On Weierstrass' monsters in the disc algebra

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To the loving memory of our friend and colleague Joe Diestel (1943–2017).

Abstract

Let Ω be a Jordan domain in the complex plane whose boundary is piecewise analytic, and let $A(\Omega)$ be the algebra of all holomorphic functions on Ω that are continuous up to the boundary. We prove the existence of dense linear subspaces and of infinitely generated subalgebras in $A(\Omega)$ all of whose nonzero members are, in a strong sense, not differentiable at almost any point of the boundary. We also obtain infinite-dimensional closed subspaces consisting of functions that are not differentiable at any point of a dense subset of the boundary. In the case of the unit disc, those dense linear subspaces can be found with their functions being nowhere differentiable in the unit circle.

1 Introduction, Notation and Preliminaries

In 1872 K. Weierstrass [44] exhibited an example of a function $f : \mathbb{R} \to \mathbb{R}$ that was continuous everywhere but differentiable nowhere. The particular example was defined as

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$, b is any odd integer and $ab > 1 + 3\pi/2$.

Functions with the property described above are called *Weierstrass monsters*. As a nice application of the Baire category theorem, S. Banach [7] and independently S. Mazurkiewicz [37] obtained in 1931 that the set of continuous functions that are nowhere differentiable is residual –that is, its complement is of first

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category– in the space of continuous functions on $\mathbb{R}$ endowed with the topology of uniform convergence in compacta (see also [38, Chapter 11] and [43]). It is our aim in this paper to shed light on the structure of the family of Weierstrass monsters, not only in the topological sense, but also in the algebraic sense. Moreover, we focus our attention on those periodic functions that can be extended holomorphically on the unit disc or, more generally, on a given domain of the complex plane. However, before going on with the history of findings and with our specific goals, let us fix some notation, mostly standard.

As usual, the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{T}$ will denote the set of positive integers, the set $\mathbb{N} \cup \{0\}$, the set of all integers, the field of rational numbers, the real line, the complex plane, the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, respectively. The symbol $\mathfrak{c}$ will represent the cardinality of the continuum. The symbol $C[0,1]$ will stand for the Banach space of all $\mathbb{R}$-valued continuous functions defined on the interval $[0,1]$, endowed with the maximum norm. If $\Omega \subset \mathbb{C}$, then $\overline{\Omega}$ and $\partial\Omega$ will stand for the closure and the boundary of Ω in $\mathbb{C}$, respectively. If Ω is an open subset of $\mathbb{C}$, $A(\Omega)$ will denote the vector space of all functions from $\overline{\Omega}$ into $\mathbb{C}$ which are continuous on $\overline{\Omega}$ and holomorphic in Ω. If Ω is bounded, $A(\Omega)$ is endowed with the topology of uniform convergence on $\overline{\Omega}$ and then $A(\Omega)$ is a Banach space. When Ω is not bounded, $A(\Omega)$ is endowed with the topology of uniform convergence on the compact subsets of $\overline{\Omega}$ and thus $A(\Omega)$ is an F-space, that is, a complete metrizable topological vector space.

By a domain we mean a nonempty connected open subset of $\mathbb{C}$. A domain $\Omega \subset \mathbb{C}$ is said to be simply connected whenever $\mathbb{C}_\infty \setminus \Omega$ is connected, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ stands for the extended complex plane. The symbols $\overline{\Omega}^\infty$ and $\partial_\infty\Omega$ will represent, respectively, the closure and the boundary of Ω in $\mathbb{C}_\infty$; that is, $\overline{\Omega}^\infty = \overline{\Omega}$ and $\partial_\infty\Omega = \partial\Omega$ if Ω is bounded, while $\overline{\Omega}^\infty = \overline{\Omega} \cup \{\infty\}$ and $\partial_\infty\Omega = \partial\Omega \cup \{\infty\}$ if the subset Ω is unbounded. A Jordan domain is a domain $\Omega \subset \mathbb{C}$ such that $\partial_\infty\Omega$ is a homeomorphic image in $\mathbb{C}_\infty$ of $\mathbb{T}$. Note that we allow unbounded domains here; for instance, an open half-plane is Jordan.

Some additional terminology, borrowed from the new theory of lineability, will be used. The corresponding concepts were coined in [2, 4, 5, 9, 14, 25, 28, 42]. For an account of results on lineability, the reader is referred to the survey [18] and the monograph [1]. Assume that X is a vector space and α is a cardinal number. Then a subset $A \subset X$ is said to be:

- *lineable* if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$.
- *α-lineable* if there exists a vector space M with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$.
- *maximal lineable* in X if A is $\dim(X)$-lineable.

If, in addition, X is a topological vector space, then the subset A is said to be:

- *spaceable* in X whenever there is a closed infinite-dimensional vector subspace M of X such that $M \setminus \{0\} \subset A$.

- *dense-lineable* in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$.

- *α-dense-lineable* in X whenever there is a dense vector subspace M of X with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$.

- *maximal dense-lineable* in X if A is $\dim(X)$-dense-lineable.

And, provided that X is a vector space contained in some (linear) algebra, then A is called:

- *algebrable* if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.

- *strongly α-algebrable* if there exists an α-generated *free* algebra M with $M \setminus \{0\} \subset A$.

We recall that if X is contained in a commutative algebra, then a set $B \subset X$ is a generating set of some free algebra contained in A if and only if for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1, \ldots, f_N \in B$, we have $P(f_1, \ldots, f_N) \neq 0$ and $P(f_1, \ldots, f_N) \in A$.

A number of implications are obvious. For instance: lineability means $\aleph_0$-lineability, where $\aleph_0 = \mathrm{card}(\mathbb{N})$, the cardinality of $\mathbb{N}$; dense-lineability implies lineability as soon as $\dim(X) = \infty$; spaceability implies lineability; strong α-algebrability implies algebrability if α is infinite. Under the continuum hypothesis (which is assumed along this paper), we obtain as a consequence of Baire's category theorem that maximal (dense) lineability equals $\mathfrak{c}$-(dense, resp.)-lineability if X a separable infinite-dimensional F-space.

Turning to our main concern, much progress have been done in the search for algebraic structures inside the class of nowhere differentiable functions. Gurariy [27] proved in 1991 the lineability of the set $ND[0,1]$ of continuous nowhere differentiable functions on $[0,1]$. In fact, Fonf, Gurariy and Kadets [23] showed that $ND[0,1]$ is spaceable in $C[0,1]$. Even more, Rodríguez-Piazza [39] proved that every separable infinite-dimensional Banach space is isometrically isomorphic to a space of continuous functions in $[0,1]$ that are, except for the null function, nowhere differentiable in $[0,1]$ (this result was generalized by Hencl [29] to nowhere hölderian functions). It turns out that the family $ND[0,1]$ is also dense-lineable in $C[0,1]$ (see [3, 13]). In fact, Bayart and Quarta [12] established the algebrability of the mentioned family (and even algebrability for the smaller class of nowhere hölderian functions), with the additional property that the existent algebra is dense in $C[0,1]$.

In all previous results, starting from the Weierstrass example, there is no problem in assuming that the functions are periodic, so we can in fact consider continuous functions on $\mathbb{T}$. In a set of recent papers, Eskenazis and Makridis [20, 21] took a step forwards on this research and studied the residuality of the family of those $F \in A(\mathbb{D})$ such that $F|_{\mathbb{T}}$ is nowhere differentiable on $\mathbb{T}$. It is well known that not every continuous function $f : \mathbb{T} \to \mathbb{K}$ can be extended to a function $F \in A(\mathbb{D})$ such that $F|_{\mathbb{T}} = f$. Such extension is possible if and only if $\int_0^{2\pi} e^{in\theta} f(e^{i\theta})\, d\theta = 0$ for all $n \in \mathbb{N}$ (in other words, the Fourier coefficients of f of negative order are all zero). Specifically, in [20, 21] it is proved that the set of $f \in A(\mathbb{D})$ such that $f|_{\mathbb{T}}$ is not differentiable at any point of $\mathbb{T}$ is residual in $A(\mathbb{D})$. In fact, it is shown that, generically, both $\Re f|_{\mathbb{T}}$ and $\Im f|_{\mathbb{T}}$ are not differentiable at any $z \in \mathbb{T}$. Analogous results for nowhere hölderian continuous functions on $\mathbb{T}$ and for $A(\mathbb{D}^I)$, where I is countable, are also obtained in [21]. In [35], these results are extended to other domains $\Omega \subset \mathbb{C}$ and their corresponding spaces $A(\Omega)$ (or spaces X related to these) so as to prove that, for a given closed subset $J \subset \partial\Omega$ without isolated points, the family

$$\Big\{ f \in X : \limsup_{\substack{z \to z_0 \\ z \in J \setminus \{z_0\}}} \Big| \frac{f(z) - f(z_0)}{z - z_0} \Big| = +\infty \text{ for every } z_0 \in J \Big\}$$

is either empty or residual in X. Recall that, in fact, in both Banach's [7] and Mazurkiewicz's [37] papers, it is obtained generically the everywhere unboundedness of incremental quotients $|\frac{f(x)-f(x_0)}{x-x_0}|$, that is, continuous functions on $[0, 1]$ are generically nowhere Lipschitz.

