Disk-cyclic and codisk-cyclic weighted pseudo-shifts

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Abstract

In this paper, we characterize disk-cyclic and codisk-cyclic weighted pseudo-shifts on Banach sequence spaces, and consider the bilateral operator weighted shifts on $\ell^2(\mathbb{Z}, \mathcal{K})$ as a special case. Moreover, we present a counter-example to show that a result in [Y. X. Liang and Z. H. Zhou], Disk-cyclicity and Codisk-cyclicity of certain shift operators, Operators and Matrices, **9**(2015), 831–846] is not correct.

1 Introduction

Let $\mathbb{N}$ denote the set of non-negative integers, $\mathbb{Z}$ denote the set of all integers. Let $L(X)$ be the space of all linear and continuous operators on a separable, infinite dimensional complex Banach space X. An operator $T \in L(X)$ is said to be *hypercyclic* if there is a vector $x \in X$ such that the orbit $\mathrm{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in X. In such a case, x is called a *hypercyclic vector* for T.

The first example of a hypercyclic operator on a Banach space was offered in 1969 by Rolewicz [15], who showed that if B is the unilateral backward shift on $\ell^2(\mathbb{N})$, then the scaled shift λB is hypercyclic if and only if $|\lambda| > 1$. Salas [16] completely characterized the hypercyclic unilateral weighted backward shifts on $\ell^p(\mathbb{N})$ with $1 \leq p < \infty$ and the bilateral weighted shifts on $\ell^p(\mathbb{Z})$ with $1 \leq p < \infty$ in terms of their weight sequences. León-Saavedra and Montes-Rodríguez [12]

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later used Salas' weight characterization to show that each type of weighted shifts is hypercyclic precisely when it satisfies the so-called Hypercyclicity Criterion. This criterion was obtained independently by Kitai [11] and by Gethner and Shapiro [4], and it provides a sufficient condition for a general operator to be hypercyclic. Using the Hypercyclicity Criterion, Grosse-Erdmann [5] extended Salas' results by obtaining a characterization for hypercyclic weighted shifts on an arbitrary F-sequence space. We refer the readers to the books by Bayart and Matheron [2], and by Grosse-Erdmann and A. Peris Manguillot [6] for more background and many examples about hypercyclic operators.

By Rolewicz's example above, λB is not hypercyclic whenever $|\lambda| \leq 1$, this led to study the disk orbit or codisk orbit notion. Disk-cyclic and codisk-cyclic operators were introduced by Zeana in her PhD thesis [8], and defined as follows:

Definition 1.1. A bounded linear operator T on X is called *disk-cyclic* if there is a vector x in X such that the set

$$\{\alpha T^n x : \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1, n \in \mathbb{N}\} \text{ is dense in } X.$$

In this case x is said to be a *disk-cyclic vector* for T.

Definition 1.2. A bounded linear operator T on X is called *codisk-cyclic* if there is a vector x in X such that the set

$$\{\alpha T^n x : \alpha \in \mathbb{C}, |\alpha| \geq 1, n \in \mathbb{N}\} \text{ is dense in } X.$$

In this case x is said to be a *codisk-cyclic vector* for T.

Remarks 1.3. (1) Every hypercyclic operator is (co)disk-cyclic;

(2) In [8], Zeana proved that the set of all disk-cyclic (respectively codisk-cyclic) vectors for a disk-cyclic (respectively codisk-cyclic) operator on Hilbert space is a dense G_δ set. With the same arguments, this conclusion is also valid in Banach spaces.

In [8] the author also proposed the disk-cyclicity criterion and codisk-cyclicity criterion in Hilbert spaces. These two criteria play a key role in this paper, now we extend them to Banach spaces and the proofs are the same as those in Hilbert spaces.

Proposition 1.4. *(Disk-Cyclicity Criterion) Let X be a separable Banach space, $T \in L(X)$ such that*

(1) There are dense sets X_0, Y_0 in X and a right inverse S of T (not necessarily bounded) such that $S(Y_0) \subset Y_0$ and $TS = I_{Y_0}$.

