Splitting Madsen-Tillmann spectra I. Twisted transfer maps

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Abstract

We record various properties of twisted Becker-Gottlieb transfer maps and study their multiplicative properties analogous to Becker-Gottlieb transfer. We show these twisted transfer maps factor through Becker-Schultz-Mann-Miller-Miller transfer; some of these might be well known. We apply this to show that $BSO(2n+1)_+$ splits off $MTO(2n)$, which after localisation away from 2, refines to a homotopy equivalence $MTO(2n) \simeq BO(2n)_+$ as well as $MTO(2n+1) \simeq *$ for all $n \geqslant 0$. This reduces the study of $MTO(n)$ to the 2-localized case. At the prime 2 our splitting allows us to identify some algebraically independent classes in mod 2 cohomology of $\Omega^\infty MTO(2n)$. We also show that BG_+ splits off MTK for some pairs (G, K) at appropriate set of primes p, and investigate the consequences for characteristic classes, including algebraic independence and non-divisibility of some universally defined characteristic classes, generalizing results of Ebert and Randal-Williams.

1 Introduction and statement of results

For $\mathbf{K} = O, U, SO, SU, Sp, Pin,$ or $Spin$, the Madsen-Tillmann spectrum $MT\mathbf{K}(n)$ ([19]) is defined to be $B\mathbf{K}(n)^{-\gamma_n}$, the Thom spectrum of $-\gamma_n$ where γ_n is the canonical bundle the classifying space for n-dimensional $\mathbf{K}$-vector bundles $B\mathbf{K}(n)$ (see Appendix A for notes on classifying spaces). One can associate to a fibre bundle $M \longrightarrow E \longrightarrow B$ whose fibre M is a n-dimensional manifolds with $\mathbf{K}$-structure the Madsen-Tillmann-Weiss map $M \longrightarrow \Omega^\infty MT\mathbf{K}(n)$ (see e. g. [14]

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for a nice account). The cohomology ring $H^*(\Omega^\infty MT\mathbf{K}(n);R)$ then contains the R-characteristic classes for such bundles, where R is some relevant ring. The decomposition $\Omega^\infty X \simeq \Omega_0^\infty X \times \pi_0(X)$ (here $\pi_0(X)$ is equipped with the discrete topology) reduces the study of any infinite loop space to that of its base-point component. It is known that $H^*(\Omega_0^\infty MT\mathbf{K}(n);\mathbb{Q})$ is the free commutative algebra generated by $H^{*>0}(MT\mathbf{K}(n);\mathbb{Q})$. As the torsion-free quotient of $H^{*>0}(\Omega_0^\infty MT\mathbf{K}(n);\mathbb{Z})$ injects to $H^{*>0}(\Omega_0^\infty MT\mathbf{K}(n);\mathbb{Q})$, this gives us a good knowledge of the torsion-free quotient of $H^{*>0}(\Omega_0^\infty MT\mathbf{K}(n);\mathbb{Z})$. To understand the remaining torsion part, we need to know the $\mathbb{Z}/p$-coefficient case, which seems rather difficult. In fact, for $p=2$, the only existing computations in the literature are due to Galatius and Randal-Williams; they have shown that there exist short exact sequences of Hopf algebras

$$H_*(\Omega_0^\infty MT\mathbf{K}(n);\mathbb{Z}/2) \longrightarrow H_*(Q_0B\mathbf{K}(n)_+;\mathbb{Z}/2) \longrightarrow H_*(\Omega_0^\infty MT\mathbf{K}(n-1);\mathbb{Z}/2)$$

where $\mathbf{K}=SO$ with $n=2$ (equivalently with $\mathbf{K}=U$ and $n=1$) [17, Theorem 1.3], and $\mathbf{K}=O$ with $n=1,2$ [41, Theorem A, Theorem B]. Here, Q denotes $\Omega^\infty\Sigma^\infty$ and the subscript 0 indicates the base point component of the associated infinite loop space. The maps are induced by maps in the cofibration of spectra below: [19, Proposition 3.1] (see also Lemma 2.6)

$$MT\mathbf{K}(n) \xrightarrow{\omega_{\mathbf{K}(n)}} B\mathbf{K}(n)_+ \xrightarrow{\tilde{t}} MT\mathbf{K}(n-1).$$

Here $\omega_{\mathbf{K}(n)}$ is the Thomification of the inclusion $-\gamma_n \rightarrow (-\gamma_n)\oplus\gamma_n$, and $\tilde{t}$ denotes the Becker-Schultz-Mann-Miller-Miller transfer discussed in Section 2.3. The case for $\mathbf{K}=Spin$ with $n=2$ has been treated in [18, Theorems 1.2, 1.3, 1.7], the results don't allow such a simple description.

At odd primes, as far as we are aware, aside from some degenerate cases, the only computation is due to Galatius for the case of $\mathbf{K}=U$, $n=1$ [17, Theorem 1.4, Corollary 1.5]. It is therefore of interest for people working in the field to proceed with further computations, or at least identify nontrivial torsion classes in (co-)homology of $\Omega^\infty MT\mathbf{K}(n)$. We are interested in splitting these spectra, so that some more familiar pieces could be identified which consequently tell us about pieces of cohomology rings $H^*(\Omega^\infty MT\mathbf{K}(n);\mathbb{Z}/p)$. We shall use standard methods of stable homotopy theory, which in this paper are mainly based on using various transfer maps, and Steinberg idempotent as well as the Whitehead conjecture in a sequel [25].

Now we summarize our main results. In many cases, we only sketch them, the detailed statement can be found in the relevant sections.

We begin by recording an observation on the twisted Becker-Gottlieb transfer map which are probably known to experts, but we don't know of any published account.

Theorem 1.1. *1. (Theorem C.4) For a fibre bundle $\pi: E \longrightarrow B$ over a "nice space" B with fibre a compact manifold F, and a vector bundle ζ over B, one can construct the "twisted" Becker-Gottlieb transfer $t_\pi^\zeta : B^\zeta \longrightarrow E^{\pi^*\zeta}$ enjoying similar properties as the usual Becker-Gottlieb transfer. Notably the composition*

$$B^\zeta \xrightarrow{t_\pi^\zeta} E^{\pi^*\zeta} \xrightarrow{Th^\zeta(\pi)} B^\zeta$$

induces multiplication by $\chi(F)$ in ordinary homology.

2. *(Proposition 2.3) For a compact Lie group G and a closed subgroup $K \subset G$, the twisted Becker-Gottlieb transfer factors through the Becker-Schultz-Mann-Miller-Miller transfer (see Subsection 2.3).*

Before proceeding further, we fix one **important terminology**. For spaces, we have two distinct notions of *splitting*. We say, when $X \cong Y \times Z$, that Y splits off X (as a direct factor). As we identify a space with its suspension spectrum, we also say, when $\Sigma^\infty X \cong \Sigma^\infty Y \vee \Sigma^\infty Z'$, that Y splits off X (as a stable wedge summand). It is easy to see that the first implies the second. Sometimes, we use the same word splitting for two notions, the meaning being clear from the context.

Note that the infinite loop space functor Ω^∞ commutes with the localisation. Thus as our main applications concern mod p (co)homology for given prime p, there is no loss of information by localizing at p. Thus we mainly work with spaces/spectra localized at a prime p, which we denote by the subscript $_{(p)}$. As we work mainly in the category of spectra, we also identify a (pointed) space X with its suspension spectrum.

Thus the above implies:

Corollary 1.2. 1. *Let $F \longrightarrow E \longrightarrow B$ be as above. If $\chi(F)$ is prime to p, then $B^{\zeta}_{(p)}$ splits off $E^{\pi^*\zeta}_{(p)}$.*

2. *(Corollary 2.4) Let G be a compact Lie group, K a closed subgroup, and η a vector bundle over BG. If $\chi(G/K)$ is prime to p, then $BG^{\eta}_{(p)}$ splits off $BK^{\eta_{|K}\oplus \mathrm{ad}_K - \mathrm{ad}_{G|K}}_{(p)}$.*

Corollary 1.2 (ii) is an important tool in proving some of our main splitting results, upon various choices of $K \subset G$ and η. Our results below provide a list of such examples, where the main task is to identify $BK^{\eta_{|K}\oplus \mathrm{ad}_K - \mathrm{ad}_{G|K}}$ as a Madsen-Tillmann spectrum.

Theorem 1.3. 1. *(Theorems 3.1, 3.3) Let G, K, p be as in Theorems 3.1 (i), (ii), or 3.3. Then $BG_{+(p)}$ splits off the Madsen-Tillmann spectra $MTK_{(p)}$.*

2. *(Lemma 3.5) If the prime p is odd, then we have*

$$MTO(2n)_{(p)} \simeq BO(2n)_{+(p)}, MTO(2n-1)_{(p)} \simeq *.$$

Thus we have, at odd primes,

$$MTO(2n)_{(p)} \simeq BSO(2n+1)_{+(p)} \simeq BO(2n)_{+(p)} \simeq BO(2n+1)_{+(p)} \simeq BSp(n)_{+(p)},$$

where the equivalences $BSO(2n+1)_{+(p)} \simeq BO(2n)_{+(p)} \simeq BO(2n+1)_{+(p)} \simeq BSp(n)_{+(p)}$ are classic.

Splitting of a spectrum E into a wedge, say $E_1 \vee E_2$, implies that the infinite loop space $\Omega^\infty E$ decomposes as a product of infinite loop spaces $\Omega^\infty E_1 \times \Omega^\infty E_2$. Thus, we have the following:

Corollary 1.4. *Let (G, K) and p be as in one of the above theorems. Then, as infinite loop spaces, $\Omega^\infty MTK_{(p)}$ decomposes as a product of $QBG_{+(p)}$ and another factor.*

Next, notice that for any pointed space X, we have $\Sigma^\infty(X_+) \cong \Sigma^\infty(X) \vee S^0$. Thus if $BG_{+(p)}$ splits off $MTK_{(p)}$, then so does $S^0_{(p)}$. At the level of infinite loop spaces, this implies that $QS^0_{(p)}$ splits off $\Omega^\infty MTK_{(p)}$. This splitting however, can also be obtained by another method. That is, the Madsen-Tillmann-Weiss map allows us to split $S^0_{(p)}$ from slightly wider class of Madsen-Tillmann spectra, including $MTSp(n)_{(p)}$'s. Thus:

Theorem 1.5 ((Theorem 2.2)). *Suppose there exists a manifold M with K-structure. Then $S^0_{(p)}$ splits off $MTK_{(p)}$ at a prime p if p doesn't divide $\chi(M)$.*

Concrete examples are given in the statement of Theorem 2.2.

We note that by either method the map from MTK to S^0 is obtained by the composition

$$MTK \xrightarrow{\omega_K} BK_+ \xrightarrow{c} S^0,$$

where ω_K is the Thomification of the inclusion $-\gamma \longrightarrow (-\gamma) \oplus \gamma$, γ denoting the appropriate universal bundle over BK, c is the "collapse" map, that is the map that sends the base point to the base point, all the rest to the other point in S^0.

At the relevant primes, Theorem 1.5 implies that $\pi_* MTK_{(p)}$ contains $\pi_*(S^0_{(p)})$, the stable homotopy groups of the sphere as a summand. It also implies that $H^*(\Omega^\infty_0 MTK; \mathbb{Z}/p)$ contains a copy of $H^*(Q_0 S^0; \mathbb{Z}/p)$ as a tensor factor. Thus all non-trivial characteristic classes in $H^*(Q_0 S^0; \mathbb{Z}/p)$ are non-trivial in $H^*(\Omega^\infty MT\mathbf{K}(n); \mathbb{Z}/p)$. Thus we can generalize [41, Theorem 6.1], or rather [41, Lemma 6.3], as we are not pulling back the characteristic classes to moduli spaces, and show:

Corollary 1.6 ((Corollary 4.1)). *Let K be as in Corollary 4.1. Then the composition*

$$MTK \xrightarrow{\omega_K} BK_+ \xrightarrow{c} S^0 \xrightarrow{\iota} KO,$$

where ι is the unit map, induces an injection in mod 2 cohomology of infinite loop spaces

$$H^*(\mathbb{Z} \times BO; \mathbb{Z}/2) \hookrightarrow H^*(\Omega^\infty MTK; \mathbb{Z}/2).$$

Thus if we define the class $\xi_i \in H^*(\Omega^\infty_0 MTK; \mathbb{Z}/2)$ by

$$\xi_i = (\omega_K \circ c \circ \iota)^*(w_i),$$

then they are algebraically independent.

Let $F \longrightarrow E \xrightarrow{\pi} B$ be a manifold (with suitable structure) bundle with the associated Madsen-Tillmann-Weiss map $f_\pi : B \longrightarrow \Omega^\infty_0 MTK$. One can define the characteristic class $\xi_i(E)$ of this bundle simply as the pull-back $\xi_i(E) = f_\pi^*(\xi_i)$. Note that as in [41, Theorem 6.2], one can give a more geometrical interpretation of these characteristic classes, with the equality $\xi_i(E) = w_i(KO^*(t_\pi)(1))$ where t_π is the Becker-Gottlieb transfer, and $KO^*(t_\pi)(1)$ is the virtual bundle given by $\Sigma(-1)^i[H^i(F_b, \mathbb{R})]$ ([5, Theorem 6.1]).

Note that in the case of $MTO(2)$, we have, $\tau(\xi_i) = \chi_i$ where the τ is the conjugation of the Hopf algebra $H^*(\Omega^\infty_0 MTO(2); \mathbb{Z}/2)$, where χ_i's are defined in

[41, Theorem C]. This is because $w_i(V)$ and $w_i(-V)$ are related by the conjugation of the Hopf algebra $H^*(BO;\mathbb{Z}/2)$, and the maps of Hopf algebra respect the conjugation.

The complex analogue of the above using the Chern classes also holds, that is, if we use KU instead of KO and $c_{i(p-1)}$ instead of w_i in the above to define $\zeta_i^{\mathbb{C}}$, then we have:

Corollary 1.7 ((Corollary 4.2))**.** *Let K and p be as in Theorem 2.2. The classes $\zeta_i^{\mathbb{C}} \in H^*(\Omega^\infty(MTK;\mathbb{Z}/p))$'s are algebraically independent.*

Again we can interpret the characteristic class $\zeta_i^{\mathbb{C}}$ geometrically as before, using appropriate Chern classes and KU-cohomology instead of Stiefel-Whitney classes and KO-cohomology.

Another family of characteristic classes, arising from the cohomology of the classifying space BG, are discussed in [41, Subsection 2.4].

Definition 1.8. *A universally defined characteristic class is an element in the image of the map*

$$H^*(BK;R) \xrightarrow{\sigma^{\infty *}} H^*(Q_0(BK_+);R) \xrightarrow{\omega_K^*} H^*(\Omega_0^\infty MTK;R).$$

We write $\overline{v}_c$ for the image of $c \in H^*(BK;R)$ in $H^*(\Omega_0^\infty MTK;R)$. For a manifold bundle $F \longrightarrow E \xrightarrow{\pi} B$ with K structure on F with the associated Madsen-Tillmann-Weiss map $f_\pi : B \longrightarrow \Omega_0^\infty MTK$, $\overline{v}_c(E)$ is defined by

$$\overline{v}_c(E) = f_\pi{}^*(\overline{v}_c) \in H^*(B;R).$$

This includes Wahl's ζ classes, Randal-Williams' μ-classes, and the Miller-Morita-Mumford κ classes, we will come back to this later. The arguments as in the proof of [41, Theorem 2.4] show that this definition agrees with the usual one. The method of [41, Example 2.6] gives some relations among them. Our splitting theorem can be used to show that, for classes arising from the summand

$$H^*(BSO(2n+1);\mathbb{Z}/2) \subset H^*(BO(2n);\mathbb{Z}/2),$$

there can be no other relations as they live in a tensor factor

$$H^*(Q_0BSO(2n+1)_+;\mathbb{Z}/2) \subset H^*(\Omega_0^\infty MTO(2n)),$$

which is understood by [47]. In subsequent work [25], we will discuss relations among other classes, and in particular establish a complete set of relations when $n = 1$. In many cases, $H^*(BK;R)$ is a polynomial algebra, and if not, it contains a polynomial algebra generated by a family of characteristic classes (Theorem A.1). So we will use following conventions for the ease of notation. If $I = (i_1, i_2, \cdots i_n)$, and $a_1, \cdots a_n$'s are some cohomology classes indexed by integers, then a^I will denote the monomial $a_1^{i_1} \cdots a_n^{i_n}$. In the case of the Stiefel-Whitney classes or Chern classes in the cohomology of $BSO(n)$ or $BSU(n)$ respectively, we simply skip the index i_1. Now we can state the following:

Theorem 1.9 ((Theorem 4.6)). *Let $\nu_I = \overline{\nu}_{Bj^*(w^I)}$ where $j : O(2n) \longrightarrow SO(2n+1)$ will be defined in Section 3. Then the only relations among these classes are the ones generated by*

$$\nu_I^2 = \nu_{2I}.$$

Thus the classes $\nu_{i_2,\cdots,i_{m+1}}$ with at least one i_k odd are algebraically independent.

Our method can also be applied to the cohomology with integer coefficient. That is, if p^I denotes the monomial in Pontryagin classes, then

Theorem 1.10 ((Theorem 4.10)). *The classes $\zeta_I = \overline{\nu}_{p^I}$ are not divisible in $H^*(\Omega_0^\infty MTO(2m);\mathbb{Z})$.*

The case $m = 1$, combined with the homological stability theorem of [46] is Theorem A of [15]; the classes ζ_I generalize Wahl classes (see Definition 4.9).