Our main aim in this paper is to pick up the baton of this study and to lead it to the setting of lineability in the framework of the algebra $A(\Omega)$. To be more precise, we will establish, for a Jordan domain Ω having smooth boundary, the existence of dense vector subspaces and of large free algebras in $A(\Omega)$ consisting, except for zero, of functions that are not differentiable at almost (in several senses) every boundary point. We even obtain unboundedness of incremental quotients at such points. In Section 3 these results will be stated for $\Omega = \mathbb{D}$, in which case we obtain nowhere differentiability in the unit circle for the existing dense vector subspace. In Section 4, our findings will be translated to the general case. The short Section 2 is devoted to furnish the appropriate approximation and lineability tools. As an appendix, genericity of this kind of functions in a measure theoretic sense is also considered.

2 Some auxiliary results

The following approximation theorem, due to Mergelyan, can be found in [40, Theorem 20.5] (see also [24, Chapter III]) and plays an important role in the sequel.

Theorem 2.1. *Let K be a compact set in $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected. If $f : K \to \mathbb{C}$ is a continuous function which is holomorphic in the interior of K and $\varepsilon > 0$, then there is a polynomial P such that $|f(z) - P(z)| < \varepsilon$ for all $z \in K$.*

The proof of the following auxiliary assertion, which has its roots in an earlier version by Aron *et al.* [3, Theorem 2.2 and Remark 2.5] and is based on the notion of "stronger than" coined by them, can be found in [17, Theorem 2.3] (see also [1, Section 7.3]).

Theorem 2.2. *Assume that X is a metrizable separable topological vector space. Suppose that A and B are subsets of X such that $A + B \subset A$, $A \cap B = \varnothing$ and B is dense-lineable. We have:*

(a) *If α is an infinite cardinal number such that A is α-lineable, then A is α-dense-lineable in X.*

(b) *In particular, if A is $\mathfrak{c}$-lineable, then A is maximal-dense-lineable in X.*

Note that since the space X is metrizable and separable, its cardinality is $\mathrm{card}(X) = \mathfrak{c}$ and hence $\dim(X) \leq \mathfrak{c}$. Therefore, if A is $\mathfrak{c}$-lineable, then in fact $\dim(X) = \mathfrak{c}$ and it follows that A is maximal-lineable.

A strong method to generate (or discover) algebras of strange functions from $[0,1]$ into $\mathbb{R}$ is the one developed by Balcerzak *et al.* in [6, Proposition 7] (see also [8, Theorem 1.5 and Section 6] and [10]). This method can be extended to complex-valued functions (see [15]), so as to yield the assertion contained in the next lemma. By $\mathcal{E}$ we denote the family of exponential-like functions on $\mathbb{C}$, that is, the functions of the form

$$\varphi(z) = \sum_{j=1}^{m} a_j e^{b_j z}$$

for some $m \in \mathbb{N}$, some $a_1, \ldots, a_m \in \mathbb{C} \setminus \{0\}$ and some distinct $b_1, \ldots, b_m \in \mathbb{C} \setminus \{0\}$.

Lemma 2.3. *Let Ω be a nonempty set and let $\mathcal{F}$ be a family of functions from Ω into $\mathbb{C}$. Assume that there exists a function $f : \Omega \to \mathbb{C}$ such that $f(\Omega)$ is uncountable and $\varphi \circ f \in \mathcal{F}$ for every $\varphi \in \mathcal{E}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More precisely, if $H \subset (0, +\infty)$ is a set with $\mathrm{card}(H) = \mathfrak{c}$ and linearly independent over the field $\mathbb{Q}$, then*

$$\{\exp \circ (rf) : r \in H\}$$

is a free system of generators of an algebra contained in $\mathcal{F} \cup \{0\}$.

In the proof of Theorem 4.1, we will need the following result about Fourier series in the disc algebra whose proof can be found in [16].

Theorem 2.4. *For each $g(z) = \sum_{n=0}^{\infty} a_n z^n \in A(\mathbb{D})$ and each $\theta \in \mathbb{R}$, let*

$$S(g|_{\mathbb{T}}, n)(e^{i\theta}) := \sum_{k=-n}^{n} a_n e^{in\theta}.$$

If E is a countable subset of $\mathbb{T}$ and the space $\mathbb{C}^E$ of all $\mathbb{C}$-valued functions on E is endowed with the topology of pointwise convergence, then the family

$$\mathcal{F}_E = \left\{ g \in A(\mathbb{D}) : \{S(g|_{\mathbb{T}}, n)\}_{n \geq 1} \text{ is dense in } \mathbb{C}^E \right\}$$

is spaceable in $A(\mathbb{D})$.

Remark 2.5. The residuality in $A(\mathbb{D})$ of the family $\mathcal{F}_E$ was proved for each countable set $E \subset \mathbb{T}$ by Herzog and Kunstmann [31].

3 Weierstrass' monsters in the disc algebra

In the disc algebra $A(\mathbb{D})$, we find a rich algebraic structure inside the set

$$ND(\mathbb{T}) := \{f \in A(\mathbb{D}) : f|_{\mathbb{T}} \text{ is not differentiable at any point of } \mathbb{T}\}.$$

Let $NL(\mathbb{T})$ denote the (smaller) class of functions f in the disc algebra such that $f|_{\mathbb{T}}$ is nowhere Lipschitz, that is,

$$NL(\mathbb{T}) := \left\{ f \in A(\mathbb{D}) : \limsup_{z \in \mathbb{T}, z \to z_0} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| = +\infty \text{ for every } z_0 \in \mathbb{T} \right\}.$$

Theorem 3.1. *The set $NL(\mathbb{T})$ is $\mathfrak{c}$-lineable. Hence $ND(\mathbb{T})$ is also $\mathfrak{c}$-lineable in $A(\mathbb{D})$.*

Proof. Some of our arguments are based on [36] and [43, Theorem 3.4]. For each $a \in (0,1)$ let

$$f_a(z) := \sum_{n=0}^{\infty} a^n z^{9^n}.$$

From the convergence of the geometrical series $\sum_{n=0}^{\infty} a^n$ and the Weierstrass M-test for uniform convergence one derives that $f_a \in A(\mathbb{D})$. Define

$$V := \operatorname{span} \left\{ f_a : \frac{7}{9} < a < \frac{8}{9} \right\}.$$

Then, of course, V is a vector subspace of $A(\mathbb{D})$. First of all, we will prove that the set $\left\{ f_a : \frac{7}{9} < a < \frac{8}{9} \right\}$ is linearly independent. Let us suppose that

$$\lambda_1 f_{a_1} + \lambda_2 f_{a_2} + \cdots + \lambda_k f_{a_k} = 0$$

for some $k \in \mathbb{N}$, $\frac{7}{9} < a_k < \cdots < a_2 < a_1 < \frac{8}{9}$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. Then

$$\sum_{n=0}^{\infty} (\lambda_1 a_1^n + \lambda_2 a_2^n + \cdots + \lambda_k a_k^n)\, z^{9^n} = 0$$

for every $z \in \overline{\mathbb{D}}$, so the uniqueness of coefficients of the Taylor series yields

$$\lambda_1 a_1^n + \lambda_2 a_2^n + \cdots + \lambda_k a_k^n = 0$$

for every $n \in \mathbb{N}_0$. For $n = 0, \ldots, k-1$, we obtain the following conditions:

$$\left\{ \begin{array}{c} \lambda_1 + \lambda_2 + \cdots + \lambda_k = 0 \\ \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k = 0 \\ \vdots \\ \lambda_1 a_1^{k-1} + \lambda_2 a_2^{k-1} + \cdots + \lambda_k a_k^{k-1} = 0. \end{array} \right.$$

These equations are equivalent to the following one:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_k \\ & & & \\ a_1^{k-1} & a_2^{k-1} & \cdots & a_k^{k-1} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The previous matrix is a Vandermonde matrix, so its determinant is not 0 because $a_1, \dots, a_k$ are different and non-zero. Therefore, $\lambda_1 = 0$, $\lambda_2 = 0, \dots, \lambda_k = 0$. This proves that the set $\left\{ f_a : \frac{7}{9} < a < \frac{8}{9} \right\}$ is linearly independent and thus $\dim (V) = \mathfrak{c}$.