(2) There is a sequence $(n_k) \subset \mathbb{N}$ such that

(a) $\lim_{k\to\infty} \|S^{n_k} y\| = 0$ *for all* $y \in Y_0$;

(b) $\lim_{k\to\infty} \|T^{n_k} x\| \, \|S^{n_k} y\| = 0$ *for all* $x \in X_0, y \in Y_0$.

Then T is disk-cyclic.

Proposition 1.5. *(Codisk-Cyclicity Criterion) Let X be a separable Banach space, $T \in L(X)$ such that*

(1) There are dense sets X_0, Y_0 in X and a right inverse S of T (not necessarily bounded) such that $S(Y_0) \subset Y_0$ and $TS = I_{Y_0}$.

(2) There is a sequence $(n_k) \subset \mathbb{N}$ such that

(a) $\lim_{k\to\infty} \|T^{n_k}x\| = 0$ *for all* $x \in X_0$;

(b) $\lim_{k\to\infty} \|T^{n_k}x\| \, \|S^{n_k}y\| = 0$ *for all* $x \in X_0, y \in Y_0$.

Then T is codisk-cyclic.

For examples of disk-cyclic operators, Zeana [10] characterized the disk-cyclic bilateral weighted shifts on $\ell^2(\mathbb{Z})$. Liang and Zhou studied the disk-cyclic and codisk-cyclic tuples of the adjoint weighted composition operators on Hilbert spaces in [14]. For more results about (co)disk-cyclic operators, we recommend papers [17], [1] and [9]. In this paper, motivated by Grosse-Erdmann's work [5], we investigate the (co)disk-cyclicity of weighted pseudo-shifts on arbitrary Banach sequence spaces.

To proceed further we recall some definitions of the sequence spaces and weighted pseudo-shifts. For a comprehensive survey we recommend Grosse-Erdmann's paper [5].

Definition 1.6. (Sequence Space) If we allow an arbitrary countably infinite set I as an index set, then a *sequence space over I* is a subspace of the space $\omega(I) = \mathbb{C}^I$ of all scalar families $(x_i)_{i\in I}$. The space $\omega(I)$ is endowed with its natural product topology.

A *topological sequence space X over I* is a sequence space over I that is endowed with a linear topology in such a way that the inclusion mapping $X \hookrightarrow \omega(I)$ is continuous or, equivalently, that every *coordinate functional* $f_i : X \to \mathbb{C}, (x_k)_{k\in I} \mapsto x_i$ $(i \in I)$ is continuous. A *Banach (Hilbert) sequence space over I* is a topological sequence space over I that is a Banach (Hilbert) space.

Definition 1.7. (OP-basis) By $(e_i)_{i\in I}$ we denote the canonical unit vectors $e_i = (\delta_{ik})_{k\in I}$ in a topological sequence space X over I. We say $(e_i)_{i\in I}$ is an *$OP-basis$* or (*Ovsepian Pelczyński basis*) if span$\{e_i : i \in I\}$ is a dense subspace of X and the family of *coordinate projections* $x \mapsto x_i e_i (i \in I)$ on X is equicontinuous. Note that in a Banach sequence space over I the family of coordinate projections is equicontinuous if and only if $\sup_{i\in I} ||e_i||||f_i|| < \infty$.

Definition 1.8. (Pseudo-shift Operators) Let X be a Banach sequence space over I. Then a continuous linear operator $T : X \to X$ is called a *weighted pseudo-shift* if there is a sequence $(b_i)_{i\in I}$ of non-zero scalars and an injective mapping $\varphi : I \to I$ such that

$$T(x_i)_{i\in I} = (b_i x_{\varphi(i)})_{i\in I}$$

for $(x_i) \in X$. We then write $T = T_{b,\varphi}$, and $(b_i)_{i\in I}$ is called the *weight sequence*.