We conclude the paper with a 'non-theorem' which tells that computations such as Galatius' and Randal-Williams' were somehow exceptional cases and for an infinite family of Madsen-Tillmann spectra, such a description in terms of short exact sequences is not available. We have the following.

Proposition 1.11 ((Proposition 4.11)). *In many cases (a precise hypothesis is given in Proposition 4.11), the sequence of Hopf algebras*

$$H_*(\Omega_0^\infty MT\mathbf{K}(m+1);\mathbb{Z}/p) \longrightarrow H_*(Q_0 B\mathbf{K}(m+1)_+;\mathbb{Z}/p) \xrightarrow{(\Omega_0^\infty \tilde{t})_*} H_*(\Omega_0^\infty MT\mathbf{K}(m);\mathbb{Z}/p)$$

induced by the cofibration for Madsen-Tillmann spectra (Lemma 2.6) is not short exact.

However, in our subsequent work [25], we will exhibit summands of $MTO(n)$'s for which such exact sequences exist. We simply mention that in the case of $MTO(2)$, we will have

Theorem 1.12 (([25])). *Let $D(n)$ be the cofibre of $Sp^{2^{n-1}}S^0 \longrightarrow Sp^{2^n}S^0$ induced by the $X \longrightarrow X^{\times 2}$ where $Sp^{2^n}S^0$ is the 2^n-th symmetric power of S^0. Then, completed at $p = 2$, we have*

$$MTO(2) \simeq BSO(3)_+ \vee \Sigma^{-2}D(2).$$

Besides the points made above, we use the following conventions. We denote by X_+ the space X with the disjoint basepoint added. We use the bold letter $\mathbf{K}$ to denote a generic family of Lie groups, that can be specialized to $\mathbf{K}(n)$. For instance for $\mathbf{K} = O$. we have $\mathbf{K}(n) = O(n)$. On the other hand, the normal letters K, G etc. will denote a particular Lie group. For a (virtual) vector bundle $\alpha \longrightarrow B$ over some CW-complex B, we write B^α for the Thom (spectrum) space of α. For a space B, we use $\mathbb{R}^k$ and $B \times \mathbb{R}^k$ interchangeably for the k-dimensional trivial vector bundle over B which will be clear from the context; the notation $\mathbb{R}^k$ also denotes the k-dimensional Euclidean space as usual. The notation $\simeq$ denotes weak homotopy equivalence of spectra. Since we work with CW-spectra, by Whitehead theorem weak homotopy equivalence coincides with homotopy equivalence. By abuse of notation, $\cong$ is used to denote homeomorphism of spaces or isomorphism of algebraic objects which will be clear from the context. We shall write $\mathbb{Z}/p$ for the cyclic group of order p, and $\mathbb{Z}_{(p)}$ for p-localisation of the ring of integers. p will always denote a (positive) prime integer.

2 Thom spectra and the transfers

2.1 Recollections on Thom spaces and Thom spectra

Here, we recall some facts about Thom spaces and Thom spectra; these are the main objects of study in this paper for which [43] is a standard reference. We refer the reader to [4] for details on the Becker-Gottlieb and Boardman transfer maps, and to [12] for further material on umkehr maps. For the construction and properties of Thom diagonals, as well as [43], we refer the reader to [13] for a detailed study on the algebraic properties of this map.

Definition Let X be a space and ζ a vector bundle over X equipped with a Riemannian metric. In our applications X will be a classifying space of a Lie group and ζ will be a bundle obtained from a representation, but what follows here will be valid for any vector bundle over any (good) space. We define the Thom space of ζ by $X^\zeta = D(\zeta)/S(\zeta)$ where $D(\zeta)$ and $S(\zeta)$ are the total spaces of disc and sphere bundles associated to ζ, respectively.

Functoriality Suppose $\xi \longrightarrow E$ and $\zeta \longrightarrow B$ are vector bundles, and there is a map of bundles $\xi \longrightarrow \zeta$ covering a map $f : E \longrightarrow B$ which is injective on the fibre. Then one can use the metric of $f^*(\zeta)$ to induce one on ξ, thus it induces a map of Thom (spectra) spaces. if $\xi = f^*\zeta$, we write $Th^\zeta(f)$ for this induced map.

Pontryagin-Thom construction Suppose for now that $f : M \longrightarrow N$ is an embedding of a compact manifold M in another manifold N. One can identify its tubular neighbourhood with the total space of the disc bundle of the normal bundle ν_f. Thus one can define a map N to M^{ν_f} by collapsing the points outside $D(\nu_f)$. As M^{ν_f} is compact, one can extend this to a map from N^+, its one point compactification. The procedure is called the Pontryagin-Thom construction, and the resulting map $f_! : N^+ \longrightarrow M^{\nu_f}$ is the *umkehr* map associated to f. Now, let ζ be a vector bundle over N. Then the embedding $D(\nu_f) \subset N$ can be extended to $D(\nu_f \oplus \zeta_{|M}) \subset D(\zeta)$. Thus by collapsing the points outside, we get the *twisted umkehr* map [12], [6, (4.4)]

$$f_!^\zeta : N^\zeta \longrightarrow M^{\nu_f \oplus \zeta_{|M}}.$$

Boardman transfer Let $F \longrightarrow E \xrightarrow{\pi} B$ be a fibre bundle of compact manifolds. The compactness of E guarantees the existence of an embedding $\iota : E \longrightarrow \mathbb{R}^k$ by Whitney's embedding theorem, which in turn allows us to find an embedding $j : E \longrightarrow B \times \mathbb{R}^k$ that extends π, that is, an embedding j whose first component $E \longrightarrow B$ is equal to π. For example, it suffices to take $j = (\pi, \iota)$. The associated umkehr map

$$j_! : (B \times \mathbb{R}^k)^+ = B^{\mathbb{R}^k} \cong \Sigma^k B_+ \longrightarrow E^{\nu_j}$$

is called the Boardman transfer, and denoted by $\bar{t}_\pi$ (see also [4, Section 4]), [7, Chapter V, Section 4]. As in the above, if ζ is a vector bundle over B, one can “twist” this construction with ζ to obtain

$$\bar{t}^{\zeta}_{\pi} : B^{\mathbb{R}^k \oplus \zeta} \longrightarrow E^{\nu_j \oplus \pi^*(\zeta)}.$$

Becker-Gottlieb transfer Keep the notations as in the above, and write $T(M)$ for the tangent bundle of a manifold M. The embedding j induces a bundle isomorphism over the identity

$$T(E) \oplus \nu_j \cong j^* T(B \times \mathbb{R}^k) \cong \pi^*(T(B)) \oplus \mathbb{R}^k.$$

As $T(\pi) : T(E) \longrightarrow T(B)$ is surjective, we get a bundle map $\nu_j \longrightarrow \mathbb{R}^k$ that is injective on the fibre. Or, if we denote $T_\pi(E)$ the vertical tangent bundle (also called fibrewise tangent bundle), the bundle given by the kernel of $T(\pi) : T(E) \longrightarrow T(B)$, then we have a direct sum decomposition

$$T_\pi(E) \oplus \nu_j \cong \mathbb{R}^k.$$

By Thomifying we get a map $E^{\nu_j} \longrightarrow E^{\mathbb{R}^k} \cong \Sigma^k E_+$. By composing with the Boardman transfer, one gets the Becker-Gottlieb transfer $\Sigma^k B_+ \longrightarrow \Sigma^k E_+$ which we denote by t_π or t when π is understood. Again as in the above, we can twist it with a vector bundle ζ over B to obtain

$$t^{\zeta}_{\pi} : \Sigma^k B^{\zeta} \cong B^{\mathbb{R}^k \oplus \zeta} \longrightarrow E^{\mathbb{R}^k \oplus \pi^*(\zeta)} \cong \Sigma^k E^{\pi^*(\zeta)}.$$

Note that by the above construction, t^{ζ}_{π} also factors through the Boardmann transfer $\bar{t}^{\zeta}_{\pi}$.

Uniqueness First note that if we replace in the above $j : E \longrightarrow B \times \mathbb{R}^k$ with

$$j' : E \longrightarrow B \times R^k = \mathbb{R}^k \oplus 0 \subset B \times R^k \oplus \mathbb{R}^m \cong \mathbb{R}^{k+m}$$

then the resulting transfer map is the m-th suspension of the original one. Thus, if $j_l : E \longrightarrow B \times \mathbb{R}^{k_l}$, $l = 1, 2$ are two embeddings extending π, one can easily construct an isotopy between the two embeddings

$$E \xrightarrow{j_1} B \times \mathbb{R}^{k_1} = B \times (\mathbb{R}^{k_1} \oplus 0) \subset B \times (\mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2}),$$
$$E \xrightarrow{j_2} B \times \mathbb{R}^{k_2} = B \times (0 \oplus \mathbb{R}^{k_2}) \subset B \times (\mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2})$$

and the isotopy between two induces an homotopy between the (suspended) transfer maps. Therefore, the stable homotopy class of the transfer maps doesn't depend on the choice of embedding. Furthermore, it is also known that if two maps π and π' are homotopic to each other, then the resulting transfer is homotopic.

Extension to non-compact base space We need to deal with the transfer associated to the fibre bundle whose base is not necessarily compact. For the ease of notations, we only deal with untwisted case, but things generalize to twisted case without problem. Consider a fibre bundle $F \longrightarrow E \xrightarrow{\pi} B$ where F is a compact manifold, and B allows a filtration by compact manifolds B_n. Let π_n denote the restriction of π to B_n, and ι_n the inclusion $\pi^{-1}(B_n) \subset E$. The paragraph above implies that $\iota_n \circ \bar{t}_{\pi_n}$ defines an element in $\varprojlim[\Sigma^k B_n, E^{\nu_j}]$. Thus the Milnor exact sequence provides us with an element in $[\varinjlim \Sigma^k (B_n)_+, E^{\nu_j}]$, unique up to the image of $\varprojlim^1[\Sigma^k(B_n)_+, E^{\nu_j}]$, thus we have a well-defined weak homotopy class of the Boardman transfer $\bar{t}_\pi : \Sigma^k B_+ \longrightarrow E^{\nu_j}$. By composing with the Thomified map $E^{\nu_j} \longrightarrow \Sigma^k E_+$, we get a well-defined weak homotopy class of the Becker-Gottlieb transfer $t_\pi : B_+ \longrightarrow E_+$.

Thom spectra for virtual bundles If X is a finite complex then $KO^0(X)$ is finite which implies that any virtual bundle ζ over X can be written as $\zeta' - \mathbb{R}^m$ with ζ' a genuine vector bundle. In this case X^ζ is defined to be $\Sigma^{-m} X^{\zeta'}$. For general X admitting a filtration by finite subcomplexes $\{X_n\}$, we see that we can define Thom spectrum of a (virtual) vector bundle $\zeta \longrightarrow X$ using the naturality arguments. One first consider the restriction $\zeta_n = \zeta|_{X_n}$. The collection of stable complexes $\{X_n^{\zeta_n}\}$ determine a spectrum which is the desired Thom spectrum X^ζ.

Thom isomorphism is stable in the sense that for bundles over finite complexes, Thom isomorphism for $X^{\zeta'} \cong \Sigma^m X^\zeta$, where $\zeta' = \zeta \oplus \mathbb{R}^m$, is given by the composition of Thom isomorphism for X^ζ and the suspension isomorphism. Consequently, Thom isomorphism also holds for Thom spectrum of a (virtual) vector bundle over arbitrary X admitting a filtration by finite subcomplexes. Note that when the vector bundle has an extra structure, its Thom space/spectrum only depends on the underlying real (unoriented) vector bundle. Many notions, including that of the functoriality or that of twisted umkehr map, can be generalized to the Thom spectra of virtual bundles. Note that the use of Thom spectra for virtual bundles may lead to simplification of notations even when we normally only need Thom spaces. For example, the (twisted) Boardman transfer can be seen as a map of spectra

$$\bar{t}^\zeta_\pi : B^\zeta \longrightarrow E^{\pi^*(\zeta)\oplus\nu_j - \mathbb{R}^k} \cong E^{\pi^*(\zeta) - T_\pi(E)}.$$

Thom diagonal The Thom isomorphism, when it holds, allows us to consider $H^*(X^\zeta; k)$ as a module over $H^*(X_+; k)$, free of rank 1. However, we would like to do so without the orientability hypothesis. For this purpose, we have the "generalized cup product" ([43, IV.5.36], [13, 2.0.1]) at hand. That is, if ζ is a (genuine) vector bundle over X, then the diagonal $X \longrightarrow X \times X$ pulls $\zeta \times 0$ back to ζ, thus induces a map of Thom spaces $X^\zeta \longrightarrow (X \times X)^{\zeta \times 0} \cong X^\zeta \wedge X_+$, called the *Thom diagonal*. The induced map in the cohomology $H^*(X_+; k) \otimes H^*(X^\zeta; k) \longrightarrow H^*(X^\zeta; k)$ is called the *generalized cup product* and turns $H^*(X^\zeta; k)$ into a $H^*(X_+; k)$-module. This construction is "stable" in the above sense, thus can be generalized to virtual bundles.

2.2 The Becker-Gottlieb transfer and the splitting

Let's start by recalling the following. Let B be a space that admits a filtration by compact manifolds. This, of course, includes the case where B itself is a compact manifold. Let $\pi : E \longrightarrow B$ be a fibre bundle whose fibre F is a smooth compact manifold.

Theorem 2.1. *[4, Theorem 5.5] The composition*

$$B_+ \xrightarrow{t_\pi} E_+ \xrightarrow{\pi} B_+$$

induces multiplication by $\chi(F)$ in $H^*(-;\Lambda)$ for any Abelian group Λ.

This implies that if $\chi(F)$ is not divisible by p then $B_{+(p)}$ splits off $E_{+(p)}$. In particular, if B is connected, $B_{(p)}$ splits off $E_{(p)}$.

Thus this theorem has been a source of various splitting results. For example, let G be a compact Lie group, and K a closed subgroup. Note that the construction by Grassmannian of BG (e.g. Appendix A.1 for the cases relevant to us) leads to a filtration of BG by compact manifolds. Furthermore, by taking EG/K as a model for BK we can apply the above theorem to the fibre bundle

$$G/K \longrightarrow BK \xrightarrow{\pi} BG$$

Thus if $\chi(G/K)$ is prime to p, then $BG_{(p)}$ splits off $BK_{(p)}$. Such phenomenon is well known and has been used extensively to study the stable homotopy type of the classifying space BG in the case where G is finite (e. g. [40]). In this case the Becker-Gottlieb transfer agrees with the classical Kahn-Priddy transfer([24]). The case when G is not finite is also well-known. For example, it has been shown [49, Lemma 1] that $BSO(2n+1)_{(p)}$ splits off $BO(2n)_{(p)}$ (this splitting actually occurs without localisation) and $BSU(n+1)_{(p)}$ splits off $BU(n)_{(p)}$ unless p divides $n+1$. Later we will show that in some cases we can refine this splitting to split classifying spaces off appropriate Thom spectra. For now, let's consider the case where B is a single point. This particular case is known as Hopf's vector field Theorem ([4, Theorem 2.4]). Note that in [4], they deal with an equivariant version, and what is relevant for us is the particular case where we have the action of the trivial group. Of course, in this case, the splitting of $B_{+(p)}$ off $E_{+(p)}$ is uninteresting. However, by considering the factorisation of the Becker-Gottlieb transfer through the Boardman transfer, we obtain Theorem 1.5.

Theorem 2.2. *(Theorem 1.5) Suppose there exists a manifold M with K-structure. Then S^0 splits off MTK at a prime p if p doesn't divide $\chi(M)$. In particular,*

1. *S^0 splits off MTK when $K = O(2n)$, $Pin^+(4n)$ or $Pin^-(4n+2)$.*
2. *$S^0_{(p)}$ splits off $MTK_{(p)}$ when $K = SO(2n)$ if p is odd.*
3. *$S^0_{(p)}$ splits off $MTK_{(p)}$ when $K = U(n)$ or $Sp(n)$ if p doesn't divide $n+1$.*

Proof. Suppose that M is a manifold with reduction of the structure group of the tangent bundle to K. Note that the vertical tangent bundle for the fibre bundle $M \longrightarrow pt$ is nothing but the tangent bundle of M. Thus the Madsen-Tillmann-Weiss map associated to it is given by the composition

$$S^0 \longrightarrow M^{-T(M)} \longrightarrow MTK = BK^{-\gamma}$$

where the first map is the Boardman transfer, the second is the Thomification of the classifying map of $T(M)$, $f : M \longrightarrow BK$, and γ is the universal vector bundle. Consider the following diagram where the arrows named c are the collapse maps.

$$\begin{tikzcd} S^0 \arrow[r] & M^{-T(M)} \arrow[r] \arrow[d] & M^{T(M)-T(M)} \arrow[r, equal] & M_+ \arrow[r, "c"] \arrow[d, "f_+"] & S^0 \arrow[d, equal] \\ & MTK = BK^{-\gamma} \arrow[r, "\omega_K"] & BK^{\gamma-\gamma} \arrow[r, equal] & BK_+ \arrow[r, "c"] & S^0 \end{tikzcd}$$

Clearly it is commutative, and the composition of the left horizontal arrows in the top row is the Becker-Gottlieb transfer. Thus the composition of the entire top row is a map of degree $\chi(M)$ by Theorem 2.1 with $E = M$, $B = pt$. Therefore, if $\chi(M)$ is prime to p, we obtain a splitting of $S^0_{(p)}$ off $MTK_{(p)}$ using the Madsen-Tillmann-Weiss map.