We will prove that every function $f \in V \setminus \{0\}$ belongs to $NL(\mathbb{T})$, that is, for every $z_0 \in \mathbb{T}$ it satisfies

$$\limsup_{z \in \mathbb{T}, z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty. \tag{3.1}$$

If $f \in V \setminus \{0\}$, then there are $k \in \mathbb{N}$, $\frac{7}{9} < a_k < \cdots < a_1 < \frac{8}{9}$ and $\lambda_1, \dots, \lambda_k \in \mathbb{C} \setminus \{0\}$ such that

$$f = \lambda_1 f_{a_1} + \cdots + \lambda_k f_{a_k}.$$

We can assume that $\lambda_1 = 1$. If that were not the case, then we would prove the property (3.1) for the function

$$g = f_{a_1} + \frac{\lambda_2}{\lambda_1} f_{a_{k-1}} + \cdots + \frac{\lambda_k}{\lambda_1} f_{a_k}$$

and then $f = \lambda_1 g$ would also satisfy (3.1).

Let z_0 be any fixed point in $\mathbb{T}$ and let $x_0 \in \mathbb{R}$ such that $z_0 = e^{i\pi x_0}$. For each $m \in \mathbb{N}$ there is $\alpha_m \in \mathbb{Z}$ such that $9^m x_0 - \alpha_m \in (-1/2, 1/2]$. We define

$$t_m := 9^m x_0 - \alpha_m \in \left(\frac{-1}{2}, \frac{1}{2} \right], \qquad x_m := \frac{\alpha_m - 1}{9^m}.$$

Then

$$x_m - x_0 = \frac{\alpha_m - 1 - 9^m x_0}{9^m} = -\frac{1 + t_m}{9^m} \underset{m \to \infty}{\longrightarrow} 0.$$

That is, $\lim_{m \to \infty} x_m = x_0$, so $\lim_{m \to \infty} e^{i\pi x_m} = z_0$.

For each $j \in \{1, \dots, k\}$ we consider the function $u_j : \mathbb{R} \to \mathbb{R}$ defined as

$$u_j(x) = \Re \left[f_{a_j} \left(e^{i\pi x} \right) \right] = \sum_{n=0}^{\infty} a_j^n \cos(9^n \pi x).$$

Then

$$\frac{u_j(x_m) - u_j(x_0)}{x_m - x_0} = \sum_{n=0}^{\infty} a_j^n \frac{\cos(9^n \pi x_m) - \cos(9^n \pi x_0)}{x_m - x_0} = S_{j,m} + T_{j,m},$$

where

$$S_{j,m} := \sum_{n=0}^{m-1} a_j^n \frac{\cos(9^n \pi x_m) - \cos(9^n \pi x_0)}{x_m - x_0}$$

and

$$T_{j,m} := \sum_{n=0}^{\infty} a_j^{m+n} \frac{\cos(9^{m+n} \pi x_m) - \cos(9^{m+n} \pi x_0)}{x_m - x_0}.$$

We will use the following relationship, which holds for every $x, y \in \mathbb{R}$:

$$\cos(x+y) - \cos(x-y) = -2\sin x \sin y.$$

Then we have

$$\begin{aligned} \left|S_{j,m}\right| &= \left| \sum_{n=0}^{m-1} a_j^n \frac{-2\sin\left(\frac{9^n\pi x_m + 9^n \pi x_0}{2}\right)\sin\left(\frac{9^n\pi x_m - 9^n\pi x_0}{2}\right)}{x_m - x_0} \right| \\ &\leq \sum_{n=0}^{m-1} \pi \left(9a_j\right)^n \left| \frac{\sin\left(9^n\pi \frac{x_m - x_0}{2}\right)}{9^n \pi \frac{x_m - x_0}{2}} \right|. \end{aligned}$$

Since $\left|\frac{\sin x}{x}\right| \leq 1$ for every $x \in \mathbb{R}$ and $a_j > \frac{7}{9}$, we obtain

$$\left|S_{j,m}\right| \leq \sum_{n=0}^{m-1} \pi \left(9a_j\right)^n = \pi \frac{\left(9a_j\right)^m - 1}{9a_j - 1} \leq \frac{\pi}{6}\left(9a_j\right)^m. \tag{3.2}$$

Now, we study the series $T_{j,m}$. On the one hand, since $\alpha_m \in \mathbb{Z}$, we have

$$\begin{aligned} \cos\left(9^{m+n}\pi x_m\right) &= \cos\left(9^{m+n}\pi\frac{\alpha_m - 1}{9^m}\right) = \cos\left(9^n\left(\alpha_m - 1\right)\pi\right) \\ &= (-1)^{\alpha_m - 1} = -\left(-1\right)^{\alpha_m}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \cos\left(9^{m+n}\pi x_0\right) &= \cos\left(9^{m+n}\pi\frac{\alpha_m + t_m}{9^m}\right) = \cos\left(9^n\pi\alpha_m + 9^n\pi t_m\right) \\ &= \cos\left(9^n\pi\alpha_m\right)\cos\left(9^n\pi t_m\right) - \sin\left(9^n\pi\alpha_m\right)\sin\left(9^n\pi t_m\right) \\ &= (-1)^{\alpha_m}\cos\left(9^n\pi t_m\right). \end{aligned}$$

Therefore,

$$\begin{aligned} T_{j,m} &= \sum_{n=0}^{\infty} a_j^{m+n} \frac{-\left(-1\right)^{\alpha_m} - \left(-1\right)^{\alpha_m}\cos\left(9^n\pi t_m\right)}{-\frac{1+t_m}{9^m}} \\ &= (-1)^{\alpha_m}\left(9a_j\right)^m \sum_{n=0}^{\infty} a_j^n \frac{1+\cos\left(9^n\pi t_m\right)}{1+t_m}. \end{aligned}$$

Now, note that

$$a_j^n \frac{1+\cos\left(9^n\pi t_m\right)}{1+t_m} \geq 0$$

for every n, m, j and that $\cos\left(\pi t_m\right) \geq 0$ because $t_m \in (-1/2, 1/2]$, so

$$\begin{aligned} \left|T_{j,m}\right| &= \left(9a_j\right)^m \sum_{n=0}^{\infty} a_j^n \frac{1+\cos\left(9^n\pi t_m\right)}{1+t_m} \geq \left(9a_j\right)^m \frac{1+\cos\left(\pi t_m\right)}{1+t_m} \\ &\geq \left(9a_j\right)^m \frac{1}{1+\frac{1}{2}} = \frac{2}{3}\left(9a_j\right)^m. \end{aligned}$$

Moreover, since $a_j < 8/9$, we have

$$\begin{aligned}\left|T_{j,m}\right| &= \left(9a_j\right)^m \sum_{n=0}^{\infty} a_j^n \frac{1+\cos\left(9^n \pi t_m\right)}{1+t_m} \\ &\leq \left(9a_j\right)^m \sum_{n=0}^{\infty} \left(\frac{8}{9}\right)^n \frac{2}{1-\frac{1}{2}} = 36\left(9a_j\right)^m .\end{aligned}$$

Consequently, we are led to

$$\frac{2}{3}\left(9a_j\right)^m \leq \left|T_{j,m}\right| \leq 36\left(9a_j\right)^m . \tag{3.3}$$

Therefore, by (3.2) and (3.3),

$$\left|\frac{u_j\left(x_m\right)-u_j\left(x_0\right)}{x_m-x_0}\right| = \left|S_{j,m}+T_{j,m}\right| \leq \frac{\pi}{6}\left(9a_j\right)^m + 36\left(9a_j\right)^m < 37\left(9a_j\right)^m$$

and

$$\begin{aligned}\left|\frac{u_j\left(x_m\right)-u_j\left(x_0\right)}{x_m-x_0}\right| &= \left|S_{j,m}+T_{j,m}\right| \geq \left|T_{j,m}\right| - \left|S_{j,m}\right| \\ &\geq \frac{2}{3}\left(9a_j\right)^m - \frac{\pi}{6}\left(9a_j\right)^m = \frac{4-\pi}{6}\left(9a_j\right)^m .\end{aligned}$$

