal{L}_V\xi,\xi)-[b(\mathcal{L}_V g)(\xi,Z)\eta(Y)+b\eta(Y)g(\mathcal{L}_V\xi,Z)]\\
&+V(b)\eta(Y)\eta(Z)+b(\mathcal{L}_V\eta)(Y)\eta(Z)+2b\eta(Y)(\mathcal{L}_V\eta)(Z)\\
&+V(a)g(Y,Z)+a(\mathcal{L}_V g)(Y,Z)\\
=&bg(Y,Z)g(\mathcal{L}_V\xi,\xi)-[2b(\lambda-a-b)\eta(Z)\eta(Y)+b\eta(Y)g(\mathcal{L}_V\xi,Z)]\\
&+V(b)\eta(Y)\eta(Z)+b[g(AY,V)+\eta(\nabla_Y V)]\eta(Z)\\
&+2b\eta(Y)[g(AZ,V)+\eta(\nabla_Z V)]+V(a)g(Y,Z)\\
&+a\Big(2\lambda g(Y,Z)-2ag(Y,Z)-2b\eta(Y)\eta(Z)\Big).
\end{aligned}\tag{5.10}$$

Finally, by comparing (5.10) with (5.7) we get

$$\begin{aligned}
&g(Z,\nabla_Y\nabla a)-g(Z,\nabla_\xi\nabla b)\eta(Y)-2fZ(b)\eta(Y)\\
&-2fY(b)\eta(Z)-2\xi(b)g(AY,Z)-2bg((\nabla_\xi A)Y,Z)-4fbg(AY,Z)\\
&+g(Z,Y)\Delta a+\eta(Y)g(A\nabla b,Z)+\eta(Y)\eta(Z)\Delta b\\
&-\Big[-g(Z,\nabla_Y\nabla b)-AY(b)\eta(Z)+\eta(Z)g(\nabla_Y\nabla b,\xi)\Big]\\
=&bg(Y,Z)g(\mathcal{L}_V\xi,\xi)-[2b(\lambda-a-b)\eta(Z)\eta(Y)+b\eta(Y)g(\mathcal{L}_V\xi,Z)]\\
&+V(b)\eta(Y)\eta(Z)+b[g(AY,V)+\eta(\nabla_Y V)]\eta(Z)\\
&+2b\eta(Y)[g(AZ,V)+\eta(\nabla_Z V)]+V(a)g(Y,Z)\\
&+a(2\lambda g(Y,Z)-2ag(Y,Z)-2b\eta(Y)\eta(Z)).
\end{aligned}$$

Replacing Y and Z by ϕY and ϕZ respectively leads to

$$\begin{aligned}
&g(\phi Z,\nabla_{\phi Y}\nabla a)-2\xi(b)g(A\phi Y,\phi Z)-2bg((\nabla_\xi A)\phi Y,\phi Z)\\
&-4fbg(A\phi Y,\phi Z)+g(\phi Z,\phi Y)\Delta a-\Big[-g(\phi Z,\nabla_{\phi Y}\nabla b)\Big]\\
=&bg(\phi Y,\phi Z)g(\mathcal{L}_V\xi,\xi)+V(a)g(\phi Y,\phi Z)+2a(\lambda-a)g(\phi Y,\phi Z).
\end{aligned}\tag{5.11}$$

Because $\sum_{i=1}^{3}(\mathcal{L}_V(e_i,e_i))=0$ (see [7, Eq.(9)]), by (5.6) we obtain the gradient field

$$\nabla(a+b)=2[\xi(b)+2fb]\xi.\tag{5.12}$$

Therefore the formula (5.11) can be simplified as

$$\begin{aligned}
&-2bg((\nabla_\xi A)\phi Y,\phi Z)+g(\phi Z,\phi Y)\Delta a\\
&=bg(\phi Y,\phi Z)g(\mathcal{L}_V\xi,\xi)+V(a)g(\phi Y,\phi Z)+2a(\lambda-a)g(\phi Y,\phi Z).
\end{aligned}\tag{5.13}$$

Moreover, using (2.5) and (5.3) we compute

$$\begin{aligned}(\nabla_\xi A)\phi Y =& -\xi(f)\phi^3 Y + \phi^2(\nabla_\xi h)Y \\ =&\xi(f)\phi Y + \Big(\frac{a+b}{2} + \xi(f) + f^2\Big)\phi Y + 2fhY + \phi h^2 Y \\ =&\Big(\frac{a+b}{2} + 2\xi(f) + f^2\Big)\phi Y + 2fhY + \phi h^2 Y.\end{aligned}$$

Substituting this into (5.13) yields

$$\begin{aligned}&-2bg(2fhY + \phi h^2 Y, \phi Z) \\ =&\Big\{bg(\mathcal{L}_V\xi, \xi) + 2b\Big[\frac{a+b}{2} + 2\xi(f) + f^2\Big] \\ &- \Delta a + V(a) + 2a(\lambda - a)\Big\}g(\phi Y, \phi Z).\end{aligned} \tag{5.14}$$

Replacing Z by ϕY in (5.14) gives

$$0 = bg(2fhY + \phi h^2 Y, \phi^2 Y) = -bg(2fhY + \phi h^2 Y, Y). \tag{5.15}$$

Next we divide into two cases.

Case I: $h = 0$. Then the Eq.(2.8) and Proposition 5.1 imply $\xi(f) = 0$, i.e., f is constant as $\nabla f = \xi(f)\xi$ followed from $df \wedge \eta = 0$, so M is an α-cosymplectic manifold.

Case II: $h \neq 0$. Suppose that $Y = e$ is an unit eigenvector corresponding to the nonzero eigenvalue λ' of h, then we may obtain from (5.15) that $fb = 0$. If the function b is not zero, there is an open neighborhood $\mathcal{U}$ such that $b|_{\mathcal{U}} \neq 0$, so $f|_{\mathcal{U}} = 0$. By Proposition 5.1 it implies $(a+b)|_{\mathcal{U}} = 0$. Notice that trace$(h^2) = -(a+b) - 2(\xi f + f^2)$, so we have trace$(h^2)|_{\mathcal{U}} = 0$, i.e., $h|_{\mathcal{U}} = 0$. It comes to a contradiction, so $b = 0$ and a is constant by (5.12). Hence by (1.3) we obtain $Ric = ag$, where $a = -2\xi(f) - 2f^2$. Moreover, we know $\lambda = a$ or $a = 0$ by (5.13). Finally we show $a \neq 0$. In fact, if $a = 0$ then $\xi(f) = -f^2$. Substituting this into (2.8) we find trace$(h^2) = -2(\xi(f) + f^2) = 0$, which is impossible because $h \neq 0$.

Thus we obtain from (5.8)

$$R(X,Y)Z = \frac{\lambda}{2}(g(Y,Z)X - g(X,Z)Y).$$

Summing up the above two cases we finish the proof Theorem 1.3.

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