Noting that $\mathbb{R}P^{2n}$ has the tangent bundle with structure group $O(2n)$, that can be lifted to $Pin^{\pm}(2n)$ according to the parity of n and $\chi(\mathbb{R}P^{2n}) = 1$, we get (i). Noting that S^{2n} has the tangent bundle with structure group $SO(2n)$ and $\chi(SO(2n)) = 2$, we get (ii). Finally, noting that $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ have the tangent bundle with structure group $O(n)$, $U(n)$ and $Sp(n)$ respectively, and $\chi(\mathbb{C}P^n) = \chi(\mathbb{H}P^n) = n + 1$, we get (iii). ∎

2.3 Becker-Schultz-Mann-Miller-Miller transfer

As promised, we will now refine the splitting of classifying spaces of compact Lie groups. Again, the Becker-Gottlieb transfer factors through the Boardman transfer. In the situation at hand, the Boardman transfer can be identified with the Becker-Schultz-Mann-Miller-Miller transfer (BSMMM transfer for short) up to a self homotopy equivalence of the target. So we start from recalling the construction of BSMMM transfer for which [6], [33], [35] are our main references.

Let G be a compact Lie group and M a smooth compact manifold with free G action. Consider $\mathfrak{g}$, the Lie algebra of G, with the adjoint G-action. Let

$$\mu_G(M) = M \times_G \mathfrak{g} \longrightarrow M/G$$

be the adjoint bundle associated to the fibre bundle $M \longrightarrow M/G$. By compactness of M, we may assume that M/G has a Riemannian metric. Note that G acts on the tangent bundle $T(M)$. By the existence of Riemannian metric on M/G, it appears that there is a decomposition of bundles over M/G [6, (3.1)], [33, Lemma 2.1]

$$T(M)/G \cong \mu_G(M) \oplus T(M/G).$$

Now, suppose $K < G$ is a closed subgroup and consider the fibre bundle $\pi : M/K \longrightarrow M/G$. Choose an embedding $j_0 : M/K \longrightarrow \mathbb{R}^k$ and let $j = (j_0, \pi) : M/K \longrightarrow \mathbb{R}^k \times M/G$. For a (virtual) vector bundle $\zeta \longrightarrow M/G$, we have a twisted Boardman transfer

$$(M/G)^{\mathbb{R}^k \oplus \zeta} \longrightarrow (M/K)^{\nu_j \oplus \pi^* \zeta}.$$

Now, we have the following commutative diagram.

$$\begin{tikzcd}
 & T(M/K) \oplus \nu_j \arrow[dl] \arrow[d] \arrow[dr] & \\
T(\mathbb{R}^k \times M/G) \arrow[d] \arrow[rr, dotted] & M/K \arrow[dl, "j"'] \arrow[dr, "\pi"] & T(M/G) \oplus \mathbb{R}^k \arrow[d] \\
\mathbb{R}^k \times M/G \arrow[rr, "\pi_2"] & & M/G
\end{tikzcd}$$

Here π_2 denotes the projection to the second factor, and all parallelograms are pull-back squares. Thus we have

$$T(M/K) \oplus \nu_j \cong j^* T(M/G \times \mathbb{R}^k) \cong \pi^*(T(M/G) \oplus \mathbb{R}^k).$$

Now, by plugging in the above decomposition for $T(M)/G$ as well as $T(M)/K$, we get a “relative framing”

$$\pi^* \mu_G(M) \oplus \nu_j = \mu_K(M) \oplus \mathbb{R}^k.$$

Hence, replacing ζ with $\mu_G(M) \oplus \alpha$ for an arbitrary virtual bundle $\alpha \longrightarrow M/G$, together with the above relative framing, we obtain a transfer map as in [6, (3.7)], [33, Section 2]

$$(M/G)^{\mu_G(M) \oplus \alpha} \longrightarrow (M/K)^{\mu_K(M) \oplus \pi^* \alpha}.$$

This is the BSMMM transfer associated to π twisted by α. Denote by $\mathrm{ad}_G = EG \times_G \mathfrak{g}$ the vector bundle associated to the adjoint representation of G over BG. We can approximate/filter EG by compact free G-manifolds, and ad_G restricts to $\mu_G(M)$ over each compact manifold $M \subset EG$. Thus we get a transfer map

$$\tilde{t} = \tilde{t}^{\alpha}_{K,G} : BG^{\mathrm{ad}_G \oplus \alpha} \longrightarrow BK^{\mathrm{ad}_K \oplus \alpha_{|K}}$$

where $\alpha_{|K} = \pi^* \alpha$. As the BSMMM transfer is just a special case of the Boardman transfer composed with an automorphism of the target Thom spectrum, we get the following factorisation of the Becker-Gottlieb transfer through the BSMMM transfer, which is presumably well-known, but we record it for the sake of reference.

Proposition 2.3. *Suppose G and K are as above. Then the Becker-Gottlieb transfer $BG_+ \longrightarrow BK_+$ admits a factorisation through the BSMMM transfer as*

$$BG_+ \xrightarrow{\tilde{t}} BK^{\mathrm{ad}_K - \mathrm{ad}_{G|K}} \longrightarrow BK_+.$$

Similarly, for arbitrary $\eta \longrightarrow BG$, the twisted Becker-Gottlieb admits a factorisation as

$$BG^{\eta} \longrightarrow BK^{\eta_{|K} \oplus \mathrm{ad}_K - \mathrm{ad}_G|_K} \longrightarrow BK^{\eta_{|K}}.$$

Proof. First, note that for $K < G$, $\pi : BK \longrightarrow BG$, and $0 \longrightarrow BG$, the twisted Becker-Gottlieb transfer

$$t^0_\pi : BG^0 \cong BG_+ \longrightarrow BK^{\pi^* 0} = BK^0 \cong BK_+$$

where 0 denotes the 0-dimensional trivial bundle, agrees with the Becker-Gottlieb transfer. Second, recall that the twisted Becker-Gottlieb transfer admits a factorisation as

$$\Sigma^k B^\zeta = B^{\mathbb{R}^k \oplus \zeta} \longrightarrow E^{\pi^*\zeta \oplus \nu_j} \longrightarrow E^{\mathbb{R}^k \oplus \pi^*\zeta} = \Sigma^k E^{\pi^*\zeta}$$

for any fibre bundle $\pi : E \longrightarrow B$ over some B admitting a filtration by compact manifolds, where $\zeta \longrightarrow B$ is some (virtual) vector bundle. By choosing M a compact manifold on which G acts freely, taking $\pi : E \longrightarrow B$ to be the fibre bundle $\pi : M/K \longrightarrow M/G$, and replacing ζ with $\mu_G(M) \oplus \alpha$ and using the relative framing $\pi^*\mu_G(M) \oplus \nu_j = \mu_K(M) \oplus \mathbb{R}^k$, we have a factorisation

$$\Sigma^k (M/G)^{\mu_G(M) \oplus \alpha} = B^{\mathbb{R}^k \oplus \mu_G(M) \oplus \alpha} \longrightarrow$$
$$(M/K)^{\mathbb{R}^k \oplus \alpha_{|K} \oplus \mu_K(M)} \longrightarrow (M/K)^{\mathbb{R}^k \oplus \mu_G(M)_{|K} \oplus \alpha_{|K}}.$$

By allowing M to approximate EG, hence eventually taking $M = EG$, we obtain a factorisation of the twisted Becker-Gottlieb transfer as

$$BG^{\mathbb{R}^k \oplus \mathrm{ad}_G \oplus \alpha} \longrightarrow BK^{\mathbb{R}^k \oplus \mathrm{ad}_K \oplus \alpha_{|K}} \longrightarrow BK^{\mathbb{R}^k \oplus \mathrm{ad}_{G|K} \oplus \alpha_{|K}}$$

which upon choosing $\alpha = -\mathrm{ad}_G$ yields the first factorisation. For the second factorisation, it follows if we simply replace α by $\eta - \mathrm{ad}_G$. ∎

The above proposition combined with Theorem C.4 (Theorem 2.1 in the untwisted case) leads to the following:

Corollary 2.4. *(Corollary 1.2 (ii)) Let G be a compact Lie group, K a closed subgroup, such that $\chi(G/K)$ is prime to p, and η a vector bundle over BG. Then $BG^\eta_{(p)}$ splits off $BK^{\eta_{|K} \oplus \mathrm{ad}_K - \mathrm{ad}_{G|K}}_{(p)}$. In particular, $BG_{+(p)}$ splits off $BK^{\mathrm{ad}_K - \mathrm{ad}_{G|K}}_{(p)}$.*

2.4 The cofibre of transfer maps

Suppose that we have a fibre bundle $F \longrightarrow E \longrightarrow B$ with $\chi(F)$ prime to p. Then the Becker-Gottlieb transfer $t : B_+ \longrightarrow E_+$ provides a stable splitting of $E_{+(p)}$ off $B_{+(p)}$. The other summand is just the cofibre of t, that is, we have a homotopy equivalence $E_{+(p)} \simeq B_{+(p)} \vee C_{t(p)}$. A similar statement holds for a twisted BSMMM transfer. Morisugi's cofibration [38, Theorem 1.3] allows us to identify this cofibre in favorable cases. By setting $E = EG$ in loc.cit. Theorem 1.3, we obtain:

Theorem 2.5. *([38, Theorem 1.3]) Let G be a compact Lie group, K a closed subgroup, such that there exists a G-representation V with $G/K = S(V)$ as G-spaces, where $S(V)$ is the sphere in V with a certain G-invariant metric. Let α be a vector bundle over BG. Denote by λ the vector bundle over BG induced by the representation V, $EG \times_G V \longrightarrow BG$. Then, there exists a cofibration of spectra:*

$$BG^{\mathrm{ad}_G \oplus \alpha - \lambda} \longrightarrow BG^{\mathrm{ad}_G \oplus \alpha} \xrightarrow{\tilde{t}} BK^{\mathrm{ad}_K \oplus \alpha|_K} \longrightarrow BG^{\mathbb{R} \oplus \mathrm{ad}_G \oplus \alpha - \lambda} \cong \Sigma BG^{\mathrm{ad}_G \oplus \alpha - \lambda}$$

The following lemma provides an application of the above theorem.

Lemma 2.6. *Let $\mathbf{K} = O, SO, Pin^+, Pin^-, Spin, U$, or Sp. Then there is a cofibration sequence of spectra*

$$MT\mathbf{K}(n+1) \xrightarrow{\omega_{\mathbf{K}(n+1)}} B\mathbf{K}(n+1)_+ \xrightarrow{\tilde{t}} \Sigma^{1-d} MT\mathbf{K}(n) \xrightarrow{j} \Sigma MT\mathbf{K}(n+1)$$

where $d = 1$ if $\mathbf{K} = O, SO, Pin^+, Pin^-$ or $Spin$, $d = 2$ if $\mathbf{K} = U$ or SU, and $d = 4$ if $\mathbf{K} = Sp$.

Proof. First we deal with the case $\mathbf{K} = O, SO, U$ and Sp. Let $\mathbb{F} = \mathbb{R}$ if $\mathbf{K} = O, SO$, $\mathbb{C}$ if $\mathbf{K} = U$, $\mathbb{H}$ if $\mathbf{K} = Sp$. Thus $\mathbb{F}$ is d-dimensional vector space over $\mathbb{R}$. The group $\mathbf{K}(k)$ admits a canonical representation $\gamma_k^{\mathbb{F}}$ on $\mathbb{F}^k$, and the corresponding group action preserves the metric. Thus the $\mathbf{K}(k)$ action on $\mathbb{F}^k$ restricts to a transitive $\mathbf{K}(k)$ action on the sphere S^V where V is $\mathbb{F}^k$ viewed as $\mathbf{K}(k)$-space. Now, set $k = n+1$. The isotropy subgroup of any unit vector is isomorphic to $\mathbf{K}(n)$, so we have $\mathbf{K}(n+1)/\mathbf{K}(n) \cong S(V)$. Thus we can apply Theorem 2.5 with $G = \mathbf{K}(n+1), K = \mathbf{K}(n), \lambda = \gamma_{n+1}^{\mathbb{F}}$. Set $\alpha = -\mathrm{ad}_G$ for the twisting bundle. It now remains to identify its restriction $\alpha|K$ or its inverse $\mathrm{ad}_G|K$. We have

$$\begin{pmatrix} X & \\ & 1 \end{pmatrix} \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \begin{pmatrix} X^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} XAX^{-1} & XB \\ -(XB)^* & D \end{pmatrix},$$

where X, A are $n \times n$ matrices, B is a $1 \times n$ matrix, and D is 1×1 matrix with coefficients in $\mathbb{F}$. Furthermore, for the matrix $\begin{pmatrix} A & B \\ -B^* & D \end{pmatrix}$ to lie in the appropriate Lie algebra, we must have $A^* = -A$, $D^* = -D$. Thus A is an element of the Lie algebra of K. The block XB corresponds to the canonical representation $\gamma_n^{\mathbb{F}}$ whereas the block D corresponds to a trivial representation of appropriate dimension. $D \in \mathbb{F}$ with $D^* = -D$, so the dimension over $\mathbb{R}$ is $d-1$. Thus we see that $ad_G|K = ad_K \oplus \gamma_n^{\mathbb{F}} \oplus \mathbb{R}^{d-1}$. This concludes the proof in the cases considered.

The cases $\mathbf{K} = (S)Pin^{\pm}$ follow from the case $\mathbf{K} = (S)O$ noting that the canonical and adjoint representations factors through those of the latter. Finally, the case $\mathbf{K} = SU$ can be handled as in the above, noting that $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathfrak{u}(1)$ as $SU(n)$-representation, the splitting map $\mathfrak{u}(1) \longrightarrow \mathfrak{u}(n)$ being given by the diagonal divided by n.

We note that when $\mathbf{K} = SO, Spin$ or SU and $n = 0$, our definition of $B\mathbf{K}(0)$ makes the sequence $S^d \longrightarrow B\mathbf{K}(n) \longrightarrow B\mathbf{K}(n+1)$ a fibre bundle. Thus we can modify the proof of [38, Theorem 1.3] to fit our case. ∎

Remark 2.7. *We note that the cofibre sequences above specialises to those of [19, Proposition 3.1] when* $\mathbf{K} = O$ *or* SO*, and give rise to the fibration of infinite loop spaces as in [18, (1.3)] when* $\mathbf{K} = Spin$ *and* $n = 1$*. This can be seen noting the fact that all these cofibrations are obtained as special cases of the cofibration [23]*

$$S(V)^{p^*W} \longrightarrow B^W \longrightarrow B^{V \oplus W} \longrightarrow \Sigma S(V)^{p^*W} \tag{1}$$

where B *is a space,* $p : V \longrightarrow B$ *is a genuine vector bundle over* B *and* $q : W \longrightarrow B$ *a vector bundle (that may be virtual) over* B*,* $S(V) \longrightarrow B$ *is the sphere bundle of* V*, and the map* $B^W \longrightarrow B^{V \oplus W}$ *is the Thomification of the embedding* $W \longrightarrow V \oplus W$*.*

As a non-example, where Morisugi's result does not apply, at least integrally, consider embedding of $K = O(n)$ in $\mathbf{G} = SO(n+1)$ by $X \mapsto (\det X)(X \oplus 1)$ with $n > 1$. We have $G/K = \mathbb{R}P^n$ which cannot be identified as a sphere in some vector space as $\pi_1 \mathbb{R}P^n \simeq \mathbb{Z}/2$ for $n > 1$.

2.5 Cohomology of Madsen-Tillmann spectra

Let $\mathbf{K} = SU, U$ or Sp and k be an arbitrary field, or $\mathbf{K} = O$ and k a field of characteristic 2, and d be as in Lemma 2.6. Then we have the following.

Lemma 2.8. *The cofibration*

$$\Sigma^{-d} MT\mathbf{K}(n-1) \longrightarrow MT\mathbf{K}(n) \longrightarrow \Sigma^\infty B\mathbf{K}(n)_+$$

gives rise to a short exact sequence in cohomology

$$H^*(\Sigma^\infty B\mathbf{K}(n)_+; k) \longrightarrow H^*(MT\mathbf{K}(n); k) \longrightarrow H^*(\Sigma^{-d} MT\mathbf{K}(n-1); k),$$

and dually to a short exact sequence in homology

$$H_*(\Sigma^{-d} MT\mathbf{K}(n-1); k) \longrightarrow H_*(MT\mathbf{K}(n); k) \longrightarrow H_*(\Sigma^\infty B\mathbf{K}(n)_+; k).$$

Therefore we have isomorphisms of graded k-vector spaces:

$$H_*(MT\mathbf{K}(n); k) \cong \oplus_{j=0}^n \Sigma^{-dj} H_*(B\mathbf{K}(j); k), H^*(MT\mathbf{K}(n); k) \cong \oplus_{j=0}^n \Sigma^{-dj} H^*(B\mathbf{K}(j); k).$$

Proof. We have $H^*(B\mathbf{K}(n); k) \cong k[z_i, \cdots, z_n]$ where $i = 2$ if $\mathbf{K} = SU$ and $i = 1$ otherwise, with degree of polynomial generators z_m being equal to dm (Theorem A.1). By the Thom isomorphism, we have

$$H^*(MT\mathbf{K}(n); k) \cong z_n^{-1} k[z_1, \cdots, z_n].$$

Here the notation means the free $k[z_1, \cdots, z_n]$ module generated by one element z_n^{-1}, and we can consider that this is included in an appropriate localisation of $H^*(B\mathbf{K}(n); k)$. Similarly we have

$$H^*(MT\mathbf{K}(n-1); k) \cong z_{n-1}^{-1} k[z_1, \cdots, z_{n-1}].$$

Since the canonical representation γ_n of $\mathbf{K}(n)$ pulls back to $\gamma_{n-1} \oplus \mathbb{R}$ over $\mathbf{K}(n-1)$, the map

$$H^*(MT\mathbf{K}(n); k) \longrightarrow H^*(\Sigma^{-1} MT\mathbf{K}(n-1); k)$$

is given by

$$z_n^{-1} f(z_1, \cdots, z_{n-1}, z_n) \mapsto \sigma^{-1} z_{n-1}^{-1} f(z_1, \cdots, z_{n-1}, 0)$$

where $f(z_1, \cdots, z_{n-1}, z_n) \in k[z_1, \cdots, z_n]$. Thus it is surjective, and we get the desired short exact sequences in cohomology from the long exact sequence for the cofibration. By dualizing we get the result in homology. The last statement follows by induction on n. ∎

3 Splitting Madsen-Tillmann spectra

In this section we deduce the splitting of Madsen-Tillmann spectra from the general theory of splitting of Thom spectra. First, we have:

Theorem 3.1. *1. Suppose (K, G) is one of the pairs*

$$(O(2n), SO(2n+1)),\ (Pin^+(4n), Spin(4n+1)),\ (Pin^-(4n+2), Spin(4n+3)).$$

Then BG_+ stably splits off MTK.