Thus,

$$\frac{4-\pi}{6}\left(9a_j\right)^m \leq \left|\frac{u_j\left(x_m\right)-u_j\left(x_0\right)}{x_m-x_0}\right| \leq 37\left(9a_j\right)^m . \tag{3.4}$$

Next, we study the function $v_j : \mathbb{R} \to \mathbb{R}$ defined as

$$v_j(x) = \Im\left[f_{a_j}\left(e^{i\pi x}\right)\right] = \sum_{n=0}^{\infty} a_j^n \sin\left(9^n \pi x\right).$$

Observe that

$$\frac{v_j\left(x_m\right)-v_j\left(x_0\right)}{x_m-x_0} = \sum_{n=0}^{\infty} a_j^n \frac{\sin\left(9^n\pi x_m\right)-\sin\left(9^n\pi x_0\right)}{x_m-x_0} = S'_{j,m}+T'_{j,m},$$

where

$$S'_{j,m} := \sum_{n=0}^{m-1} a_j^n \frac{\sin\left(9^n\pi x_m\right)-\sin\left(9^n\pi x_0\right)}{x_m-x_0}$$

and

$$T'_{j,m} := \sum_{n=0}^{\infty} a_j^{m+n} \frac{\sin\left(9^{m+n}\pi x_m\right)-\sin\left(9^{m+n}\pi x_0\right)}{x_m-x_0}.$$

We will first study the finite sum $S'_{j,m}$. To do that, we will apply the following relationship:

$$\sin(x+y) - \sin(x-y) = 2\cos x \sin y.$$

Then

$$\begin{aligned}\left|S'_{j,m}\right| &= \left|\sum_{n=0}^{m-1} a_j^n \frac{2\cos\left(\frac{9^n\pi x_m+9^n\pi x_0}{2}\right)\sin\left(\frac{9^n\pi x_m-9^n\pi x_0}{2}\right)}{x_m-x_0}\right| \\ &\leq \sum_{n=0}^{m-1}\pi\left(9a_j\right)^n\left|\frac{\sin\left(9^n\pi\frac{x_m-x_0}{2}\right)}{9^n\pi\frac{x_m-x_0}{2}}\right|.\end{aligned}$$

Since $\left|\frac{\sin x}{x}\right| \leq 1$ for every $x \in \mathbb{R}$ and $a_j > \frac{7}{9}$, we obtain

$$\left|S'_{j,m}\right| \leq \sum_{n=0}^{m-1}\pi\left(9a_j\right)^n = \pi\frac{\left(9a_j\right)^m-1}{9a_j-1} < \frac{\pi\left(9a_j\right)^m}{6} < \left(9a_j\right)^m. \tag{3.5}$$

Now, we consider the series $T'_{j,m}$. On the one hand, since $\alpha_m \in \mathbb{Z}$, we have that

$$\sin\left(9^{m+n}\pi x_m\right) = \sin\left(9^{m+n}\pi\frac{\alpha_m-1}{9^m}\right) = \sin\left(9^n\pi\left(\alpha_m-1\right)\right) = 0.$$

On the other hand,

$$\begin{aligned}\sin\left(9^{m+n}\pi x_0\right) &= \sin\left(9^{m+n}\pi\frac{\alpha_m+t_m}{9^m}\right) = \sin\left(9^n\pi\alpha_m+9^n\pi t_m\right) \\ &= \sin\left(9^n\pi\alpha_m\right)\cos\left(9^n\pi t_m\right)+\cos\left(9^n\pi\alpha_m\right)\sin\left(9^n\pi t_m\right) \\ &= (-1)^{\alpha_m}\sin\left(9^n\pi t_m\right).\end{aligned}$$

Therefore

$$\begin{aligned}T'_{j,m} &= \sum_{n=0}^{\infty} a_j^{m+n}\frac{-(-1)^{\alpha_m}\sin\left(9^n\pi t_m\right)}{-\frac{1+t_m}{9^m}} \\ &= (-1)^{\alpha_m}\left(9a_j\right)^m\sum_{n=0}^{\infty}a_j^n\frac{\sin\left(9^n\pi t_m\right)}{1+t_m}.\end{aligned}$$

Since $a_j < \frac{8}{9}$ and $t_m \in (-1/2, 1/2]$, we obtain

$$\left|T'_{j,m}\right| \leq \left(9a_j\right)^m\sum_{n=0}^{\infty}\left(\frac{8}{9}\right)^n\frac{1}{1-\frac{1}{2}} = 18\left(9a_j\right)^m. \tag{3.6}$$

It follows from (3.5) and (3.6) that, for every $j \in \{1,\ldots,k\}$,

$$\left|\frac{v_j\left(x_m\right)-v_j\left(x_0\right)}{x_m-x_0}\right| = \left|S'_{j,m}+T'_{j,m}\right| \leq 19\left(9a_j\right)^m. \tag{3.7}$$

We recall that

$$f = f_{a_1}+\lambda_2 f_{a_2}+\cdots+\lambda_k f_{a_k},$$

where $\frac{7}{9} < a_k < \cdots < a_1 < \frac{8}{9}$ and $\lambda_2,\ldots,\lambda_k \in \mathbb{C}\setminus\{0\}$. For each $j = 2,\ldots,k$, let $p_j, q_j \in \mathbb{R}$ be such that $\lambda_j = p_j + iq_j$. Then

$$f\left(e^{i\pi x}\right) = u_1(x)+iv_1(x)+\sum_{j=2}^{k}\left(p_j+iq_j\right)\left(u_j(x)+iv_j(x)\right)$$

$$= \left[u_1(x) + \sum_{j=2}^{k} \left(p_j u_j(x) - q_j v_j(x)\right)\right] + i\left[v_1(x) + \sum_{j=2}^{k} \left(q_j u_j(x) + p_j v_j(x)\right)\right].$$

Hence

$$\Re \frac{f\left(e^{i\pi x_m}\right) - f\left(e^{i\pi x_0}\right)}{x_m - x_0} = \frac{u_1(x_m) - u_1(x_0)}{x_m - x_0} + \sum_{j=2}^{k} p_j \frac{u_j(x_m) - u_j(x_0)}{x_m - x_0} - \sum_{j=2}^{k} q_j \frac{v_j(x_m) - v_j(x_0)}{x_m - x_0}.$$

By (3.4) and (3.7),

$$\left|\frac{f\left(e^{i\pi x_m}\right) - f\left(e^{i\pi x_0}\right)}{x_m - x_0}\right| \geq \left|\Re \frac{f\left(e^{i\pi x_m}\right) - f\left(e^{i\pi x_0}\right)}{x_m - x_0}\right|$$

$$\geq \left|\frac{u_1(x_m) - u_1(x_0)}{x_m - x_0}\right| - \sum_{j=2}^{k} \left|p_j \frac{u_j(x_m) - u_j(x_0)}{x_m - x_0}\right| - \sum_{j=2}^{k} \left|q_j \frac{v_j(x_m) - v_j(x_0)}{x_m - x_0}\right|$$

$$\geq \frac{4-\pi}{6} (9a_1)^m - \sum_{j=2}^{k} 37 \left(9a_j\right)^m |p_j| - \sum_{j=2}^{k} 19 \left(9a_j\right)^m |q_j|$$

$$= (9a_1)^m \left[\frac{4-\pi}{6} - \sum_{j=2}^{k} 37 |p_j| \left(\frac{a_j}{a_1}\right)^m - \sum_{j=2}^{k} 19 |q_j| \left(\frac{a_j}{a_1}\right)^m\right].$$

Finally, since $0 < a_j/a_1 < 1$ for every $j = 2, \ldots, k$ and $9a_1 > 1$, we deduce that

$$\lim_{m\to\infty} \left|\frac{f\left(e^{i\pi x_m}\right) - f\left(e^{i\pi x_0}\right)}{x_m - x_0}\right| = +\infty. \tag{3.8}$$

Since

$$\lim_{m\to\infty} \frac{x_m - x_0}{e^{i\pi x_m} - e^{i\pi x_0}} = \frac{1}{i\pi e^{i\pi x_0}} \neq 0, \tag{3.9}$$

we obtain

$$\lim_{m\to\infty} \left|\frac{f\left(e^{i\pi x_m}\right) - f(z_0)}{e^{i\pi x_m} - z_0}\right| = \lim_{m\to\infty} \left|\frac{x_m - x_0}{e^{i\pi x_m} - e^{i\pi x_0}}\right| \cdot \left|\frac{f\left(e^{i\pi x_m}\right) - f\left(e^{i\pi x_0}\right)}{x_m - x_0}\right|,$$

and the last limit equals $+\infty$ thanks to (3.8) and (3.9). This implies the desired property (3.1) and the proof is concluded. ∎

With Theorem 2.2 at hand, it is possible to extract maximal-dense-lineability from mere lineability.