Remarks 1.9. (1) If $T = T_{b,\varphi} : X \to X$ is a weighted pseudo-shift, then each $T^n (n \geq 1)$ is also a weighted pseudo-shift as follows

$$T^n(x_i)_{i\in I} = (b_{n,i}x_{\varphi^n(i)})_{i\in I}$$

where

$$\varphi^n(i) = (\varphi \circ \varphi \circ \cdots \circ \varphi)(i) \quad (n - \text{fold})$$

$$b_{n,i} \;=\; b_i b_{\varphi(i)} \cdots b_{\varphi^{n-1}(i)} = \prod_{v=0}^{n-1} b_{\varphi^v(i)}.$$

(2) We consider the inverse $\psi = \varphi^{-1} : \varphi(I) \to I$ of the mapping φ. We also set

$$b_{\psi(i)} = 0 \quad \text{and} \quad e_{\psi(i)} = 0 \quad \text{if } i \in I \setminus \varphi(I),$$

i.e. when $\psi(i)$ is " undefined ". Then for all $i \in I$,

$$T_{b,\varphi}e_i = b_{\psi(i)}e_{\psi(i)}.$$

(3) We denote $\psi^n = \psi \circ \psi \circ \cdots \circ \psi$ (n-fold), and we set $b_{\psi^n(i)} = 0$ and $e_{\psi^n(i)} = 0$ when $\psi^n(i)$ is " undefined ".

Definition 1.10. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ of mappings $\varphi_n : I \to I$ is called a *run-away sequence* if for each pair of finite subsets $I_0 \subset I$ and $J_0 \subset I$ there exists an $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $\varphi_n(J_0) \cap I_0 = \emptyset$.

We usually apply this definition to the sequence of iterates of the mapping $\varphi : I \to I$. Specifically, if we denote $\varphi^n := \varphi \circ \varphi \circ \cdots \circ \varphi$ (n-fold), we call $(\varphi^n)_n$ a *run-away sequence* if for each pair of finite subsets $I_0 \subset I$ and $J_0 \subset I$, there exists an $n_0 \in \mathbb{N}$ such that $\varphi^n(J_0) \cap I_0 = \emptyset$ for every $n \geq n_0$.

The rest of the paper is organized as follows: Equivalent conditions for disk-cyclic and codisk-cyclic pseudo-shifts on arbitrary Banach sequence spaces are given in Section 2. In Section 3, we illustrate the result about disk-cyclic pseudo-shifts in Section 2 with operator weighted shifts on $\ell^2(\mathbb{Z}, \mathcal{K})$. As a consequence, we point out a mistake in [13] by a simple counter-example. Motivated by Feldman's work in [3], we derive that the characterizations are far simplified when the operator weighted shifts are invertible in Section 4.

2 Disk-cyclic and Codisk-cyclic weighted pseudo-shifts

In this section let X be a Banach sequence space over I in which $(e_i)_{i\in I}$ is an OP-basis. We are concerned with the (co)disk-cyclicity of weighted pseudo-shifts on X. For the characterization of hypercyclic weighted pseudo-shifts on X Grosse-Erdmann established the following result in [5].

Theorem 2.1. [5, *Theorem* 5] *Let $T = T_{b,\varphi} : X \to X$ be a weighted pseudo-shift. Then the following assertions are equivalent:*

(i) T is hypercyclic;

(ii) (α) The mapping $\varphi : I \to I$ has no periodic point;

(β) There exists an increasing sequence (n_k) of positive integers such that, for every $i \in I$,

$$(H1) \qquad \left\| \left(\prod_{v=0}^{n_k-1} b_{\varphi^v(i)} \right)^{-1} e_{\varphi^{n_k}(i)} \right\| \to 0,$$

$$(H2) \qquad \left\| \left(\prod_{v=1}^{n_k} b_{\psi^v(i)} \right) e_{\psi^{n_k}(i)} \right\| \to 0,$$

as $k \to \infty$.

Remark 2.2. In paper [5], Theorem 2.1 holds for weighted pseudo-shifts on F-sequence space.

The following theorem is our main result in this section.