2. Let p be an odd prime. Let (K, G) be one of the pairs $(SO(2n), SO(2n+1))$, equivalently $(Spin(2n), Spin(2n+1))$, or $(O(2n), O(2n+1))$. Then we have $MTK_{(p)} \simeq BG_{+(p)} \vee \Sigma MTG_{(p)}$. Furthermore, the splitting of $MTO(n)_{(p)}$ reduces to

$$MTO(2n)_{(p)} \simeq BO(2n)_{+(p)},\ MTO(2n-1)_{(p)} \simeq *.$$

First, we record the following.

Lemma 3.2. *Let $j : O(2n) \longrightarrow SO(2n+1)$ be as above. Then we have*

$$Bj^*(w_2) = w_2 + n w_1^2$$

in mod 2 cohomology.

Proof. Consider the following commutative diagram where all unnamed arrows are the obvious inclusions, and φ is given by $\varphi(a_1, \ldots, a_{2n}) = (aa_1, \ldots, aa_{2n}, a)$ with $a = \Pi_{i=1}^{2n} a_i$.

$$\begin{tikzcd}
O(1)^{2n} \arrow[rr, "\varphi"] \arrow[d] & & O(1)^{2n+1} \arrow[d] \\
O(2n) \arrow[r, "j"] & SO(2n+1) \arrow[r] & O(2n+1)
\end{tikzcd}$$

Thus to determine $Bj^*(w_2)$, it suffices to compute $B\varphi^*(\sigma_2(t_1,\ldots,t_{2n+1}))$ by Theorems A.1 and A.2. Now, let's note that in general, we have

$$\begin{aligned}\sigma_2(\alpha+x_1,\ldots,\alpha+x_{2n}) &= \Sigma_{1\le i<k\le 2n}(\alpha^2+(x_i+x_k)\alpha+x_ix_k)\\ &= n(2n-1)\alpha^2+(2n-1)\alpha\sigma_1(x_1,\ldots,x_{2n})+\sigma_2(x_1,\ldots,x_{2n})\end{aligned}$$

Thus we get, noting that we are working modulo 2,

$$\begin{aligned}B\varphi^*(\sigma_2(t_1,\ldots,t_{2n+1})) &= B\varphi^*(\sigma_2(t_1,\ldots,t_{2n})+t_{2n+1}\Sigma_{i=1}^{2n}t_i)\\ &= \sigma_2(B\varphi^*(t_1),\ldots,B\varphi^*(t_{2n}))+B\varphi^*(t_{2n+1})\Sigma_{i=1}^{2n}(B\varphi^*(t_i))\\ &= \sigma_2(t+t_1,\ldots,t+t_{2n})+t\cdot\Sigma_{i=1}^{2n}(t+t_i)\\ &= n(2n-1)t^2+t\cdot(2n-1)t+\sigma_2(t_1,\ldots,t_{2n})+t^2\\ &= nt^2+\sigma_2(t_1,\ldots,t_{2n})\\ &= n\sigma_1(t_1,\ldots,t_{2n})^2+\sigma_2(t_1,\ldots,t_{2n})\end{aligned}$$

as required, where $t=\Sigma_{i=1}^{2n}t_i=\sigma_1(t_1,\ldots,t_{2n})$. ■

of Theorem 3.1. (i) Consider the embedding

$$O(2n)\ni X\mapsto j(X)=(\det X)(X\oplus 1)\in SO(2n+1).$$

One sees that the fibre of Bj is

$$SO(2n+1)/O(2n)\cong\mathbb{R}P^{2n}$$

with $\chi(\mathbb{R}P^{2n})=1$ (with any coefficient). Furthermore, we have

$$\begin{pmatrix} X & \\ & W\end{pmatrix}\begin{pmatrix} A & B\\ -B^* & D\end{pmatrix}\begin{pmatrix} X^{-1} & \\ & W^{-1}\end{pmatrix}=\begin{pmatrix} XAX^{-1} & W^{-1}XB\\ -(W^{-1}XB)^* & D\end{pmatrix},$$

where X and A are $2n\times 2n$ matrices, W and D are 1×1 matrices and B is a $1\times 2n$ matrix. Replacing X and W with $det(X)\cdot X$ and $det(X)$ respectively, we see that

$$ad_{SO(2n+1)}|_{O(2n)} = j^*ad_{SO(2n+1)} = ad_{O(2n)}\oplus\gamma_{2n} \text{ i.e.,}$$
$$-\gamma_{2n}=ad_{O(2n)}-ad_{SO(2n+1)|_{O(2n)}}.$$

Applying Corollary 2.4 to the embedding $j: O(2n)\longrightarrow SO(2n+1)$ proves Theorem 3.1(i) for the pair $(O(2n),SO(2n+1))$.

For the remaining two pairs we proceed as follows. The definition of $Pin^{\pm}()$ groups (see Appendix A.2) together with Lemma 3.2 implies that j induces a map of double covers $Pin^{\pm}(2n)\longrightarrow Spin(2n+1)$, where the sign $\pm$ is $+$ if n is even, $-$ if n is odd. Thus with the choice of appropriate sign, we get the following commutative square

$$\begin{tikzcd} Pin^{\pm}(2n) \arrow[r, "\tilde{j}"] \arrow[d] & Spin(2n+1) \arrow[d]\\ O(2n) \arrow[r, "j"] & SO(2n+1)\end{tikzcd}$$

where the vertical arrows are the canonical projections. Thus we get a diffeomorphism

$$Spin(2n+1)/\tilde{j}(Pin^{\pm}(2n)) \cong SO(2n+1)/j(O(2n)) \cong \mathbb{R}P^{2n+1}.$$

On the other hand, by definition the canonical representations of $Pin^{\pm}(2n)$ and $Spin(2n+1)$ are the pull-back of the canonical representations of $O(2n)$ and $SO(2n+1)$ by the canonical projection. Furthermore, the adjoint representations of $Pin^{\pm}(2n)$ and $Spin(2n+1)$ are the pull-back of the adjoint representations of $O(2n)$ and $SO(2n+1)$ by the canonical projection, since the kernel of the canonical projection is the centre. Thus we can apply Proposition 2.4 to prove Theorem 3.1(i) for the other two pairs.

(ii) Through the usual embeddings $O(2n) \subset O(2n+1)$ and $SO(2n) \subset SO(2n+1)$ we have diffeomorphisms

$$O(2n+1)/O(2n) \cong SO(2n+1)/SO(2n) \cong S^{2n}.$$

Moreover, by passing to the $\mathbb{Z}/2$-central extension, we see that

$$Spin(2n+1)/Spin(2n) \cong S^{2n}.$$

Since $\chi(S^{2n}) = 2$ and p is odd, by Corollary 2.4 $BG_{+(p)}$ splits off $BK_{(p)}^{(\mathrm{ad}_K - \mathrm{ad}_{G_{|K}})}$ by the transfer. The cofibre of transfer maps associated to these embeddings is identified in Lemma 2.6. The result then follows by the discussion of Section 2.4. The identification of $MTO(n)$ at odd primes is postponed to the end of the section (Lemma 3.5). This completes the proof. ∎

For the unitary and special unitary groups, we need somewhat odd looking condition on p, and we have

Theorem 3.3. *Let $K = U(n)$, $G = SU(n+1)$. Suppose that p doesn't divide $n+1$. Then $BG_{+(p)}$ splits off $MTK_{(p)}$.*

We begin with a lemma.

Lemma 3.4. *Let $p \nmid n+1$. Then the homomorphism $\varphi : A \mapsto \det(A)A$ induces a self homotopy equivalence of $BU(n)$, as well as a homotopy equivalence*

$$BU(n)_{(p)}^{-\det\otimes\gamma_n} \simeq BU(n)_{(p)}^{-\gamma_n} = MTU(n)_{(p)}.$$

Proof. It suffices to show that it induces an automorphism on $H^*(BU(n);\mathbb{Z}/p)$, as $BU(n)$ is of finite type. Consider the following commutative diagram

$$\begin{array}{ccc} U(1)^n & \xrightarrow{\overline{\varphi}} & U(1)^n \\ \downarrow & & \downarrow \\ U(n) & \xrightarrow{\varphi} & U(n) \end{array}$$

where the vertical arrows are the inclusions of the diagonal matrices with entries in $U(1)$, $\overline{\varphi}$ is given by

$$\overline{\varphi}(e^{i\theta_1}, \cdots, e^{i\theta_n}) = (e^{i(\theta_1+\theta)}, \cdots e^{i(\theta_n+\theta)}) \text{ where } \theta = \theta_1 + \cdots + \theta_n.$$

Now we see that $H^*(B\overline{\varphi})$ on $H^*(BU(1)^n; \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, \cdots, x_n]$ is given by

$$H^*(B\overline{\varphi})(x_i) = x_i + c_1 \text{ with } c_1 = x_1 + \cdots + x_n.$$

Thus by restricting to $H^*(BU(n); \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[c_1, \cdots, c_n]$, we see that

$$H^*(B\varphi)(c_1) = (1+n)c_1, H^*(B\varphi)(c_i) \equiv c_i \bmod (c_1) \text{ for } i > 1.$$

Thus $H^*(B\varphi)$ is an automorphism if and only if p doesn't divide $n+1$. Now, we note that the pull-back by φ of the canonical representation γ_n is just $\det \otimes \gamma_n$, so using the same notation for the bundle and representation, we get a bundle map $\det \otimes \gamma_n \longrightarrow \gamma_n$ over the map φ, and thus $-\det \otimes \gamma_n \longrightarrow -\gamma_n$ as well. Since φ is a homotopy equivalence, we see that the map between the Thom spectra $BU(n)^{-\det \otimes \gamma_n} \longrightarrow BU(n)^{-\gamma_n} = MTU(n)$ is also a homotopy equivalence. ■

of Theorem 3.3. Consider the embedding

$$U(n) \ni X \mapsto X \oplus (\det X)^{-1} \in SU(n+1).$$

The fibre of the map of classifying spaces $BU(n) \longrightarrow BSU(n+1)$ is given by the diffeomorphism

$$SU(n+1)/U(n) \cong \mathbb{C}P^n.$$

As in the above,

$$\begin{pmatrix} X & \\ & W \end{pmatrix} \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \begin{pmatrix} X^{-1} & \\ & W^{-1} \end{pmatrix} = \begin{pmatrix} XAX^{-1} & W^{-1}XB \\ (W^{-1}XB)^* & D \end{pmatrix},$$

where X, A are $m \times m$ matrices, W, D are 1×1 matrices and B is a $1 \times m$ matrix. By setting $W = det(X)^{-1}$ we see that the representation $\mathrm{ad}_{G|K} - \mathrm{ad}_K$ is isomorphic to the tensor product (over $\mathbb{C}$) of the canonical representation with the determinant representation. The proof is complete by Lemma 3.4. ■

We conclude the section by identifying the $MTO(n)$ spectra at odd primes. The following generalizes the known cases of $MTO(1)$ and $MTO(2)$, c. f. [41, subsection 5.1]. The standard equivalences $BO(2n)_{+(p)} \simeq BSO(2n+1)_{+(p)} \simeq BSp(n)_{+(p)}$ is included here for the convenience of possible reference.

Lemma 3.5. *Suppose that p is odd. For all $n \geqslant 0$, there are homotopy equivalences*

$$MTO(2n)_{(p)} \simeq BO(2n)_{+(p)} \simeq BSO(2n+1)_{+(p)} \simeq BSp(n)_{+(p)}, \; MTO(2n+1)_{(p)} \simeq *.$$

Proof. The proof is by strong induction. Note that the induction starts since

$$MTO(0) \simeq S^0 \simeq BO(0)_+.$$

Suppose now we have $MTO(2n)_{(p)} \simeq BO(2n)_{+(p)}$ and consider the commutative square

$$\begin{tikzcd} BO(2n+1)_+ \arrow[r] \arrow[d, "="] & MTO(2n) \arrow[d, "\omega_{O(2n)}"] \\ BO(2n+1)_+ \arrow[r, "t_{B\iota}"] & BO(2n)_+ \end{tikzcd}$$

corresponding to the factorisation of the Becker-Gottlieb transfer through the BSMMM transfer (Corollary 2.4). By the induction hypothesis, the right vertical arrow is a p-local homotopy equivalence. Since the inclusion $\iota : O(2n) \subset O(2n+1)$ induces an equivalence

$$B\iota_{(p)} : BO(2n)_{(p)} \simeq BO(2n+1)_{(p)}$$

by [45, Theorem 1.6], and the composition

$$BO(2n+1)_+ \xrightarrow{t_{B\iota}} BO(2n)_+ \xrightarrow{B\iota} BO(2n+1)_+$$

induces the multiplication by $\chi(S^{2n}) = 2$ in homology (Theorem 2.1), we see that the bottom horizontal arrow is a p-local homotopy equivalence. Thus the top horizontal row is also a p-local homotopy equivalence. However, by Lemma 2.6, $MTO(2n+1)$ is its fibre, thus p-locally contractible.

Next, the cofibration (Lemma 2.6)

$$MTO(2(n+1)) \xrightarrow{\omega_{O(2(n+1))}} BO(2(n+1))_+ \longrightarrow MTO(2n+1)$$

together with $MTO(2n+1)_{(p)} \simeq *$ will show that

$$\omega_{O(2(n+1))} : MTO(2(n+1)) \longrightarrow BO(2(n+1))_+$$

is a p-local homotopy equivalence. This finishes the induction.

The standard maps $BO(2n+1) \longrightarrow BSO(2n+1)$ and $BSp(n) \longrightarrow BO(2n)$ are well-known to be p-local homotopy equivalences. This complete the proof. ∎

4 Cohomology of infinite loop spaces associated to the Madsen-Tillmann spectra

The splitting of Madsen-Tillmann spectra discussed previously (Theorems 2.2, 3.1 and 3.3) implies the splitting of associated infinite loop space. Thus we can derive some information on their (co)homology, including information on various characteristic classes that live in their cohomology rings. In this section we see some examples.

4.1 Polynomial families in $H^*(\Omega^\infty MTK;\mathbb{Z}/p)$

We start with the following corollary of Theorem 2.2, identifying a polynomial family in the cohomology ring of $\Omega^\infty MTK$ for numerous groups K.

Corollary 4.1. *(Corollary 1.6) Let K be $O(2n)$, $U(2n)$, $Sp(2n)$, $Pin^+(4n)$ or $Pin^-(4n+2)$. The composition*

$$MTK \xrightarrow{\omega_K} BK_+ \xrightarrow{c} S^0 \xrightarrow{\iota} KO,$$

where ι is the unit map, induces an injection in mod 2 cohomology of infinite loop spaces

$$H^*(\mathbb{Z}\times BO;\mathbb{Z}/2) \hookrightarrow H^*(\Omega^\infty MTK;\mathbb{Z}/2).$$

Thus if we define the class $\xi_i \in H^*(\Omega_0^\infty MTK;\mathbb{Z}/2)$ by

$$\xi_i = (\omega_K \circ c \circ \iota)^*(w_i),$$

then we have

$$\mathbb{Z}/2[\xi_1,\ldots,\xi_k,\ldots] \subset H^*(\Omega_0^\infty MTG;\mathbb{Z}/2).$$

Proof. Consider the composition

$$\Omega^\infty MTK \xrightarrow{\Omega^\infty\omega_K} QBK_+ \xrightarrow{Qc} QS^0 \longrightarrow \mathbb{Z}\times BO.$$

We first show that this composition induces an injection in cohomology. By Lemma B.1 the map

$$H_*(QS^0;\mathbb{Z}/2) \longrightarrow H_*(\mathbb{Z}\times BO;\mathbb{Z}/2)$$

is surjective in homology. By the hypothesis and Corollary 2.2 the map

$$\Omega^\infty MTK \longrightarrow QBK_+ \xrightarrow{Qc} QS^0$$

splits, so it is also surjective in homology. Thus by composing and dualising, we see that

$$H^*(BO;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1,\ldots,w_k,\ldots]$$

injects to $H^*(MTK;\mathbb{Z}/2)$. Noting that the image of w_k is ξ_k, we get the desired result. ∎

In the special case $K = O(2)$, the family discussed above agrees with the one defined in [41, Section 6] up to conjugation, and generates the same subalgebra. We now discuss its complex analogue. That is:

Corollary 4.2. *(Corollary 1.7) Let K and p be as in Theorem 2.2. The composition*

$$MTK \xrightarrow{\omega_K} BK_+ \xrightarrow{c} S^0 \xrightarrow{\iota_{KU_{(p)}}} KU_{(p)}$$

where $\iota_{KU_{(p)}}$ is the unit for $KU_{(p)}$ factors through the Adams summand $E(1)$, and induces an injection in mod p cohomology of infinite loop spaces

$$H^*(\Omega^\infty E(1);\mathbb{Z}/p) \cong \mathbb{Z}/p[c_{p-1},c_{2(p-1)},\ldots] \hookrightarrow H^*(\Omega^\infty MTK;\mathbb{Z}/p).$$

Thus if we define the class $\xi_i \in H^*(\Omega_0^\infty MTK;\mathbb{Z}/p)$ *by*

$$\xi_i^{\mathbb{C}} = (\omega_K \circ c \circ \iota_{KU_{(p)}})^*(c_{i(p-1)}),$$

then we have

$$\mathbb{Z}/p[\xi_1^{\mathbb{C}},\ldots,\xi_k^{\mathbb{C}},\ldots] \subset H^*(\Omega_0^\infty MTK;\mathbb{Z}/p).$$

Proof. Note that the unit map of $KU_{(p)}$ factors through that of $E(1)$, for degree reasons, as $\pi_0(\Sigma^{2i}E(1)) = 0$ if $0 \leq i \leq p-2$. Thus the result follows by Lemma B.2, Proposition B.3 and Corollary 2.2 as in the proof of Theorem 4.1. ∎

4.2 Recollections on homology suspension

We use our splitting results to obtain information on the universally defined characteristic classes. An important ingredient in defining these classes is the (co)homology of the evaluation map $\Sigma^\infty\Omega^\infty E \longrightarrow E$, where E is a suitable spectrum, known as the (co)homology suspension. For this reason, we record some useful properties of the (co)homology suspension maps.