Theorem 3.2. *The set $NL(\mathbb{T})$ is $\mathfrak{c}$-dense-lineable, so maximal dense-lineable in $A(\mathbb{D})$. Hence $ND(\mathbb{T})$ is also maximal dense-lineable in $A(\mathbb{D})$.*

Proof. Let $\mathcal{P}$ denote the set of restrictions to $\overline{\mathbb{D}}$ of the family of all polynomials on $\mathbb{C}$. By Mergelyan's theorem, $\mathcal{P}$ is dense in $A(\mathbb{D})$ In fact, $\mathcal{P}$ is dense-lineable, because it is a vector space. Consequently, the set of all polynomials with rational coefficients is dense in $A(\mathbb{D})$ and thus $A(\mathbb{D})$ is a separable Banach space. Moreover, if $g \in NL(\mathbb{T})$, $P \in \mathcal{P}$ and $z_0 \in \mathbb{T}$, there exists a sequence $(z_m)_{m=1}^{\infty}$ in $\mathbb{T}$ such that

$$\lim_{m\to\infty}\left|\frac{g(z_m)-g(z_0)}{z_m-z_0}\right| = +\infty.$$

Then

$$\begin{aligned}&\lim_{m\to\infty}\left|\frac{(g+P)(z_m)-(g+P)(z_0)}{z_m-z_0}\right| \\ &\geq \lim_{m\to\infty}\left|\frac{g(z_m)-g(z_0)}{z_m-z_0}\right| - \left|P'(z_0)\right| = +\infty.\end{aligned}$$

This proves that $g+P \in NL(\mathbb{T})$, that is, $NL(\mathbb{T})+\mathcal{P} \subset NL(\mathbb{T})$. Moreover, of course, $NL(\mathbb{T}) \cap \mathcal{P} = \varnothing$. It follows from Theorem 2.2 (with $X = A(\mathbb{D})$, $A = NL(\mathbb{T})$ and $B = \mathcal{P}$) that $NL(\mathbb{T})$ is maximal-dense-lineable in $A(\mathbb{D})$. ∎

In the next and final theorem of this section we show the existence of large free algebras consisting of functions in $A(\mathbb{D})$ presenting non-differentiability at all points of $\mathbb{T}$, except for a small set of them. Denote by m the normalized Lebesgue measure on $\mathbb{T}$. Moreover, let

$$\begin{aligned}\widetilde{ND}(\mathbb{T}) := \{f \in A(\mathbb{D}) &: \text{there is a closed subset } A_f \subset \mathbb{T} \text{ such that} \\ &m(A_f) = 0 \text{ and } f|_{\mathbb{T}} \text{ is not differentiable at any point of } \mathbb{T}\setminus A_f\},\end{aligned}$$

$$\begin{aligned}\widetilde{NL}(\mathbb{T}) := \{f \in A(\mathbb{D}) &: \text{ there is a closed subset } A_f \subset \mathbb{T} \text{ such that} \\ &m(A_f) = 0 \text{ and } \limsup_{z\in\mathbb{T},\, z\to z_0}\left|\tfrac{f(z)-f(z_0)}{z-z_0}\right| = +\infty \text{ for every } z_0 \in \mathbb{T}\setminus A_f\}.\end{aligned}$$

Of course, $ND(\mathbb{T}) \subset \widetilde{ND}(\mathbb{T})$, $NL(\mathbb{T}) \subset \widetilde{NL}(\mathbb{T})$ and $\widetilde{NL}(\mathbb{T}) \subset \widetilde{ND}(\mathbb{T})$. Note that the above sets $\mathbb{T}\setminus A_f$ where differentiability fails are dense, open (in particular, residual) and full-measure in $\mathbb{T}$, that is, $m(\mathbb{T}\setminus A_f) = 1 = m(\mathbb{T})$. Hence they are both topologically and metrically large in $\mathbb{T}$.

Theorem 3.3. *The set $\widetilde{NL}(\mathbb{T})$ is strongly $\mathfrak{c}$-algebrable. Consequently, the family $\widetilde{ND}(\mathbb{T})$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let us choose any function $f \in NL(\mathbb{T})$ and an exponential-like function φ (see Section 2). Then φ is entire and nonconstant, so φ' is not identically zero. By the Identity Principle for analytic functions, the set $Z := \{z \in \mathbb{C} : \varphi'(z) = 0\}$ lacks accumulation points in $\mathbb{C}$. By continuity, the set $f(\mathbb{T})$ is compact. In particular, the intersection $Z \cap f(\mathbb{T})$ is finite, say

$$Z \cap f(\mathbb{T}) = \{w_1, \ldots, w_k\}.$$

Now, since $f \in NL(\mathbb{T}) \subset A(\mathbb{D})$, we have that f is a non-constant function which belongs to the Hardy space H^∞ of holomorphic functions that are bounded on $\mathbb{D}$. Therefore, f cannot be constant on any subset of $\mathbb{T}$ of positive measure (see [40, Theorem 17.18]). That is, for each $w \in \mathbb{C}$, the set $B_w := \{z \in \mathbb{T} : f(z) = w\}$ satisfies $m(B_w) = 0$. The continuity of f implies that each B_w is closed.

Let us define the function $g := \varphi \circ f$, which trivially belongs to $A(\mathbb{D})$. Let

$$A_g := B_{w_1} \cup \cdots \cup B_{w_k}.$$

Then A_g is a closed subset of $\mathbb{T}$ such that $m(A_g) = 0$. Fix a point $z_0 \in \mathbb{T} \setminus A_g$. Then $f(z_0) \notin Z$, and so $\varphi'(f(z_0)) \neq 0$. Since $f \in NL(\mathbb{T})$, there exists a sequence $(z_m) \subset \mathbb{T} \setminus \{z_0\}$ such that $z_m \to z_0$ and

$$\lim_{m\to\infty} \left| \frac{f(z_m) - f(z_0)}{z_m - z_0} \right| = +\infty.$$

Without loss of generality we can assume $f(z_m) \neq f(z_0)$ for all $m \in \mathbb{N}$. The continuity of f at z_0 entails $f(z_m) \to f(z_0)$. Therefore

$$\lim_{m\to\infty} \left| \frac{\varphi(f(z_m)) - \varphi(f(z_0))}{f(z_m) - f(z_0)} \right| = |\varphi'(f(z_0))| > 0.$$

Consequently,

$$\begin{aligned} \lim_{m\to\infty} \left| \frac{g(z_m) - g(z_0)}{z_m - z_0} \right| &= \lim_{m\to\infty} \left| \frac{\varphi(f(z_m)) - \varphi(f(z_0))}{f(z_m) - f(z_0)} \right| \cdot \left| \frac{f(z_m) - f(z_0)}{z_m - z_0} \right| \\ &= +\infty. \end{aligned}$$

But this implies

$$\limsup_{z\in\mathbb{T}, z\to z_0} \left| \frac{g(z) - g(z_0)}{z - z_0} \right| = +\infty.$$

That is, $g = \varphi \circ f \in \widetilde{NL}(\mathbb{T})$. Observe that since f is holomorphic on $\mathbb{D}$ and non-constant, $f(\mathbb{D})$ is a nonempty open set and hence it is uncountable. To complete the proof, it is enough to invoke Lemma 2.3 with $\Omega = \mathbb{D}$ and $\mathcal{F} = \widetilde{NL}(\mathbb{T})$. ∎

Remark 3.4. Concerning the first part of the last proof, it is worth mentioning that, thanks to a theorem due to Fatou (see [32, p. 80]), given $w \in \mathbb{C}$ and a closed set $K \subset \mathbb{T}$ with $m(K) = 0$, there exists $f \in A(\mathbb{D})$ such that $\{z \in \overline{\mathbb{D}} : f(z) = w\} = \{z \in \mathbb{T} : f(z) = w\} = K$.

We finish this section by posing the following natural problems.

Problems 3.5.
1. Are $ND(\mathbb{T})$ or $NL(\mathbb{T})$ spaceable in $A(\mathbb{D})$?
2. Are $ND(\mathbb{T})$ or $NL(\mathbb{T})$ (strongly) algebrable? Can the corresponding algebras be found dense in $A(\mathbb{D})$?

Concerning Problem 3.5.1, see Theorem 4.1(c) in the next section.

4 Nowhere differentiability on the boundary of more general domains

In this section, it will be shown that, in an algebraic sense, many boundary-almost-nowhere differentiable functions can be obtained, provided that the structure of the boundary is smooth enough. With this aim, we now recall some concepts and facts. Let us recall that if $\Omega \subset \mathbb{C}$, then the symbols $\overline{\Omega}^{\infty}$ and $\partial_\infty \Omega$ represent, respectively, the closure and the boundary of Ω in $\mathbb{C}_\infty$. For the following, we refer the reader to, e.g., [30, Chapter 14] and [40, Chapter 16]. According to the Riemann mapping theorem, for each simply connected domain $\Omega \subset \mathbb{C}$ we can fix an isomorphism $F_\Omega : \Omega \to \mathbb{D}$, that is, a bijective biholomorphic mapping. If (and only if) Ω is, in addition, a Jordan domain, the Osgood–Carathéodory theorem states that F_Ω can be homeomorphically extended from $\overline{\Omega}^{\infty}$ onto $\overline{\mathbb{D}}$ and thus $\mathbb{T}$ corresponds to $\partial_\infty \Omega$.

If Ω is a bounded Jordan domain, then the Jordan curve $\partial\Omega$ is called *piecewise analytic* if there is a parametric representation $z : [\alpha, \beta] \to \mathbb{C}$ and a finite subdivision $\alpha = \tau_0 < \tau_1 < \cdots < \tau_n = \beta$ such that the restriction of $z = z(\tau)$ to each subinterval $[\tau_{k-1}, \tau_k]$ agrees with a function z_k that is holomorphic on a domain of the complex plane containing $[\tau_{k-1}, \tau_k]$ and whose derivative is never zero. Similarly, if Ω is an unbounded Jordan domain, then $\partial\Omega$ is *piecewise analytic* if there is a parametric representation $z : (\alpha, \beta) \to \mathbb{C}$, with $\lim_{\tau\to\alpha} z(\tau) = \infty = \lim_{\tau\to\beta} z(\tau)$, and a finite subdivision $\alpha < \tau_0 < \tau_1 < \cdots < \tau_n < \beta$ such that the restriction of $z = z(\tau)$ to each subinterval $[\tau_{k-1}, \tau_k]$, $(\alpha, \tau_0]$ and $[\tau_n, \beta)$ agrees with a function z_k that is holomorphic on a domain of the complex plane containing the corresponding subinterval and whose derivative is never zero.

The points $z_k(\tau_k)$ are called the *corners* of $\partial\Omega$. We represent by C_Ω the set of such corners. In the case $C_\Omega = \varnothing$, we simply say that $\partial\Omega$ is analytic. Then the Osgood–Caratheodory extension F_Ω can be holomorphically continued to a neighborhood of every point $z_0 \in (\partial\Omega) \setminus C_\Omega$ (see [30, Section 16.4]).

At this point, some further notation is needed. Let H^1 stand for the 1-dimensional Hausdorff measure on the Borel sets of $\mathbb{R}^2$ (see, e.g., [22, Chapter 2]). For a domain $\Omega \subset \mathbb{C}$ and a subset $Z \subset \partial\Omega$ we set:

$$ND(\partial\Omega) := \{f \in A(\Omega) : \ f|_{\partial\Omega} \text{ is nowhere differentiable}\},$$

$$NL(\partial\Omega) := \{f \in A(\Omega) : \ f|_{\partial\Omega} \text{ is nowhere Lipschitz}\},$$

$$ND_Z(\partial\Omega) := \{f \in A(\Omega) : \ f|_{\partial\Omega} \text{ is not differentiable at any } z_0 \in (\partial\Omega) \setminus Z\},$$

$$NL_Z(\partial\Omega) := \{f \in A(\Omega) : \ f|_{\partial\Omega} \text{ is not Lipschitz at any } z_0 \in (\partial\Omega) \setminus Z\},$$

$$\begin{aligned}\widetilde{ND}(\partial\Omega) :=\{f \in A(\Omega) :& \text{ there is a closed subset } A_f \subset \partial\Omega \text{ such that} \\ & H^1(A_f) = 0 \text{ and } f|_{\partial\Omega} \text{ is not differentiable at any } z_0 \in \partial\Omega \setminus A_f\},\end{aligned}$$

$$\begin{aligned}\widetilde{NL}(\partial\Omega) :=\{f \in A(\Omega) :& \text{ there is a closed subset } A_f \subset \partial\Omega \text{ such that} \\ & H^1(A) = 0 \text{ and } f|_{\partial\Omega} \text{ is not Lipschitz at any } z_0 \in \partial\Omega \setminus A_f\}.\end{aligned}$$

Of course, we have a number of trivial relations, such as $NL(\partial\Omega) \subset ND(\partial\Omega)$, $\widetilde{NL}(\partial\Omega) \subset \widetilde{ND}(\partial\Omega)$ and many others.

We are now ready to establish and prove our result.

Theorem 4.1. *Let $\Omega \subset \mathbb{C}$ be a Jordan domain with piecewise analytic boundary. Let $Z = Z_\Omega := C_\Omega \cup \{z \in (\partial\Omega) \setminus C_\Omega : F'(z) = 0\}$, where $F : \Omega \to \mathbb{D}$ is a fixed biholomorphic mapping. We have:*

(a) *The set $NL_Z(\partial\Omega)$ is maximal dense-lineable in $A(\Omega)$. Hence $ND_Z(\partial\Omega)$ is also maximal dense-lineable in $A(\Omega)$.*

(b) *The set $\widetilde{NL}(\partial\Omega)$ is strongly $\mathfrak{c}$-algebrable. Hence $\widetilde{ND}(\partial\Omega)$ is also strongly $\mathfrak{c}$-algebrable.*

(c) *There is a dense subset D_Ω in $\partial\Omega$ such that the family*

$$\{f \in A(\Omega) : f|_{\partial\Omega} \text{ is not differentiable at any point of } D_\Omega\}$$

is spaceable in $A(\Omega)$.

Proof. Let S be any subset of $\partial\Omega$ such that

$$\liminf_{z\in\partial\Omega,\, z\to z_0} \left| \frac{F(z) - F(z_0)}{z - z_0} \right| > 0$$

for all $z_0 \in (\partial\Omega) \setminus S$ (of course, that property happens if there exists $F'(z_0) \neq 0$). Let us fix any function $g \in NL_{F(S)}(\mathbb{T})$ and any point $z_0 \in (\partial\Omega) \setminus S$. It is evident that $f := g \circ F \in A(\Omega)$. By injectivity and bicontinuity, respectively, we have $F(z) \neq F(z_0)$ if $z \in \mathbb{T} \setminus \{z_0\}$ and $F(z) \to F(z_0)$ if and only if $z \to z_0$. Therefore,

$$\limsup_{z\in\partial\Omega, z\to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \geq$$

$$\geq \limsup_{w\in\mathbb{T}, w\to F(z_0)} \left| \frac{g(w) - g(F(z_0))}{w - F(z_0)} \right| \cdot \liminf_{z\in\partial\Omega, z\to z_0} \left| \frac{F(z) - F(z_0)}{z - z_0} \right| = +\infty.$$

Hence f is not Lipschitz at z_0. With our terminology, we have proved that

$$NL_S(\partial\Omega) \supset \{g \circ F : g \in NL_{F(S)}(\mathbb{T})\} \supset \{g \circ F : g \in NL(\mathbb{T})\}, \tag{4.1}$$

the second inclusion being trivial.

According to Theorem 3.1, there is a $\mathfrak{c}$-dimensional vector space $V_0 \subset A(\mathbb{D})$ such that $V_0 \setminus \{0\} \subset NL(\mathbb{T})$. Let us define $V := \{g \circ F : g \in V_0\}$, that is a vector subspace of $A(\Omega)$. If $\{g_i\}_{i\in I}$ (with $\mathrm{card}(I) = \mathfrak{c}$) is an algebraic basis for V_0, then the fact $F(\overline{\Omega}) = \overline{\mathbb{D}}$ shows that the functions $f_i := g_i \circ F$ form an algebraic basis of V. Since the set Z satisfies that there exists $F'(z_0) \neq 0$ for all $z \in (\partial\Omega) \setminus Z$, it follows from (4.1) that

$$\begin{aligned} V \setminus \{0\} &= \{g \circ F : g \in V_0 \setminus \{0\}\} \\ &\subset \{g \circ F : g \in NL(\mathbb{T})\} \subset NL_Z(\partial\Omega). \end{aligned}$$

This shows that $NL_Z(\partial\Omega)$ is maximal lineable. Now, we invoke Mergelian's Theorem 2.1 to get the density of the set of polynomials in $A(\Omega)$. Then we proceed as in the proof of Theorem 3.2 with $\mathbb{D}$ and $NL(\mathbb{T})$ replaced, respectively, by Ω and $NL_Z(\partial\Omega)$. This proves (a).