Let E be a spectrum in the sense of [1], that is, a sequence of pointed spaces E_j, with maps $\Sigma E_j \longrightarrow E_{j+1}$. Its homology with coefficients in k is defined to be $H_*(E;k) = \operatorname{colim}\,_j H_{*+j}(E_j;k)$. Note that inside the colimit, the homology of the basepoint suspending trivially, one can use interchangeably unreduced or reduced homology, although it is customary to use the reduced homology. When $E = \Sigma^\infty X$ for a pointed space X, we get the isomorphism $H_*(\Sigma^\infty X;k) \cong \widetilde{H}_*(X;k)$, the reduced homology of the space X. The elementary decomposition of unreduced homology to the direct sum of reduced homology and the coefficient ring, from our point of view, reflects the splitting of spectra $\Sigma^\infty X_+ \simeq \Sigma^\infty X \vee S^0$, and we have

$$H_*(\Sigma^\infty X_+;k) \cong \widetilde{H}_*(X_+;k) \cong H_*(X;k) \cong \widetilde{H}_*(X) \oplus H_*(pt).$$

The stable homology suspension homomorphism $\sigma_*^\infty : H_*(\Omega^\infty E;k) \longrightarrow H_*(E;k)$ is the standard map to the colimit above obtained by replacing E by an equivalent Ω-spectrum. It can also be defined by the map induced by the evaluation map $\Sigma^\infty\Omega^\infty E \longrightarrow E$ with the latter being adjoint to the identity map $\Omega^\infty E \longrightarrow \Omega^\infty E$. When X is a suspension spectrum, the generalities of adjoint functors ([32, Chapter IV, Theorem 1(8)]) imply the following:

Lemma 4.3. *For a pointed topological space X, the composition $\Sigma^\infty X \longrightarrow \Sigma^\infty QX \longrightarrow \Sigma^\infty X$ is the identity, i.e., $\Sigma^\infty X$ splits off $\Sigma^\infty QX$. Thus stable homology suspension*

$$\sigma_*^\infty : H_*(QX;k) \longrightarrow H_*(\Sigma^\infty X;k) \cong \widetilde{H}_*(X;k)$$

is an epimorphism.

However, this is not sufficient for our purpose, since we often have to deal with the map from $H_*(Q_0X;k)$ which is slightly smaller if X is not connected because of the decomposition

$$QX \simeq Q_0(X) \times \pi_0(QX), \pi_0(QX) \cong \lim \pi_0(\Omega^n\Sigma^n X) \cong \pi_0^S(X).$$

Fortunately, the spaces X we deal with have the form Y_+ with Y connected. Thus we have the decomposition

$$QX = QY_+ \simeq QY \times QS^0 = Q_0Y \times QS^0.$$

Noting that $H_*(QS^0)$ suspends to $\widetilde{H}^*(S^0) \cong H_*(pt)$, we deduce the following:

Lemma 4.4. *For a connected topological space Y, the composition*

$$\sigma_*^\infty : H_*(Q_0Y_+;k) \longrightarrow \widetilde{H}_*(Y_+;k) \longrightarrow \widetilde{H}_*(Y;k)$$

is onto.

It is known that homology suspension kills decomposable elements; this for example follows from [48, Corollary 3.4] applied to the path-loop fibration. An immediate corollary of this observation is the following well-known fact about the homology suspension

Lemma 4.5. *Let k be a field. The homology suspension*

$$\sigma_* : H_*(\Omega X;k) \longrightarrow H_{*+1}(X;k)$$

factors through the module of indecomposables (with respect to the Pontryagin product) $QH_*(\Omega X;k)$. In particular,

$$\sigma_*^\infty : H_*(QX;k) \longrightarrow H_*(\Sigma^\infty X;k)$$

factors through $QH_*(QX;k)$. Dually,

$$\sigma^{\infty *} : \widetilde{H}^*(X;k) \longrightarrow H^*(QX;k)$$

factors through the set of primitives $PH^*(QX;k)$.

Finally, we note that if $f : E \longrightarrow F$ is a map of spectra then there is a commutative diagram as

$$\begin{tikzcd}
H_*(\Omega^\infty E;k) \arrow[r, "\Omega^\infty f"] \arrow[d, "\sigma_*^\infty"'] & H_*(\Omega^\infty F;k) \arrow[d, "\sigma_*^\infty"'] \\
H_*(E;k) \arrow[r, "f"] & H_*(F;k)
\end{tikzcd}$$

that is $\sigma_*^\infty(\Omega^\infty f)_* = f_*\sigma_*^\infty$.

4.3 The universally defined characteristic classes in modulo p cohomology

We will now discuss how our splitting results can be used to analyse the universally defined characteristic classes (Definition 1.8) in modulo p cohomology.

We will restrict ourselves to the case of $H^*(\Omega_0^\infty MTO(m);\mathbb{Z}/2)$ for the sake of concreteness.

The composition

$$H^*(BSO(m+1);\mathbb{Z}/2) \xrightarrow{\sigma^{\infty *}} H^*(Q_0BSO(m+1)_+;\mathbb{Z}/2) \longrightarrow \\ H^*(\Omega_0^\infty MTO(m);\mathbb{Z}/2)$$

is injective because the first map is injective by dualising Lemma 4.4, and the second is so by Corollary 1.4. Of course, it is not a ring map as $\sigma^{\infty *}$ is not, but $\sigma^{\infty *}$'s natural right inverse is, being induced by a map of spaces $BSO(m+1) \longrightarrow QBSO(m+1)$. Thus the universally defined characteristic classes that are images of the standard polynomial generators of $H^*(BSO(m+1);\mathbb{Z}/2)$ are algebraically independent. Unfortunately the standard polynomial generators of $H^*(BSO(m+1);\mathbb{Z}/2)$ do not map to polynomial generators of $H^*(BO(m);\mathbb{Z}/2)$, which makes things a little bit complicated.

For example, let's take the case of $\Omega_0^\infty MTO(2)$, $p = 2$. Then we have

$$H^*(BSO(3);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3],\ H^*(BO(2);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2],$$

and the map $BO(2) \longrightarrow BSO(3)$ induces a map $w_2 \mapsto w_2 + w_1^2$ by Lemma 3.2, and by similar arguments we get $w_3 \mapsto w_1w_2$. One can derive from this the classes $\mu_{0,1} + \mu_{1,0}^2 = \overline{\nu}_{w_2+w_1^2}$ and $\mu_{1,1} = \overline{\nu}_{w_1w_2}$ as defined in [41] (see also Remark 4.7) are algebraically independent. A more detailed analysis of the homology suspension map leads to the following:

Theorem 4.6. *(Theorem 1.9) Let ν_I be the image of $w^I \in H^*(BSO(2m+1);\mathbb{Z}/2)$ in $H^*(\Omega_0^\infty MTO(2m);\mathbb{Z}/2)$ under the composition*

$$H^*(BSO(2m+1);\mathbb{Z}/2) \longrightarrow H^*(BO(2m);\mathbb{Z}/2) \longrightarrow \\ H^*(Q_0(BO(2m));\mathbb{Z}/2) \longrightarrow H^*(\Omega_0^\infty MTO(2m);\mathbb{Z}/2).$$

In other words, $\nu_I = \overline{\nu}_{Bj^*w^I}$ where $j : O(2m) \longrightarrow SO(2m+1)$ was defined in Section 3. Then the only relations among these classes are the ones generated by

$$\nu_I^2 = \nu_{2I}.$$

Thus the classes ν_I, $I = (i_2, \ldots, i_{2m+1})$ with at least one i_k odd are algebraically independent.

Proof. Consider the following diagram, which commutes by the naturality of the homology suspension.

$$\begin{tikzcd}
H^*(BSO(2m+1);\mathbb{Z}/2) \arrow[r, "Bj^*"] \arrow[d, "\sigma^{\infty *}"] & H^*(BO(2m);\mathbb{Z}/2) \arrow[r, "\omega^*"] & H^*(MTO(2m);\mathbb{Z}/2) \arrow[d, "\sigma^{\infty *}"] \\
H^*(Q_0BSO(2m+1)+;\mathbb{Z}/2) \arrow[rr] & & H^*(\Omega_0^\infty MTO(2m);\mathbb{Z}/2)
\end{tikzcd}$$

The bottom horizontal map is injective by Corollary 1.4, thus it suffices to show the corresponding results, with ν_I replaced with $\sigma^{\infty *}(w^I)$ in $H^*(Q_0BSO(2m+1)_+;\mathbb{Z}/2)$.

Now, let X be any space. Consider the following diagram.

$$\begin{tikzcd}
H^d(X;\mathbb{Z}/2) \arrow[r, "\sigma^{\infty *}"] \arrow[d, "Sq^d=(-)^2"] & H^d(Q_0X;\mathbb{Z}/2) \arrow[r] \arrow[d, "Sq^d=(-)^2"] & H^d(X;\mathbb{Z}/2) \arrow[d, "Sq^d=(-)^2"] \\
H^{2d}(X;\mathbb{Z}/2) \arrow[r, "\sigma^{\infty *}"] & H^{2d}(Q_0X;\mathbb{Z}/2) \arrow[r] & H^{2d}(X;\mathbb{Z}/2)
\end{tikzcd}$$

By the properties of the Steenrod squares, this diagram is commutative. Moreover, by Lemmata 4.3 and 4.4 the horizontal compositions are the identities. Therefore, an element of $H^*(X;\mathbb{Z}/2)$ is a square if and only if its image in $H^*(Q_0X;\mathbb{Z}/2)$ is a square. Furthermore, Lemma 4.5 provides the factorisation of the map $\sigma^{\infty *}$ as

$$\begin{tikzcd}
 & PH^*(Q_0X;\mathbb{Z}/2) \arrow[d, hook] \\
H^*(X;\mathbb{Z}/2) \arrow[r, "\sigma^{\infty *}"] \arrow[ur, dotted] & H^*(Q_0X;\mathbb{Z}/2)
\end{tikzcd}$$

and by [36, Proposition 4.21], the only elements in the kernel of the map

$$PH^*(Q_0X;\mathbb{Z}/2) \longrightarrow QH^*(Q_0X;\mathbb{Z}/2)$$

are squares. From now on, suppose that $H^*(X;\mathbb{Z}/2)$ is polynomial. Then so is $H^*(Q_0X;\mathbb{Z}/2)$ by [47, Theorem 3.11] (see also [17, Lemma 7.2]). Now, consider the dotted arrow in the following diagram.

$$\begin{tikzcd}
H^*(X;\mathbb{Z}/2) \arrow[r, "\sigma^{\infty *}"] \arrow[d, hook] & H^*(Q_0X;\mathbb{Z}/2) \\
Sym(H^*(X;\mathbb{Z}/2)) \arrow[ur, dotted] &
\end{tikzcd}$$

By the above arguments, its kernel is the ideal generated by the elements $[x^2]-[x]^2$ where $x \in H^*(X;\mathbb{Z}/2)$, $[x]$ is the corresponding element in $Sym(H^*(X;\mathbb{Z}/2))$. Theorem now follows noting that $H^*(BSO(m+1);\mathbb{Z}/2)$ is polynomial. ∎

Remark 4.7. 1. *We denote $\mu_I = \overline{v}_{w^I}$ for $H^*(BO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1,\dots,w_n]$. This generalises the classes $\mu_{i,j}$'s defined in [41, Example 2.6]. It is easy to see that once we express ν's in terms of μ's, the relations $\nu_I^2 = \nu_{2I}$ follow from the ones $\mu_J^2 = \mu_{2J}$ and the latter relations were essentially found in [41].*

2. *The arguments above also apply to other pairs G, K and at any prime satisfying the hypothesis of Corollary 1.4, as long as $H^*(BK;\mathbb{Z}/p)$ is polynomial. The proof is completely similar, and at odd primes, we remark that we only have to work with the subalgebras of $H^*(Q_0BK_+;\mathbb{Z}/p)$ generated by the elements dual to that of $H^*(BK;\mathbb{Z}/p)$, which is polynomial by [47, Theorem 3.11].*

Example 4.8. *Let's consider the case $n = 1$.*
As $Bj^* : H^*(BSO(3); \mathbb{Z}/2) \longrightarrow H^*(BO(2); \mathbb{Z}/2)$ is a ring homomorphism, we have $(w_2)^i \mapsto (w_2 + w_1^2)^i$, $w_3^j \mapsto w_1^j w_2^j$. Thus in low degrees, we have following algebraically independent elements in degrees less than or equal to 9. We show the detail of computation for first few elements.

degree	*elements in terms of ν*	*elements in terms of μ*
2	$\nu_{1,0} = \overline{\nu}_{w_2}$	$\mu_{0,1} + \mu_{1,0}^2 = \mu_{0,1} + \mu_{2,0} = \overline{\nu}_{w_2+w_1^2}$[1]
3	$\nu_{0,1} = \overline{\nu}_{w_3}$	$\mu_{1,1} = \overline{\nu}_{w_1 w_2}$
4	$N.A.$	$N.A.$
5	$\nu_{1,1} = \overline{\nu}_{w_2 w_3}$	$\mu_{1,2} + \mu_{3,1} = \overline{\nu}_{w_1 w_2^2 + w_1^3 w_2}$
6	$\nu_{3,0}$	$\mu_{0,3} + \mu_{1,1}^2 + \mu_{4,1} + \mu_{3,0}^2$
7	$\nu_{1,2}$	$\mu_{2,3} + \mu_{2,1}^2$
8	$\nu_{2,1}$	$\mu_{2,3} + \mu_{2,1}^2$
9	$\nu_{3,1},\ \nu_{0,3}$	$\mu_{1,4} + \mu_{3,3} + \mu_{5,2} + \mu_{7,1},\ \mu_{3,3}$ *(resp.)*

4.4 Cohomology with integer coefficients

In this section, we discuss the implication of our splitting theorems to the cohomology of the infinite loop spaces associated to MTK spectra with p-local integer coefficients or integer coefficients. Let (K, G) be a pair satisfying hypotheses of Theorems 3.1 or 3.3, and choose p accordingly. Then p-locally BG_+ splits off MTK, so $QBG_{+(p)}$ is a retract of $\Omega^\infty MTK_{(p)}$ and $H^*(QBG_+; \mathbb{Z}_{(p)})$ is a summand of $H^*(\Omega^\infty MTK; \mathbb{Z}_{(p)})$ even in the absence of the Künneth isomorphism. Since $H^*(QBG_+; \mathbb{Z}_{(p)})$ can be described completely in terms of $H^*(BG_+; \mathbb{Z}_{(p)})$, which is completely known in all cases ($H^*(BSpin(n); \mathbb{Z})$ which we have not discussed in Appendix A is known by [30]) we have a complete knowledge of this summand. Unfortunately $H^*(QBG_+; \mathbb{Z}_{(p)})$ as well as $H^*(\Omega^\infty MTK; \mathbb{Z}_{(p)})$ only have the structure of algebras, and not coalgebras, because of the lack of the Künneth isomorphism caused by the presence of torsion, which makes it rather difficult to work with them concretely. However, we can still get some information on them. For example, it follows immediately from [34, Theorem 4.13] that they contain a summand of order p^i for any i.

Without localisation, even less can be said. Still, we can assert the splitting of $H^*(BG; \mathbb{Z})$ off $H^*(\Omega^\infty MTK; \mathbb{Z})$ under the hypotheses of Theorem 3.1(i). As a matter of fact, a similar statement holds for any generalised cohomology. We show that in the case of ordinary cohomology with integer coefficients, this implies that the non-divisibility of generalised Wahl classes (Theorem 4.10). Let's start with a definition.

Definition 4.9. *$\zeta_I \in H^*(MTO(2m); \mathbb{Z})$ is the universally defined characteristic class associated to the monomial in the Pontryagin classes p^I, $\overline{\nu}_{p^I}$.*

Thus given a $2m$-dimensional manifold bundle $E \longrightarrow B$ with associated Madsen-Tillmann-Weiss map $f : B \longrightarrow \Omega_0^\infty MTO(2m)$, one can define

[1] In [25] we will show that $\mu_{1,0} = 0$, thus this class is equal to $\mu_{0,1}$.

$\zeta_I(E) = f^*(\zeta_I) \in H^*(B;\mathbb{Z})$. When $m = 1$, by writing $i_1 = i$, we recover Wahl's classes ζ_i. Given a surface bundle $E \longrightarrow B$, Wahl defines $\zeta_i \in H^{4i}(B;\mathbb{Z})$ to be the image of $p_1(T_\pi(E))^i$ by the transfer $H^*(E;\mathbb{Z}) \longrightarrow H^*((B;\mathbb{Z})$ where $T_\pi(E) \longrightarrow E$ is the vertical tangent bundle ([46, p.391]). Although our definition differs from hers, as in [41, Theorem 2.4] one can prove that the both definitions agree [41, Example 2.5].

Theorem 4.10. *(Theorem 1.10) The classes $\zeta_I \in H^*(MTO(2m);\mathbb{Z})$ are not divisible in $H^*(\Omega_0^\infty MTO(2m);\mathbb{Z})$.*

Proof. By the naturality of the cohomology suspension, the following square commutes.

$$\begin{tikzcd}
H^*(BSO(2m+1);\mathbb{Z}) \arrow[r] \arrow[d] & H^*(BO(2m);\mathbb{Z}) \arrow[r] & H^*(MTO(2m);\mathbb{Z}) \arrow[d] \\
H^*(Q_0BSO(2m+1);\mathbb{Z}) \arrow[rr] & & H^*(\Omega_0^\infty MTO(2m);\mathbb{Z})
\end{tikzcd}$$

Note that by Theorem 3.1, $BSO(2m+1)_+$ splits off $MTO(2m)$, thus $QBSO(2m+1)_+$ splits off $\Omega^\infty MTO(2m)$. By Lemma 4.3 $\Sigma^\infty BSO(2m+1)_+$ splits off $\Sigma^\infty QBSO(2m+1)_+$. Combining these we see that $\Sigma^\infty BSO(2m+1)_+$ splits off $\Sigma^\infty\Omega^\infty MTO(2m)$.
Thus the composition $H^*(BSO(2m+1);\mathbb{Z}) \longrightarrow H^*(\Omega^\infty MTO(2m);\mathbb{Z})$ is a split monomorphism of abelian groups.

On the other hand, $H^*(BSO(2m+1);\mathbb{Z})$ is also a direct summand of $H^*(BO(2m);\mathbb{Z})$, with the quotient group consisting only of torsion elements. Thus we have a sequence of maps

$$\begin{aligned}\mathbb{Z}[p_1,\dots,p_m] \subset H^*(BSO(2m+1);\mathbb{Z}) \longrightarrow H^*(BO(2m);\mathbb{Z}) \cong \\ \mathbb{Z}[p'_1,\dots,p'_m] \oplus T \longrightarrow \mathbb{Z}[p_1,\dots,p_m]\end{aligned}$$

where the composition is an isomorphism. Here we used the notation p'_i to distinguish the Pontryagin classes in $H^*(BO(2m);\mathbb{Z})$ from those in $H^*(BSO(2m+1);\mathbb{Z})$. In other words, a monomial in p''s is, up to torsion elements, the image of a non-divisible element in $H^*(BSO(2m+1);\mathbb{Z})$. But by definition the ζ-classes are the images of monomials in p''s, thus up to torsion elements, they are images of non-divisible element in $H^*(BSO(2m+1);\mathbb{Z})$. Since

$$H^*(BSO(2m+1);\mathbb{Z}) \longrightarrow H^*(\Omega^\infty MTO(2m);\mathbb{Z})$$

is a split mono, a non-divisible element in the former maps to a non-divisible element in the latter. Now, as in the proof of Lemma 4.3, we can replace $\Omega^\infty MTO(2m)$ with $\Omega_0^\infty MTO(2m)$ which completes the proof. ∎

We remark that in general, there is no reason to expect that a monomial in p''s in $H^*(BO(2m);\mathbb{Z})$ is actually the image of an element in $H^*(BSO(2m+1);\mathbb{Z})$. As a matter of fact, Chern classes in $H^*(BSO(2m+1);\mathbb{Z})$ map to Chern classes in $H^*(BO(2m);\mathbb{Z})$, but c_{2i} can pull back to a polynomial involving c_{2j+1} with $j < i$.

We also note that in the above, we started with non-divisible elements in $H^*(BG;\mathbb{Z})$. In some cases, it may happen that the characteristic class is already divisible in $H^*(BG;\mathbb{Z})$, in which case its image in $H^*(\Omega^\infty MTK;\mathbb{Z})$ will have the same divisibility. Thus if we take the pair (K,G) to be $(Pin^-(2), Spin(3))$, we get the first part of [42, Proposition 5.3], modulo the homological stability [42, Theorem 4.19].

To conclude the subsection, let's remark that as is noted in [46], if the fibre of the surface bundle $E \longrightarrow B$ is orientable, $\zeta_i(E)$ agrees with $\kappa_{2i}(E)$, where κ_{2i} is the well-known $2i$-th Mumford-Miller-Morita class. As a matter of fact $p_1' \in H^*(BO(2);\mathbb{Z})$ restricts to $p_1 \in H^*(BSO(2);\mathbb{Z})$, and if we let c_1 denote the first Chern class of the standard representation of $SO(2)$ considered as 1-dimensional complex representation, then we have

$$c_1 = \chi \in H^*(BSO(2);\mathbb{Z}),$$

thus we have

$$p_1 = c_1^2 \in H^*(BSO(2);\mathbb{Z}).$$

Therefore $\overline{\nu}_{p_1'^i} = \overline{\nu}_{c_1^{2i}}$. In other words, $\zeta_i(E)$ is the transfer of $c_1^{2i} \in H^*(E;\mathbb{Z})$ to $H^*(B;\mathbb{Z})$. Usually the classes κ_i's are defined to be the push-forward of $c_1^{2i+1} \in H^*(E;\mathbb{Z})$ to $H^*(B;\mathbb{Z})$, but as is mentioned in [16, Section 3], by Theorem C.1, the push-forward of c_1^{2i+1} agrees with the transfer of c_1^{2i}. Thus we have $\zeta_i(E) = \kappa_{2i}(E)$.

4.5 The failure of the exactness

We now proceed to prove the following:

Proposition 4.11. *(Proposition 1.11) Suppose the pair of groups $(\mathbf{K}(m), G)$, and the prime p satisfies hypotheses of Theorem 3.1 or Theorem 3.3, so that $BG_{+(p)}$ splits off $MT\mathbf{K}(m)_{(p)}$, and that G is non-trivial. Suppose further if $K = O$, then $p = 2$. Then, the sequence of Hopf algebras*

$$H_*(\Omega_0^\infty MT\mathbf{K}(m+1);\mathbb{Z}/p) \longrightarrow H_*(Q_0B\mathbf{K}(m+1)_+;\mathbb{Z}/p) \xrightarrow{(\Omega_0^\infty \tilde{f})_*} H_*(\Omega_0^\infty MT\mathbf{K}(m);\mathbb{Z}/p)$$

induced by the cofibration for Madsen-Tillmann spectra (Lemma 2.6) is not short exact.

Proof. Let $\mathbf{K}$, m, G and p be as in hypothesis of Proposition. We show that the sequence of Hopf algebras induced by the cofibre sequence $MT\mathbf{K}(m+1) \longrightarrow \Sigma^\infty B\mathbf{K}(m)_+ \longrightarrow MT\mathbf{K}(m)$

$$H_*(\Omega_0^\infty MT\mathbf{K}(m+1);\mathbb{Z}/p) \longrightarrow H_*(Q_0B\mathbf{K}(m+1)_+;\mathbb{Z}/p) \xrightarrow{(\Omega_0^\infty \tilde{f})_*} H_*(\Omega_0^\infty MT\mathbf{K}(m);\mathbb{Z}/p)$$

is not short exact. More precisely, we will show that $(\Omega_0^\infty \tilde{t})_*$ is not surjective. By naturality of the homology suspension, the following square is commutative,

$$\begin{tikzcd}
H_*(Q_0 B\mathbf{K}(m+1)_+;\mathbb{Z}/p) \arrow[r, "(\Omega_0^\infty \tilde{t})_*"] \arrow[d, "\sigma_*^\infty"'] & H_*(\Omega_0^\infty MT\mathbf{K}(m);\mathbb{Z}/p) \arrow[d, "\sigma_*^\infty"'] \\
H_*(\Sigma^\infty B\mathbf{K}(m+1)_+;\mathbb{Z}/p) \arrow[r, "\tilde{t}_*"] & H_*(MT\mathbf{K}(m);\mathbb{Z}/p).
\end{tikzcd}$$

Suppose that $(\Omega_0^\infty \tilde{t})_*$ is onto. By Lemma 4.4. the left vertical map is onto. On the other hand, Lemma 2.8 implies that $\tilde{t}_* = 0$. Combining these, we see that the right vertical map is trivial. However, our splitting results imply that $H_*(\Omega_0^\infty MT\mathbf{K}(m); \mathbb{Z}/p))$ contains a tensor factor isomorphic to $H_*(Q_0 BG_+;\mathbb{Z}/p)$, on which the homology suspension is nontrivial again by Lemma 4.4, which is a contradiction. ∎

A Recollection on Lie groups, and characteristic classes

In this section, we collect some preliminary materials on classical Lie groups, their cohomology, their extension. We mostly intend to fix our notation.

A.1 Cohomology of classifying spaces and characteristic classes

For a moment, let's write $\mathrm{Gr}^{\mathbf{G}}(d,+\infty)$ for the classifying space of $\mathbf{G}(d)$-vector bundles. Often, the notation $B\mathbf{G}(d) = E\mathbf{G}(d)/\mathbf{G}(d)$ denotes the classifying space of a $\mathbf{G}(d)$ which most of the time coincides $\mathrm{Gr}^{\mathbf{G}}(d,+\infty)$. However, for some choices of $\mathbf{G}$, in the case of $d=0$, there are a few exceptions. For instance, for $\mathbf{G} = SO$, $\mathrm{Gr}^{\mathbf{SO}}(d,+\infty) = \cup_k G^+(k, d+k)$ where $G^+(k,d+k)$ is the Grassmann manifold of oriented d-codimensional linear subspaces of $\mathbb{R}^{d+k}$, yields $\mathrm{Gr}^{\mathbf{G}}(d,+\infty) = S^0$ which is *not* homotopy equivalent to $BSO(0) \simeq B1 \simeq *$. This occurs because of existence of $+$ and $-$ orientations for a point. Similarly, $\mathrm{Gr}^{\mathbf{Spin}}(0,+\infty) = B\mathbb{Z}/2 \times S^0$ and $\mathrm{Gr}^{\mathbf{SU}}(0,+\infty) = S^1$, do not agree with $BSpin(0)$ and $BSU(0)$, respectively. By abuse of notation, we keep writing $B\mathbf{G}(d)$ for the classifying space of $\mathbf{G}(d)$-vector bundles as we declared in Section 1.

The following is well-known: the ring structure is given by [9, Proposition 23.2] for $\mathbf{K} = SO$, Theorem 19.1 loc.cit in other cases. The computation for $\mathbf{K} = SO, O$ with integral coefficient is [45, Theorem A, Theorem 12.1]. The identification of generators with characteristic classes follow, for example, from [8, Section 9].

Theorem A.1. *Let k be any ring if $\mathbf{K} = U, Sp$ or SU, an algebra over $\mathbb{Z}/2$ if $\mathbf{K} = O$ or SO, k' be a ring in which 2 is invertible. Then, for $n \geqslant 1$, $H^*(B\mathbf{K}(n);k)$ is given as follows: for $n \geq 0$ we have*

$$\begin{array}{rcl}
H^*(BO(n);k) & \cong & k[w_1, w_2, \ldots, w_n] \\
H^*(BU(n);k) & \cong & k[c_1, c_2, \ldots, c_n] \\
H^*(BSp(n);k) & \cong & k[p_1, p_2, \ldots, p_n]
\end{array}$$

and for $n \geq 1$ we have

$$\begin{aligned} H^*(BSO(n);k) &\cong k[w_2,\ldots,w_n] \\ H^*(BSU(n);k) &\cong k[c_2,\ldots,c_n] \end{aligned}$$

and further for $m \geq 1$ we have

$$\begin{aligned} H^*(BSO(2m);\mathbb{Z}) &\cong \mathbb{Z}[p_1,\cdots,p_m,\chi]/(\chi^2 - p_m) \oplus T \\ H^*(BSO(2m);k') &\cong k'[p_1,\cdots,p_m,\chi]/(\chi^2 - p_m) \end{aligned}$$

$$\begin{aligned} H^*(BSO(2m+1);\mathbb{Z}) \cong H^*(BO(2m+1);\mathbb{Z}) &\cong H^*(BO(2m);\mathbb{Z}) \\ &\cong \mathbb{Z}[p_1,\cdots,p_m] \oplus T \\ H^*(BSO(2m+1);k') \cong H^*(BO(2m+1);k') &\cong H^*(BO(2m);k') \\ &\cong k'[p_1,\cdots,p_m] \end{aligned}$$

where w_i, the i-th Stiefel-Whitney class, has degree i, c_i, the i-th Chern class, has degree $2i$, and $p_i \in H^{4i}(BSp(n);k)$, the i-th symplectic Pontryagin class, $p_i \in H^{4i}(BSO(n);k')$, the i-th Pontryagin class, T is an elementary abelian 2-group. Furthermore, the standard inclusions $SO(n) \subset O(n)$ induce the obvious projections sending w_1 to 0 and other w_i's to w_i's with k coefficients, and similar statement holds for the standard inclusions $SU(n) \subset U(n)$. The inclusions $O(n) \subset U(n)$, sends c_i to w_i^2 when the characteristic of k is 2, otherwise c_{2i} to p_i.

This can be stated in a more economical way by saying that for $\mathbf{K} = O, U$ or Sp, $H^*(B\mathbf{K}(n);k) \cong k[x_1,\ldots,x_n]$ with the degree of x_i equal to di, where $d = 1,2$ or 4 depending on whether $\mathbf{K} = O, U$ or Sp, similarly for $H^*(B\mathbf{S}G(n);k)$. Then the standard inclusions $\mathbf{K}(n-1) \subset \mathbf{K}(n)$ induce the obvious projections sending x_n to 0, and other x_i's to x_i.

As the names suggest, these polynomial generators are characteristic classes, more precisely the characteristic classes for universal bundles, or the universal characteristic classes. That is, for example, if V is a real n-dimensional vector bundle over the base space X with classifying map $f : X \longrightarrow BO(n)$, that is, V is the pull-back of the universal n-dimensional vector bundle over $BO(n)$ via f, then the i-th Stiefel-Whitney class of V is given by $w_i(V) = f^*(w_i)$.

We will need the following property of these classes (the injectivity is given by [9, Proposition 29.2], the image of characteristic classes in [8, 9.1, 9.2, and 9.6]):

Theorem A.2. *Let $\mathbf{K} = O, U$ or Sp, k be any ring if $\mathbf{K} = U$ or Sp, a $\mathbb{Z}/2$-algebra if $\mathbf{K} = O$, d as above. The usual inclusion $j : \mathbf{K}(1)^n \longrightarrow \mathbf{K}(n)$ induces an injection in cohomology, such that we have*

$$Bj^*(x_i) = \sigma_i(t_1,\ldots,t_n) \in H^*(B\mathbf{K}(1)^n) \cong k[t_1,\ldots,t_n]$$

where t_i's have degree d, and σ_i denotes the i-th elementary symmetric polynomial.

A.2 Pin groups, Pin-bundles, and Pin-structures

The orthogonal group $O(n)$ admits several double covers, notably we have central extensions $\mathbb{Z}/2 \longrightarrow Pin^+(n) \longrightarrow O(n)$ corresponding to w_2 and $\mathbb{Z}/2 \longrightarrow Pin^-(n) \longrightarrow O(n)$ corresponding to $w_2 + w_1^2$ in $H^2(BO(n); \mathbb{Z}/2)$. Similarly the special orthogonal group $SO(n)$ admits a central extension $\mathbb{Z}/2 \longrightarrow Spin(n) \longrightarrow SO(n)$ corresponding to w_2 ([26, p.434]). These groups can also be defined directly using Clifford algebras [3, 27, 31].

Given a real vector bundle V over X, one can ask whether the structure map can be lifted through the *canonical projection* $Pin^{\pm}(n) \longrightarrow O(n)$. Such a lift is called $Pin^{\pm}(n)$-bundle structure. V admits a Pin^+ (Pin^- respectively) structure if and only if $w_2(V)$ ($w_2(V) + w_1(V)^2$ resp.) vanishes ([27, Lemma 1.3]). For a n-dimensional manifold M, we say that M admits a $Pin^{\pm}(n)$ structure if its tangent bundle admits a $Pin^{\pm}(n)$ structure. Here we note that this is about a factorisation through particular maps $Pin^{\pm}(n) \longrightarrow O(n)$. Thus although as abstract Lie groups, $Pin^+(4n)$ and $Pin^-(4n)$ are isomorphic (c.f. [11, example 3 in 1.7, pp. 25-27], (communicated to us by Theo Johnson-Freyd,) where they are called $Pin(4n, 0)$ and $Pin(0, 4n)$), they are not isomorphic as double covers of $O(4n)$, thus the notion of $Pin^+(4n)$ bundle structure and that of $Pin^-(4n)$ structure don't agree.

The following is well-known (e.g. [26] p.434):

Proposition A.3. *$\mathbb{R}P^{4k}$ has a Pin^+ structure and $\mathbb{R}P^{4k+2}$ has a Pin^- structure.*

The proof is left as an exercise to the interested reader. One can use, for example, the relationship between the tangent bundle and the canonical line bundle c. f. [22, Chapter 2, Example 4.8].