In order to prove (b), take a freely $\mathfrak{c}$-generated algebra M_0 such that $M_0 \setminus \{0\} \subset \widetilde{NL}(\mathbb{T})$, which is furnished by Theorem 3.3. Let us define $M := \{g \circ F : g \in M_0\}$, that is a subalgebra of $A(\Omega)$. Again, the fact $F(\overline{\Omega}) = \overline{\mathbb{D}}$ implies that if $\{g_i\}_{i\in I}$ (with $\mathrm{card}(I) = \mathfrak{c}$) is a free generator system for M_0 then the functions $f_i := g_i \circ F$ form a free generator system of M. It suffices to prove that $M \setminus \{0\} \subset \widetilde{NL}(\partial\Omega)$. Fix $f \in M \setminus \{0\}$. Then there is $g \in M_0 \setminus \{0\}$ such that $f = g \circ F$. Therefore, $g \in \widetilde{NL}(\mathbb{T})$, so there is a closed subset $A_0 \subset \mathbb{T}$ with $m(A_0) = 0$ (equivalently, with $H^1(A_0) = 0$) such that g is non-Lipschitz at any point of $\mathbb{T} \setminus A_0$. The set

$$A := (F|_{\partial\Omega})^{-1}(A_0) \cup Z$$

is a closed subset of $\partial\Omega$ because Z is closed (see Remark 4.2 below, where the countability of Z is also shown) and F is continuous. In addition, there exists $F'(z) \neq 0$ for all $z \in (\partial\Omega) \setminus A$. It follows from (4.1) and the fact $g \in NL_{A_0}(\mathbb{T}) \subset NL_{F(A)}(\mathbb{T})$ that $f = g \circ F \in NL_A(\partial\Omega)$. That is, $f|_{\partial\Omega}$ is not Lipschitz at any point of $(\partial\Omega) \setminus A$.

Since F is analytic on $(\partial\Omega) \setminus Z$, with F' never zero on this set, it possesses analytic local inverse F^{-1} at every point. Then F^{-1} is locally Lipschitz on $\mathbb{T} \setminus F(Z)$ and, consequently, on $A_0 \setminus F(Z)$. It is known that the image under a locally Lipschitz mapping of a set with null H^1-measure also has null H^1-measure (see, e.g., [22, Theorem 2.8]). Hence $H^1((F|_{\partial\Omega})^{-1}(A_0 \setminus F(Z))) = 0$. Moreover, $H^1(Z) = 0$ because Z is countable, so we obtain

$$H^1(A) = H^1(Z \cup (F|_{\partial\Omega})^{-1}(A_0 \setminus F(Z))) = 0.$$

Consequently, $f \in \widetilde{NL}(\partial\Omega)$, as required.

With the aim of proving (c), let us choose

$$E := \{e^{i\pi x} : x \in \mathbb{Q}\} \setminus F(Z),$$

which is a countable subset of $\mathbb{T}$. It is also dense because $\{e^{i\pi x} : x \in \mathbb{Q}\}$ is dense in $\mathbb{T}$ and Z is discrete in $\partial\Omega$, so $F(Z)$ is discrete in $\mathbb{T}$ if Ω is bounded, while $F(Z)$ is discrete in $\mathbb{T} \setminus \{F(\infty)\}$ if Ω is unbounded. In any case, the set

$$D_\Omega := \{z \in \partial\Omega : F(z) \in E\}$$

is dense in $\partial\Omega$. By Theorem 2.4, the set

$$\mathcal{F}_E = \left\{g \in A(\mathbb{D}) : \{S(g|_{\mathbb{T}}, n)\}_{n\geq 1} \text{ is dense in } \mathbb{C}^E\right\}$$

is spaceable in $A(\mathbb{D})$, where $\{S(g|_{\mathbb{T}}, n)\}_{n\geq 1}$ denotes the sequence of Fourier partial sums of a function g. Thus, there is a closed infinite-dimensional subspace $Y_0 \subset A(\mathbb{D})$ with $Y_0 \setminus \{0\} \subset \mathcal{F}_E$. Define

$$Y := \{g \circ F : g \in Y_0\}.$$

As in the first part of the proof, we have that Y is an infinite-dimensional vector subspace of $A(\Omega)$. That Y is closed follows at once because Y_0 is closed and F is an homeomorphism. Fix $f \in Y \setminus \{0\}$ and $z_0 \in D_\Omega$. There exists $g \in Y_0$ with $f = g \circ F$. It remains to prove that $f|_{\partial\Omega}$ is not differentiable at z_0. Assume, by way of contradiction, that it is differentiable at z_0. Since $z_0 \notin Z$, there exists $F'(z_0) \neq 0$. Then the inverse mapping theorem would entail that $g|_{\mathbb{T}}$ is differentiable at $F(z_0)$. Hence the Fourier series of $g|_{\mathbb{T}}$ would converge at $F(z_0) \in E$ (see, e.g., [41, Theorem 1.4.1]). Consequently, $\{S(g|_{\mathbb{T}}, n)\}_{n\geq 1}$ could not be dense in $\mathbb{C}^E$, which is absurd. This contradiction yields the conclusion. ∎

Remark 4.2. Observe that, as in the case of the unit disc, the sets

$$E_\Omega := (\partial\Omega) \setminus (Z \cup A_f)$$

where differentiability fails are residual in $\partial\Omega$ because they are open and dense. Indeed, C_Ω is finite and, since F_Ω is nonconstant, the set $\{z \in (\partial\Omega) \setminus C_\Omega : F'(z) = 0\}$ lacks accumulation points in $(\partial\Omega) \setminus C_\Omega$ (in particular, it is countable). Hence $Z = C_\Omega \cup \{z \in (\partial\Omega) \setminus C_\Omega : F'(z) = 0\}$ is closed and has empty $\partial\Omega$-interior. Moreover, A_f is closed with $H^1(A_f) = 0$, so it also has empty $\partial\Omega$-interior.

In addition, the sets E_Ω are full-measure in $\partial\Omega$, that is, $H^1(E_\Omega) = H^1(\partial\Omega)$. Indeed, $H^1(A_f) = 0$ and both C_Ω and $\{z \in (\partial\Omega) \setminus C_\Omega : F'(z) = 0\}$ are countable, so their H^1-measure is zero. Hence the sets E_Ω are both topologically and metrically large in $\partial\Omega$.

Example 4.3. Let Π be an open halfplane. Then there is a motion $\varphi : z \mapsto \alpha z + \beta$ of $\mathbb{C}$ (with $|\alpha| = 1$, $\beta \in \mathbb{C}$) taking Π onto the open right halfplane $\Pi_+ = \{z \in \mathbb{C} : \Re z > 0\}$. Now, the function $F_0(z) = \frac{-z+1}{z+1}$ performs an isomorphism $\Pi_+ \to \mathbb{D}$ which can be extended homeomorphically to the $\mathbb{C}_\infty$-boundaries $i\mathbb{R} \cup \{\infty\}$ and $\mathbb{T}$. Then the same holds for the isomorphism $F := F_0 \circ \varphi$ between Π and $\mathbb{D}$. Since $\partial\Pi$ is a straight line (so analytic) and $F'(z) = \frac{-2\alpha}{(\varphi(z)+1)^2} \neq 0$ for all $z \in \partial\Pi$, Theorem 4.1 tells us that $NL(\partial\Pi)$ and $ND(\partial\Pi)$ are maximal dense-lineable in $A(\Pi)$ and that $\widetilde{NL}(\partial\Pi)$ and $\widetilde{ND}(\partial\Pi)$ are strongly $\mathfrak{c}$-algebrable.

Of course, problems similar to those posed at the end of Section 3 can be formulated for domains different from the unit disc.

Appendix: Prevalence of $ND(\mathbb{T})$

As a complement to the results obtained in the previous sections, we will briefly consider in this appendix the large size of $ND(\mathbb{T})$ –and its Ω-analogues– under a measure-theoretical point of view. With this aim, we first recall the concept of prevalence, that was coined by Hunt, Sauer and Yorke in 1992 (see [34]). We also refer the reader to Bastin et al. [11] for a paper including both lineability and prevalence results.