B Homology of infinite loop spaces

For an infinite loop space X, since $X \simeq \Omega^2 X_2$, where X_2 is the second space of the associated omega-spectrum, the homology $H_*(X; \mathbb{Z}/p)$ is a graded commutative ring under the Pontryagin product. Moreover, there are Kudo-Araki-Dyer-Lashof homology operations that we will call Dyer-Lashof operations, $\beta^\epsilon Q^i$ which act on $H_*(X; \mathbb{Z}/p)$. The operation Q^i is a group homomorphism, is natural with respect to infinite loop maps, and raises degrees by $2(p-1)i - \epsilon$ [34, Theorem 1.1]. These operations satisfy Adem relations, various Cartan formulae, and Nishida relations [34, Theorem 1.1]. The algebra wherein these operations live is the Dyer-Lashof algebra R; it is the free algebra generated by these operations, modulo Adem relations and excess relations. The homology of $H_*(X; \mathbb{Z}/p)$ then becomes an R-module. In some cases, these operations allow a neat description of $H_*(X; \mathbb{Z}/p)$. For instance, if $X = QY$ with Y some path connected space, then $H_*(X; \mathbb{Z}/p)$ is a free algebra generated by Dyer-Lashof allowable operations on $\widetilde{H}_*(Y; \mathbb{Z}/p)$ ([34, Chapter 1, Lemma 4.10]). Furthermore, when $Y = S^0$, the Dyer-Lashof operations act on the fundamental class of $\widetilde{H}_0(S^0; \mathbb{Z}/p)$ in such a way that $\{\beta^\epsilon Q^i[1]; i \in N, \epsilon = 0, 1\}$ is precisely the image of $H_*(B\Sigma_p)$ by the "standard inclusion" $H_*(B\Sigma_p; \mathbb{Z}/p) \longrightarrow H_*(QS^0; \mathbb{Z}/p)$. This latter also coincides with the map induced by the adjoint of the transfer associated to the inclusion of the trivial group in Σ_p.

The other cases that we shall consider in this paper, are the spaces $\mathbb{Z} \times BO$ and $\mathbb{Z} \times BU$ which are infinite loop spaces under Bott periodicity; the monoid structure coming from the Whitney sum is compatible with Bott periodicity. They correspond to ring spectra KO and KU, thus there are maps $QS^0 \longrightarrow \mathbb{Z} \times BO$ and $QS^0 \longrightarrow \mathbb{Z} \times BU$.

We have the following isomorphisms.

$$\begin{array}{lclcl}
H^*(BO;\mathbb{Z}/2) & \cong & \mathbb{Z}/2[w_1, w_2, \ldots, w_n, \ldots] & \cong & \varprojlim H^*(BO(n);\mathbb{Z}/2) \\
H_*(BO;\mathbb{Z}/2) & \cong & \mathbb{Z}/2[a_1, a_2, \ldots, a_n, \ldots] & \cong & \varinjlim H_*(BO(n);\mathbb{Z}/2) \\
H^*(BU;\mathbb{Z}) & \cong & \mathbb{Z}[c_1, c_2, \ldots, c_n, \ldots] & \cong & \varprojlim H^*(BU(n);\mathbb{Z}) \\
H_*(BU;\mathbb{Z}) & \cong & \mathbb{Z}[b_1, b_2, \ldots, b_n, \ldots] & \cong & \varinjlim H_*(BU(n);\mathbb{Z})
\end{array}$$

The elements w_i's and c_i's are as in Theorem A.1. As BO classifies stable virtual bundles, this means that we can define the Stiefel-Whitney class $w_i(V)$ for a stable virtual bundle over X with classifying map $f : X \longrightarrow BO$ by $w_i(V) = f^*(w_i)$. We note that the multiplication by (-1) on the set of virtual bundles corresponds to the "multiplication by (-1)" self-map on BO, thus the conjugation τ on $H^*(BO;\mathbb{Z}/2)$ satisfies $f^*\tau(w_i) = w_i(-V)$. Similar statements hold for BU. As for the homology computations, we will use the fact that the elements a_i's and b_i's are respectively the image of a generator of $H_i(BO(1);\mathbb{Z}/2)$ and $H_{2i}(BU(1);\mathbb{Z})$.

The map induced in homology by the unit map was determined in [39], in particular, we have

Lemma B.1. *[39, Proposition 4.10, $n = 1$ case] The map*

$$H_*(QS^0;\mathbb{Z}/2) \longrightarrow H_*(\mathbb{Z} \times BO;\mathbb{Z}/2)$$

is an epimorphism.

Basically this is because $H_*(BO;\mathbb{Z}/2)$ is generated by $H_*(BO(1);\mathbb{Z}/2)$, $H_*(BO(1);\mathbb{Z}/2)$ is "contained" in $H_*(Q_0S^0;\mathbb{Z}/2)$, and the inclusions $H_*(BO(1);\mathbb{Z}/2) \subset H_*(\mathbb{Z} \times BO;\mathbb{Z}/2)$ and $H_*(BO(1);\mathbb{Z}/2) \subset H_*(Q_0S^0;\mathbb{Z}/2)$ are compatible.

Now we would like to generalise to the "complex" case. Although it is still true that the homology of BU is generated by that of $BU(1)$, for odd prime p the homology of $BU(1)$ contains elements that are unrelated to the Dyer-Lashof operations. That is, the homology of $BU(1)$ is the even degree part of the homology of $B\mathbb{Z}/p$, which is much larger than that of $B\Sigma_p$ for an odd prime p, which is related to the Dyer-Lashof operations. However, as is well-known, (stably) $B\Sigma_{p(p)}$ splits off $B\mathbb{Z}/p$ (e.g. [40, Example 2]), and it turns out that we can also split $BU(1)_{(p)}$ in a compatible way. That is ([37])

$\mathbb{C}P^{\infty\;\wedge}_{(p)} \cong \vee_{i=0}^{p-2} X(i)$

where $X(i)$'s are spectra with $H^*(X(i)) = 0$ unless $* \equiv 2i \bmod 2(p-1)$.

As a matter of fact, the method can be applied to split $B\mathbb{Z}/p$ into $p-2$ pieces, one of which is $B\Sigma_{p(p)}$, refining the above-mentioned splitting. The piece $X(0)$

corresponds to the piece $B\Sigma_{p(p)}$. Thus, $X(0)$ will be the spectrum "representing" the Dyer-Lashof operations of length 1 surviving in $H_*(BU)$, i.e., the ones without Bocksteins, and it will play the role of $BO(1)$.

Now, as $X(0)$ being sensibly smaller than $BU(1)$, we see that BU certainly can't play the role of BO. But the space BU also admits a similar splitting. Denote by KU the complex K-theory spectrum. After localising at p, KU splits as ([1, Lecture 4])

$$KU_{(p)} \simeq \vee_{i=0}^{p-2}\Sigma^{2i}E(1)$$

where $\pi_*(E(1)) \cong \mathbb{Z}_{(p)}[v_1, v_1^{-1}]$ with the degree of v_1 equal to $2(p-1)$.

Denote j_E the resulting splitting map $E(1) \longrightarrow KU_{(p)}$.

Since $BU_{(p)} \times \mathbb{Z}_{(p)}$ is the infinite loop space associated to $KU_{(p)}$, it also splits as a product of spaces

$$BU_{(p)} \times \mathbb{Z}_{(p)} \simeq \Pi_{i=0}^{p-2}\Omega^\infty\Sigma^{2i}E(1).$$

Thus $\Omega^\infty E(1)$ is a direct factor of $BU_{(p)} \times \mathbb{Z}_{(p)}$, and it will be the correct analog of BO. Its cohomology can be described as follows:

Lemma B.2. *Let $k = \mathbb{Z}_{(p)}, \mathbb{Q}$ or $\mathbb{Z}/p$.*

$$H^*(\Omega^\infty E(1), k) \cong k[c_{p-1}, c_{2(p-1)}, \ldots, c_{m(p-1)}, \ldots],$$

and j_E^* sends $c_{m(p-1)} \in H^{2m(p-1)}(BU;k)$ to $c_{m(p-1)}$ and other c_i's to 0.

Proof. This can be shown using [20], but here we follow rather the arguments in [21]. Let's start with the case $k = \mathbb{Q}$. For k-vector spaces V, denote by $Sym_k(V)$ the symmetric algebra generated by V, i.e., $\oplus_q V^{\otimes_k q}/\Sigma_q$ with the product induced by the concatenation. It is well known that for any spectrum X with $\pi_{odd}(X) = 0$, we have natural isomorphisms

$$H_*(\Omega_0^\infty X;\mathbb{Q}) \cong H_*((\Omega_0^\infty X)_\mathbb{Q};\mathbb{Q}) \cong H_*(\Omega_0^\infty(X_\mathbb{Q});\mathbb{Q}) \cong Sym_\mathbb{Q}(\pi_{*>0}(X) \otimes \mathbb{Q}).$$

Since $\pi_*(j_E)$ is bijective for $* = 2m(p-1)$ and 0 otherwise, we get the desired result in this case. As $\Omega^\infty E(1)$ is a direct factor of $BU_{(p)} \times \mathbb{Z}_{(p)}$, $H_*(\Omega^\infty E(1);\mathbb{Z}_{(p)})$ is torsion-free. Therefore $H_*(\Omega^\infty E(1);\mathbb{Z}_{(p)})$ injects to $H_*(\Omega^\infty E(1);\mathbb{Q})$. Similarly for $BU_{(p)} \times \mathbb{Z}_{(p)}$. Thus we get the result when $k = \mathbb{Z}_{(p)}$. Finally, one can derive the case $k = \mathbb{Z}/p$ follows from this by the universal coefficient theorem. ∎

There is another splitting involving $\mathbb{C}P^\infty$ and BU, namely Segal's splitting. Consider the orientation map for the KU-theory $\mathbb{C}P^\infty \longrightarrow BU$. Since the target is an infinite loop space, by the adjointness, it factors through $s : Q\mathbb{C}P^\infty \longrightarrow BU$. Then it splits as a map of spaces, that is, there is a space F such that $Q\mathbb{C}P^\infty \simeq BU \times F$ [44, Theorem]. It turns out that the Adams' splittings of KU and $\mathbb{C}P^\infty$ interacts nicely enough with Segal's, so that the last splitting can be refined to the splitting of corresponding Adams' pieces [29, Theorem 1.1]. We have

Proposition B.3. *The map $\Omega^\infty X(0) \longrightarrow \Omega^\infty E(1)$ splits, that is we have a space F' such that $\Omega^\infty X(0) \simeq \Omega^\infty E(1) \times F'$. In particular, it induces a surjection in homology with any coefficient.*

Now we are ready to prove the following.

Lemma B.4. *The unit map of the ring spectrum $E(1)$ induces a surjection $H_*(QS^0;\mathbb{Z}/p) \longrightarrow H_*(\Omega^\infty E(1);\mathbb{Z}/p)$.*

This can be proved in many different ways. However, any reasonable proof would consist in two steps. First, we show that $H_*(\Omega^\infty E(1);\mathbb{Z}/p)$ is "generated by the image of" $H_*(X(0);\mathbb{Z}/p)$. To make sense of this notion, we will consider the composition $B\Sigma_p \longrightarrow X(0) \longrightarrow E(1)$ and the corresponding map of spaces $B\Sigma_p \longrightarrow \Omega^\infty E(1)$, which induces a map in homology

$$H_*(B\Sigma_p;\mathbb{Z}/p) \longrightarrow H_*(\Omega^\infty E(1);\mathbb{Z}/p).$$

As $H_*(\Omega^\infty E(1);\mathbb{Z}/p)$ is concentrated in even degrees and $H_{2i}(B\Sigma_p;\mathbb{Z}/p)$ vanishes unless i is a multiple of $p-1$, this map factors through $H_*(X(0);\mathbb{Z}/p)$, so it makes sense to talk about the image of $H_*(X(0);\mathbb{Z}/p)$ in $H_*(\Omega^\infty E(1);\mathbb{Z}/p)$. Second, we show that this image is contained in the image of $H_*(QS^0;\mathbb{Z}/p)$. By combining the two, we see that $H_*(QS^0;\mathbb{Z}/p)$ surjects to $H_*(\Omega^\infty E(1);\mathbb{Z}/p)$. The proof we will present here is neither the shortest or the easiest. However, it makes most use of transfers, which is one of the main subjects of this paper.

Proof. Consider the composition

$$\begin{aligned} &H_*(QB\mathbb{Z}/p;\mathbb{Z}/p) \longrightarrow H_*(Q\mathbb{C}P^\infty;\mathbb{Z}/p) \longrightarrow \\ &\qquad\qquad H_*(\Omega^\infty X(0);\mathbb{Z}/p) \longrightarrow H_*(\Omega^\infty E(1);\mathbb{Z}/p). \end{aligned}$$

The leftmost arrow is surjective because $B\mathbb{Z}/p \longrightarrow \mathbb{C}P^\infty$ induces a surjection in mod p homology. The middle arrow is also surjective since $X(0)$ is a retract of $\mathbb{C}P^\infty$. Finally, Proposition B.3 implies the surjectivity of the rightmost arrow. Thus the composition is surjective. However, we have the following commutative diagram.

$$\begin{tikzcd} B\mathbb{Z}/p \arrow[r] \arrow[d] & \mathbb{C}P^\infty \arrow[d] \\ B\Sigma_p \arrow[r] & X(0) \end{tikzcd}$$

Thus the surjection $H_*(QB\mathbb{Z}/p;\mathbb{Z}/p) \longrightarrow H_*(\Omega^\infty E(1);\mathbb{Z}/p)$ factors through another surjection

$$H_*(QB\Sigma_p) \longrightarrow H_*(\Omega^\infty E(1);\mathbb{Z}/p).$$

This is the first part. For readers familiar with the homology of infinite loop spaces, we remark that this implies that $H_*(\Omega_0^\infty E(1);\mathbb{Z}/p)$ is generated by the image of $H_*(X(0);\mathbb{Z}/p)$ as an algebra over Dyer-Lashof algebra. We also note that a much stronger result, that is the former is generated by the latter as an algebra, can be proved just by degree consideration combined with the fact that $H_*(BU)$ is generated as an algebra by $H_*(BU(1))$.

The proof will be complete once we achieve the second part by proving that the map $QB\Sigma_p \longrightarrow \Omega^\infty E(1)$ factors through the unit map $QS^0 \longrightarrow \Omega^\infty E(1)$. By the properties of adjoint functors, it suffices to prove that the composition $B\Sigma_p \longrightarrow QB\Sigma_p \longrightarrow \Omega^\infty E(1)$ factors through the unit map $QS^0 \longrightarrow \Omega^\infty E(1)$. Let's consider the following diagram of *spaces*.

$$\begin{tikzcd}
B\Sigma_p \arrow[r] \arrow[d, "t"'] & \Omega^\infty X(0) \arrow[r] \arrow[d] & \Omega^\infty E(1) \\
QB\mathbb{Z}/p \arrow[r] & Q\mathbb{C}P^\infty_{(p)} \arrow[r] & BU_{(p)} \arrow[u]
\end{tikzcd}$$

Here, the two left vertical arrows are the right inverse to the corresponding arrows in the previous diagram, in particular, t is the transfer associated to the inclusion $\mathbb{Z}/p \subset \Sigma_p$. We actually defined the transfer as a map from $B\Sigma_{p+}$ to $B\mathbb{Z}/p_+$, but of course, we can consider the composition

$$B\Sigma_p \longrightarrow B\Sigma_{p+} \longrightarrow B\mathbb{Z}/p_+ \longrightarrow B\mathbb{Z}/p.$$

The left square is easily seen to commute, the right one does by the construction of the map $\Omega^\infty X(0) \longrightarrow \Omega^\infty E(1)$ in [29, Theorem 3.2]. We also note the horizontal arrows in the bottom row are defined before the p-localisation. Thus it only remains to show that this composition from $B\Sigma_p$ to BU extends to the unit map $QS^0 \longrightarrow BU \times \mathbb{Z}$ (composed with the projection $BU \times \mathbb{Z} \longrightarrow BU$). For this it suffices to prove that it factors as $B\Sigma_p \longrightarrow BU(p) \longrightarrow BU$ where both maps are the standard ones. Now, consider the following diagram of map of *spaces*

$$\begin{tikzcd}
B\mathbb{Z}/p \arrow[r] & B\Sigma_p \arrow[r] \arrow[d, "\rho"'] & QB\mathbb{Z}/p \arrow[r] & Q\mathbb{C}P^\infty \arrow[d, "s"] \\
 & BU(p) \arrow[rr] & & BU
\end{tikzcd}$$

where ρ denotes the standard permutation representation of Σ_p, s is Segal's map. Now, it is well-known that the composition $B\mathbb{Z}/p \longrightarrow B\Sigma_{p(p)} \longrightarrow QB\mathbb{Z}/p$ is the adjoint to the self map of $B\mathbb{Z}/p$ given by $\Sigma_{i=1}^{p-1} B(i \times (-))$ e.g. [40, Example 2]. Note that [40, Example 2] is proved using Theorem 1 *loc.cit.* which in turn depends on Corollary 4 *loc.cit.*, where the equality only takes place after taking (co)homology. However, as the only proper subgroup of $\mathbb{Z}/p$ is the trivial group, the terms that appear in the double coset formula [40, Proposition 3 (4)] which don't contribute to the homology factors through the classifying space of the trivial group, so they are actually trivial. The decomposition of the regular representation of $\mathbb{Z}/p$ into one-dimensional representations allows us to write down the other composition as a similar sum, so the two compositions from $B\mathbb{Z}/p$ to BU agree. As $B\Sigma_p$ is a summand of $B\mathbb{Z}/p$, the square is commutative. In particular, the composition $B\Sigma_p \longrightarrow BU$ factors through the unit map $QS^0 \longrightarrow BU$, which concludes the proof.