Let X be a metrizable topological vector space over $\mathbb{R}$ or $\mathbb{C}$. A subset $A \subset X$ is called *prevalent* in X provided that there exist a Borel set $S \subset X$ and a measure μ on the Borel subsets of X satisfying the following conditions:

(i) $A \supset X \setminus S$,

(ii) $\mu(S+v) = 0$ for every $v \in X$,

(iii) $0 < \mu(K) < \infty$ for some compact subset $K \subset X$.

A sufficient (but not necessary) condition for a set to be prevalent is the one given in the next lemma. Its content is given in [34, p. 225, after Definition 6] for the real case, but it is immediate that it also holds in the complex case, that is the setting in which it is stated here.

Lemma. *Let us assume that X is a metrizable topological vector space over $\mathbb{C}$, $N \in \mathbb{N}$, and that $\varphi_1, \ldots, \varphi_N$ are linearly independent vectors in X. Let μ represent the $2N$-dimensional Lebesgue measure supported on* $\operatorname{span}\{\varphi_1, \ldots, \varphi_N\}$*, that is, if B is a Borel subset of X, then*

$$\mu(B) := m_{2N}\Big(\Big\{(a_1, b_1, \ldots, a_N, b_N) \in \mathbb{R}^{2N} : \sum_{j=1}^{N}(a_j + ib_j)\varphi_j \in B\Big\}\Big).$$

Let us suppose that $A \subset X$ and that there is a Borel subset $S \subset X$ such that $A \supset X \setminus S$ and $\mu(S+v) = 0$ for every $v \in X$. Then the set A is prevalent in X.

For instance, a subset $A \subset \mathbb{R}^N$ is prevalent if and only if $\mathbb{R}^N \setminus A$ has N-dimensional Lebesgue measure zero, while $A \subset \mathbb{C}^N$ is prevalent if and only if $\mathbb{C}^N \setminus A$ has $2N$-dimensional Lebesgue measure zero (see [34]). Hunt [33] proved the prevalence in $C[0,1]$ of the family $ND[0,1]$ of Weierstrass monsters. We will take advantage of the methods developed by Hunt in order to obtain prevalence for the family $ND(\mathbb{T})$ of Weierstrass monsters in the disc algebra. Let us recall that $Z_\Omega = C_\Omega \cup \{z \in (\partial\Omega) \setminus C_\Omega : F'_\Omega(z) = 0\}$, where $F_\Omega : \Omega \to \mathbb{D}$ is a biholomorphic mapping.

Theorem. *Assume that $\Omega \subset \mathbb{C}$ is a Jordan domain whose boundary is piecewise analytic. Then the set $NL_{Z_\Omega}(\partial\Omega)$ is prevalent in $A(\Omega)$. Consequently, the family $ND_{Z_\Omega}(\partial\Omega)$ is prevalent in $A(\Omega)$.*

Proof. It is enough to consider the special case $\Omega = \mathbb{D}$. The general case can be handled as in the proof of Theorem 4.1, by considering the homeomorphism $F_\Omega : \overline{\Omega}^\infty \to \overline{\mathbb{D}}$ and taking into account that $NL_Z(\partial\Omega) \supset \{f \circ F_\Omega : f \in NL(\mathbb{T})\}$.

Let us deal with the case of the unit disc, so $Z_\mathbb{D} = \varnothing$ and $NL_{Z_\mathbb{D}}(\mathbb{T}) = NL(\mathbb{T})$. It is proved in [33, Proposition 2] that, for all $f \in C[0,1]$, the function $f + \alpha g + \beta h$ is nowhere Lipschitz for Lebesgue almost $(\alpha, \beta) \in \mathbb{R}^2$, where

$$g(x) := \sum_{n=1}^{\infty} \frac{1}{n^2}\cos(2^n \pi x) \quad \text{and} \quad h(x) := \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(2^n \pi x).$$

A straightforward calculation reveals that the same holds if g and h are replaced by $g_0(x) := g(2x)$ and $h_0(x) := h(2x)$, respectively. In other words, the set

$$N_f := \{(\alpha, \beta) \in \mathbb{R}^2 : f + \alpha g_0 + \beta h_0 \text{ is Lipschitz at some point of } [0,1]\}$$

satisfies $m_2(N_f) = 0$ for all $f \in C[0,1]$.

For $M > 0$, we say that a function $f : \mathbb{T} \to \mathbb{C}$ is M-Lipschitz at a point $z_0 \in \mathbb{T}$ provided that $|f(z) - f(z_0)| \leq M|z - z_0|$ for all $x \in \mathbb{T}$. Denote

$$S_M := \{f \in A(\mathbb{D}) : f|_{\mathbb{T}} \text{ is } M\text{-Lipschitz at some point of } \mathbb{T}\}.$$

A compactness argument reveals that S_M is closed in $A(\mathbb{D})$. Hence the set

$$S := A(\mathbb{D}) \setminus NL(\mathbb{T}) = \bigcup_{M \in \mathbb{N}} S_M$$

is a Borel subset of $A(\mathbb{D})$. Consider the function

$$\varphi(z) = \sum_{n=1}^{\infty} \frac{z^{2^n}}{n^2}.$$

Again, Weierstrass M-test yields that $\varphi \in A(\mathbb{D})$. Observe that $g_0(x) = \Re\varphi(e^{2\pi i x})$ and $h_0(x) = \Im\varphi(e^{2\pi i x})$ for all $x \in [0,1]$.

Take $X := A(\mathbb{D})$, $A := NL(\mathbb{T})$, $N := 1$, and define the measure

$$\mu(B) := m_2\big(\{(a,b) \in \mathbb{R}^2 : (a+ib)\varphi \in B\}\big)$$

for each Borel subset B of $A(\mathbb{D})$. According to the previous Lemma, it is enough to prove that $\mu(S+\phi) = 0$ for all $\phi \in A(\mathbb{D})$. To do that, we take any $\phi \in A(\mathbb{D})$ and define the functions $u_0 : [0,1] \to \mathbb{R}$ and $v_0 : [0,1] \to \mathbb{R}$ as

$$u_0(x) = \Re\phi(e^{2\pi i x}), \quad v_0(x) = \Im\phi(e^{2\pi i x}).$$

Assume that $(a,b) \in \mathbb{R}^2$ satisfies $(a+ib)\varphi \in S+\phi$. Then there exists $x_0 \in [0,1]$ such that $((a+bi)\varphi - \phi)|_{\mathbb{T}}$ is Lipschitz at $z_0 := e^{2\pi i x_0}$, so there is $M > 0$ satisfying $|\Phi(z) - \Phi(z_0)| \leq M|z - z_0|$ for all $z \in \mathbb{T}$, where we have set $\Phi := (a+bi)\varphi - \phi$. We now use the inequality $|e^{i\alpha} - e^{i\beta}| \leq |\alpha - \beta|$ for every $\alpha, \beta \in \mathbb{R}$ to obtain that if $x \in [0,1]$ then

$$\begin{aligned}
&|[-u_0(x) + ag_0(x) - bh_0(x)] - [-u_0(x_0) + ag_0(x_0) - bh_0(x_0)]| \\
&\quad = |\Re\left[\Phi(e^{2\pi i x}) - \Phi(e^{2\pi i x_0})\right]| \leq |\Phi(e^{2\pi i x}) - \Phi(e^{2\pi i x_0})| \\
&\quad \leq M|e^{2\pi i x} - e^{2\pi i x_0}| \leq 2\pi M|x - x_0|.
\end{aligned}$$

That is, the function $-u_0 + ag_0 - bh_0$ is Lipschitz at x_0. With our previous notation, we get $(a,b) \in \sigma(N_{-u_0})$, where σ denotes the axial symmetry $(\alpha,\beta) \mapsto (\alpha,-\beta)$ on $\mathbb{R}^2$. Thus, we have proved that

$$\{(a,b) \in \mathbb{R}^2 : (a+ib)\varphi \in S+\phi\} \subset \sigma(N_{-u_0}).$$

Consequently, since σ is an isometry, we obtain

$$\begin{aligned}
0 \ &\leq \ \mu(S+\phi) = m_2\big(\{(a,b) \in \mathbb{R}^2 : (a+ib)\varphi \in S+\phi\}\big) \\
&\leq \ m_2(\sigma(N_{-u_0})) = m_2(N_{-u_0}) = 0.
\end{aligned}$$

Therefore, $\mu(S+\phi) = 0$ for every $\phi \in A(\mathbb{D})$, which concludes the proof. ∎

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