For readers familiar to homology of infinite loop spaces, let's remark that the above implies that the image of fundamental classes of $B\Sigma_p$ in $H_*(\Omega^\infty E(1);\mathbb{Z}/p)$ agrees with $Q^i[1]$'s. This second step can also be proven using the fact that completed at p, the $E(1)$ cohomology of $B\Sigma_p$ is free of rank 1. Alternatively, noting that we only need to know an equivalent fact after passing to the homology, we can prove this step using [28, Theorem 3]. ■

To conclude the section, let's note that the splitting of $B\mathbb{Z}/p$ above is the simplest case of the splitting of $(B\mathbb{Z}/p)^n$ via the Steinberg idempotents, one of the main subjects of [25].

C Twisted Becker-Gottlieb transfer maps

We presented various transfer maps, among which the twisted Becker-Gottlieb transfer. It appears that this particular family of transfers is not quite well-documented in the literature, so we use this occasion to record some of its properties. The first example is the relationship between the Gysin homomorphisms (also known as the integration along the fibre) and the transfer. Let $F \longrightarrow E \xrightarrow{\pi} B$ and ζ be as in Subsection 2.1, the paragraph on the Boardman transfer. We keep the notations there. Then assuming that F is n-dimensional, ν_j is $(k-n)$-dimensional. suppose that π is R-orientable for a ring spectrum R, that is, if the vertical tangent bundle $T_\pi E$ is R-orientable in the usual sense. Then using the Thom isomorphism $Th : R^*E^{-T_\pi(E)} \longrightarrow R^{*+n}E_+$ and the Boardman transfer $t_\pi : B_+ \longrightarrow E^{-T_\pi(E)}$ we can define the Gysin homomorphism $\int_F : R^{*+n}E_+ \longrightarrow R^*B_+$ by the composition

$$R^{*+n}E_+ \xrightarrow{Th^{-1}} R^*E^{-T_\pi(E)} \xrightarrow{R^*(\bar{t}_\pi)} R^*B_+$$

(see also [2, Section 4]). Furthermore, if ζ is an R-orientable m-dimensional vector bundle, then we can twist the above to get $\int_F^\zeta$:

$$R^{*+n}E^{\pi^*\zeta} \xrightarrow{Th_1} R^{*+n-m}E_+ \xrightarrow{(Th_2)^{-1}} R^*E^{\pi^*\zeta - T_\pi(E)} \xrightarrow{R^*(\bar{t}_\pi^\zeta)} R^*B_+^\zeta$$

where Th_1 is the Thom isomorphism for $\pi^*\zeta$ and Th_2 the one for $\pi^*\zeta - T_\pi(E)$.

Let $e = e(T_\pi E) \in R^nE_+$ denote the Euler class of $T_\pi E$ and $e^\zeta \in R^{n+\dim\zeta}E^{\pi^*\zeta}$ its image under the Thom isomorphism $R^nE_+ \longrightarrow R^{n+\dim\zeta}E^\zeta$. We then have the following.

Theorem C.1. *1. ([4, Theorem 4.3]) For the Becker-Gottlieb transfer t_π we have $t_\pi^*(x) = \int_F(x \cup e)$.*

2. Suppose $\zeta \longrightarrow B$ is a vector bundle for which Thom isomorphism in R-homology holds. Then, for the twisted Becker-Gottlieb transfer $t_\pi^\zeta : B^\zeta \longrightarrow E^{\pi^*\zeta}$ we have

$$t_\pi^{\zeta\,*}(x) = \int_F^\zeta (x \cup e^\zeta)$$

This can be proved using the factorisation of the twisted Becker-Gottlieb transfer through the Boardman transfer, and noting that the inverse of the Thom isomorphism is essentially the multiplication by the Euler class. The details are left to the reader.

Example C.2. *Let G be a compact Lie group, $K \subset G$ a closed subgroup, V a (virtual) representation of G. Then, $BG^V_{(p)}$ splits off $BK^{V|_K}_{(p)}$ if $p \nmid \chi(G/K)$. In particular, if we denote by $N_G(T)$ the normaliser of a maximal torus T, one has $\chi(G/N_G(T)) = 1$ [4, Section 6]. Thus BG^V splits off $BN_G(T)^{V|_{N_G(T)}}$.*

Next, we record some multiplicative properties of twisted transfer maps, analogous to those of the usual Becker-Gottlieb transfer ([4, Sections 3,5]). They follow immediately from the construction.

1. Suppose $\pi_i : E_i \longrightarrow B_i$, $i = 1, 2$ are fibre bundles as above, with ζ_i (virtual) vector bundles over B_i. Suppose further that we have a map of fibre bundles given by the following commutative square

$$\begin{tikzcd} E_1 \arrow[r, "h_E"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"'] \\ B_1 \arrow[r, "h_B"] & B_2 \end{tikzcd}$$

so that the maps h_E and h_B are covered by bundle maps $\pi_1^*\zeta_1 \longrightarrow \pi_2^*\zeta_2$ and $\zeta_1 \longrightarrow \zeta_2$. This yields a commutative square as

$$\begin{tikzcd} B_1^{\zeta_1} \arrow[r, "h_B"] \arrow[d, "t^{\zeta_1}_{\pi_1}"'] & B_2^{\zeta_2} \arrow[d, "t^{\zeta_2}_{\pi_2}"'] \\ E_1^{\pi_1^*\zeta_1} \arrow[r, "h_E"] & E_2^{\pi_2^*\zeta_2} \end{tikzcd}$$

where we have retained h_E and h_B for the Thomified maps. This is analogous to [4, (3.2)].

2. Next, note that for a fibre bundle $\pi : E \longrightarrow B$ and a CW complex X admitting a filtration by finite subcomplexes (compact subspaces), we may consider the fibre bundle $\pi \times 1_X : E \times X \longrightarrow B \times X$, as well as the vector bundle $\zeta \times 0 \longrightarrow B \times X$. We then have

$$t^{\zeta\times 0}_{\pi\times 1_X} = t^{\zeta}_{\pi} \wedge 1 : B^{\zeta} \wedge X_+ \longrightarrow E^{\pi^*\zeta} \wedge X_+.$$

This generalises to $t^{\zeta_1\times\zeta_2}_{\pi_1\times\pi_2} = t^{\zeta_1}_{\pi_1} \wedge t^{\zeta_2}_{\pi_2}$ as [10, (2.2)], but we only use the special case of $t^{\zeta\times 0}_{\pi\times 1_X}$.

3. Finally, for the trivial bundle $\pi : F \longrightarrow \{0\}$, identifying $\{0\}_+ = S^0$, the composition $\pi \circ t_\pi : S^0 \longrightarrow S^0$ has degree $\chi(F)$ [4, (3.4)]. Note that this is just Hopf's vector field theorem [4, Theorem 2.4]

As an application, properties (1) and (2) can be used to prove a multiplicative formula for the (co)homology of twisted transfer maps. We have the following.

Lemma C.3. *Suppose $\pi : E \longrightarrow B$ and $\zeta \longrightarrow B$ are as above. For $x \in H^*B^\zeta$ and $y \in H^*E$, we have*

$$t_\pi^{\zeta\,*}(Th^\zeta(\pi)^*(x) \cup y) = x \cup t_\pi{}^*(y).$$

Here $\cup$ on the left is a 'generalised' cup product

$$H^*(E^{\pi^*\zeta}) \otimes H^*(E) \longrightarrow H^*(E^{\pi^*\zeta})$$

whereas the $\cup$ on the right is the usual cup product on H^*B^ζ induced by the usual diagonal.

Proof. For a fibre bundle $\pi : E \longrightarrow B$, and a twisting vector bundle $\zeta \longrightarrow B$, consider $1_B \times \pi : B \times E \longrightarrow B \times B$. Note that the diagonal map $d_B : B \longrightarrow B \times B$ and $(\pi \times 1_E) \circ d_E$ induce a map of fibre bundles

$$\begin{tikzcd}
E \arrow[r, "d_E"] \arrow[d, "\pi"'] & E \times E \arrow[r, "\pi\times 1_E"] & B \times E \arrow[d, "1_B\times\pi"] \\
B \arrow[rr, "d_B"] & & B \times B
\end{tikzcd}$$

covered by bundle maps

$$\begin{tikzcd}
\pi^*\zeta \arrow[r] \arrow[d] & \pi^*\zeta \times 0 \arrow[r] & \zeta \times 0. \arrow[d] \\
\zeta \arrow[rr] & & \zeta \times 0.
\end{tikzcd}$$

Thus by applying properties (1) and (2) we get a commutative diagram.

$$\begin{tikzcd}
B^\zeta \arrow[rr, "d_B"] \arrow[d, "t^\zeta_\pi"'] & & B^\zeta \wedge B_+ \arrow[d, "1_{B^\zeta}\wedge t_\pi = t^{\zeta\times 0}_{1_B\times\pi}"] \\
E^{\pi^*\zeta} \arrow[r, "d_E"] & E^{\pi^*\zeta} \wedge E_+ \arrow[r, "Th^\zeta(\pi)\wedge 1_E"] & B^\zeta \wedge E_+
\end{tikzcd}$$

The lemma follows by comparing the images of $x \otimes y \in H^*(B^\zeta \wedge E_+)$ by the two compositions of induced maps, noting that d_B and d_E are the Thom diagonals (Subsection 2.1). ∎

As an application of the above, we show the following generalisation of Theorem 2.1.

Theorem C.4. *(Theorem 1.1 (i)) The composition*

$$B^{\zeta} \xrightarrow{t_{\pi}^{\zeta}} E^{\pi^*\zeta} \xrightarrow{Th^{\zeta}(\pi)} B^{\zeta}$$

induces multiplication by $\chi(F)$ in $H^*(-;\Lambda)$ for any Abelian group Λ where $Th^{\zeta}(\pi)$ denotes the induced map among Thom spectra. Consequently, if $\chi(F)$ is prime to p, B^{ζ} splits off $E^{\pi^*\zeta}$.

Proof. By Lemma C.3

$$t_{\pi}^{\zeta\,*}(Th^{\zeta}(\pi)^*(x) \cup y) = x \cup t_{\pi}^*(y)$$

for all $x \in H^*B^{\zeta}$ and $y \in H^*E$ where $\cup$ on the left side of the equation denotes the generalised cup product $H^*(E^{\pi^*\zeta}) \otimes H^*(E) \longrightarrow H^*(E^{\pi^*\zeta})$ (c.f. Subsection 2.1). The rest follows by setting $y = 1$, and noting that $t_{\pi}^*(1) = \chi(F)$ as in the proof of [4, Theorem 5.5]. ∎

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References

[1] J. F. Adams. Lectures on generalised cohomology. In *Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three)*, pages 1–138. Springer, Berlin, 1969.

[2] Mathew Ando, Andrew J. Blumberg, and David Gepner. Parametrized spectra, multiplicative thom spectra, and the twisted umkehr map. arXiv:1112.2203.

[3] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3 (suppl. 1):3–38, 1964.

[4] J. C. Becker and D. H. Gottlieb. The transfer map and fiber bundles. *Topology*, 14:1–12, 1975.

[5] James C. Becker and Reinhard E. Schultz. The real semicharacteristic of a fibered manifold. *Quart. J. Math. Oxford Ser. (2)*, 33(132):385–403, 1982.

[6] J.C. Becker and R.E. Schultz. Equivariant function spaces and stable homotopy theory. I. *Comment. Math. Helv.*, 49:1–34, 1974.

[7] J. Michael Boardman. Stable homotopy theory (mimeographed). 1966.

[8] A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces. I. *Amer. J. Math.*, 80:458–538, 1958.

[9] Armand Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953.

[10] G. Brumfiel and Ib Madsen. Evaluation of transfer and universal classes. *Inv. Math.*, 32:133-170, 1976.

[11] Yvonne Choquet-Bruhat and Cécile DeWitt-Morette. *Analysis, manifolds and physics. Part II*. North-Holland Publishing Co., Amsterdam, 1989. 92 applications.

[12] Ralph L. Cohen and John R. Klein. Umkehr maps. *Homology Homotopy Appl.*, 11(1):17–33, 2009.

[13] Adrian P. Dobson. KO-theory of Thom complexes. PhD thesis, University of Manchester, 2007.

[14] Johannes Ebert. A vanishing theorem for characteristic classes of odd-dimensional manifold bundles. *J. Reine Angew. Math.*, 684:1–29, 2013.

[15] Johannes Ebert and Oscar Randal-Williams. On the divisibility of characteristic classes of non-oriented surface bundles. *Topology Appl.*, 156(2):246–250, 2008.

[16] Johannes Ebert and Oscar Randal-Williams. Generalised Miller-Morita-Mumford classes for block bundles and topological bundles. *Algebr. Geom. Topol.*, 14(2):1181–1204, 2014.

[17] Søren Galatius. Mod p homology of the stable mapping class group. *Topology*, 43(5):1105 – 1132, 2004.

[18] Søren Galatius. Mod 2 homology of the stable spin mapping class group. *Math. Ann.*, 334(2):439–455, 2006.

[19] Søren Galatius, Ib Madsen, Ulrike Tillmann, and Michael Weiss. The homotopy type of the cobordism category. *Acta Math.*, 202(2):195–239, 2009.

[20] M. J. Hopkins and J. R. Hunton. On the structure of spaces representing a Landweber exact cohomology theory. *Topology*, 34(1):29–36, 1995.

[21] John Hunton and Nigel Ray. A rational approach to Hopf rings. *J. Pure Appl. Algebra*, 101(3):313–333, 1995.

[22] Dale Husemoller. *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1994.

[23] I. James. *Topology of Stiefel manifolds, Lecture Notes Series*, volume 76. London Mathematical Society, 1976.

[24] Daniel S. Kahn and Stewart B. Priddy. Applications of the transfer to stable homotopy theory. *Bull. Am. Math. Soc.*, 78:981–987, 1972.

[25] Takuji Kashiwabara and Hadi Zare. Splitting Madsen-Tillmann spectra II. the Steinberg idempotents and Whitehead conjecture. In preparation.

[26] R. C. Kirby and L. R. Taylor. A calculation of Pin^+ bordism groups. *Comment. Math. Helv.*, 65(3):434–447, 1990.

[27] R. C. Kirby and L. R. Taylor. Pin structures on low-dimensional manifolds. In *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, volume 151 of *London Math. Soc. Lecture Note Ser.*, pages 177–242. Cambridge Univ. Press, Cambridge, 1990.

[28] Stanley O. Kochman. The homology of the classical groups over the dyer-lashof algebra. *Bull. Amer. Math. Soc.*, 77(1):142–147, 01 1971.

[29] Akira Kono. On the mod p decomposition of $Q(\mathbf{C}P^\infty)$. *Publ. Res. Inst. Math. Sci.*, 20(4):893–896, 1984.

[30] Akira Kono. On the integral cohomology of $B\mathrm{Spin}(n)$. *J. Math. Kyoto Univ.*, 26(3):333–337, 1986.

[31] T. Y. Lam. *The algebraic theory of quadratic forms*. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1980. Revised second printing, Mathematics Lecture Note Series.

[32] Saunders MacLane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[33] Benjamin M. Mann, Edward Y. Miller, and Haynes R. Miller. S^1-equivariant function spaces and characteristic classes. *Trans. Amer. Math. Soc.*, 295(1):233–256, 1986.

[34] J. P. May. *Homology of E^∞-Spaces, The*, volume 533 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[35] Haynes Miller. Universal Bernoulli numbers and the S^1-transfer. In *Current trends in algebraic topology, Part 2 (London, Ont., 1981)*, volume 2 of *CMS Conf. Proc.*, pages 437–449. Amer. Math. Soc., Providence, RI, 1982.

[36] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[37] Mamoru Mimura, Goro Nishida, and Hirosi Toda. Localization of CW-complexes and its applications. *J. Math. Soc. Japan*, 23(4):593–624, 1971.

[38] Kaoru Morisugi. Graph constructions and transfer maps. *Bull. Fac. Ed. Wakayama Univ. Natur. Sci.*, (59):1–9, 2009.

[39] Stewart Priddy. Dyer-Lashof operations for the classifying spaces of certain matrix groups. *Q. J. Math., Oxf. II. Ser.*, 26:179–193, 1975.

[40] Stewart Priddy. Lectures on the stable homotopy of BG. In *Proceedings of the School and Conference in Algebraic Topology*, volume 11 of *Geom. Topol. Monogr.*, pages 289–308. Geom. Topol. Publ., Coventry, 2007.

[41] Oscar Randal-Williams. The homology of the stable nonorientable mapping class group. *Algebraic and Geometric Topology*, (8):1811–1832, 2008.

[42] Oscar Randal-Williams. Homology of the moduli spaces and mapping class groups of framed, r-Spin and Pin surfaces. *J. Topol.*, 7(1):155–186, 2014.

[43] Yuli B. Rudyak. *On Thom spectra, orientability, and cobordism.* Berlin: Springer, 1998.

[44] Graeme Segal. The stable homotopy of complex projective space. *Quart. J. Math. Oxford Ser. (2)*, 24:1–5, 1973.

[45] Emery Thomas. On the cohomology of the real Grassmann complexes and the characteristic classes of n-plane bundles. *Trans. Amer. Math. Soc.*, 96:67–89, 1960.

[46] Nathalie Wahl. Homological stability for the mapping class groups of non-orientable surfaces. *Invent. Math.*, 171(2):389–424, 2008.

[47] Robert J. Wellington. The unstable Adams spectral sequence for free iterated loop spaces. *Mem. Amer. Math. Soc.*, 36(258):viii+225, 1982.

[48] George W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978.

[49] Dung Yung Yan. Stable splittings of $BO(2n)$ and $BU(2n)$. *Proc. Amer. Math. Soc.*, 124(6):1913–1915, 1996.

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