Theorem 2.3. *Let $T = T_{b,\varphi}$ be a weighted pseudo-shift on X. If $(\varphi^n)_n$ is a run-away sequence, then the following assertions are equivalent:*

(1) T is disk-cyclic;

(2) There exists an increasing sequence (n_k) of positive integers such that, for every $i, j \in I$,

(a) $\lim_{k\to\infty} \left\| \left(\prod_{v=0}^{n_k-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^{n_k}(j)} \right\| = 0;$

(b) $\lim_{k\to\infty} \left\| \left(\prod_{v=0}^{n_k-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^{n_k}(j)} \right\| \left\| \left(\prod_{v=1}^{n_k} b_{\psi^v(i)} \right) e_{\psi^{n_k}(i)} \right\| = 0.$

(3) T satisfies the Disk-Cyclicity Criterion.

Proof. (1) $\Rightarrow$ (2). Assume T is disk-cyclic. To prove (2), we need the following fact.

Fact *For every finite subset I_0 of I, any $0 < \varepsilon \leq 1$ and $N \in \mathbb{N}$ there exists an integer $n > N$ such that*

$$\left\| \left(\prod_{v=0}^{n-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^{n}(j)} \right\| < \varepsilon, \text{ for all } j \in I_0, \tag{2.1}$$

and

$$\left\| \left(\prod_{v=0}^{n-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^{n}(j)} \right\| \left\| \left(\prod_{v=1}^{n} b_{\psi^v(i)} \right) e_{\psi^{n}(i)} \right\| < \varepsilon, \text{ for all } i, j \in I_0. \tag{2.2}$$

Proof of the fact. Let $0 < \varepsilon \leq 1$, finite subset $I_0 \subset I$ and $N \in \mathbb{N}$ be given. Since (φ^n) is a run-away sequence, there exists an $n_0 \in \mathbb{N}$ such that for every $m \geq n_0$,

$$\varphi^m(I_0) \cap I_0 = \emptyset. \tag{2.3}$$

By the equicontinuity of the coordinate projections in X, there is some $\delta > 0$ so that for $x = (x_i)_{i\in I} \in X$

$$\|x_i e_i\| < \frac{\varepsilon}{2} \text{ for all } i \in I, \text{ if } ||x|| < \delta. \tag{2.4}$$

Since the set of disk-cyclic vectors for T is dense in X, there exist a disk-cyclic vector $x \in X$, a complex number α with $0 < |\alpha| \leq 1$ and $n \in \mathbb{N}$ with $n >$ max $\{N, n_0\}$ such that

$$\left\| x - \sum_{i\in I_0} e_i \right\| < \delta \text{ and } \left\| \alpha T^n x - \sum_{j\in I_0} e_j \right\| < \delta. \tag{2.5}$$

(Here we prove that the selection of n in the second inequality of (2.5) can be arbitrarily large. Let $A = \{\alpha T^n x : \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1, n \in \mathbb{N}\}$, $B = \{y : \|y - \sum_{j\in I_0} e_j\| < \delta\}$. For every $p \in \mathbb{N}$, let $B_p = \{\alpha T^n x : \alpha \in \mathbb{C}$, $0 \leq |\alpha| \leq 1, n \in \mathbb{N}, n \leq p\}$. It is enough to show that $B \cap (A \setminus B_p) \neq \emptyset$. Since X is an infinite dimensional Banach space, for every $p \in \mathbb{N}$, $B \setminus B_p$ is a non-empty open subset of X. It follows that $B \cap (A \setminus B_p) = (B \setminus B_p) \cap A \neq \emptyset$, because A is dense in X.)
By the continuous inclusion of X into $\omega(I)$, we can in addition obtain that

$$\sup_{i\in I_0} |x_i - 1| \leq \frac{1}{2} \text{ and } \sup_{j\in I_0} |\alpha y_j - 1| \leq \frac{1}{2}, \tag{2.6}$$

where $T^n x = (y_j)_{j\in I} = \left(\left(\prod_{v=0}^{n-1} b_{\varphi^v(j)} \right) x_{\varphi^n(j)} \right)_{j\in I}$.
(2.4) and the first inequality in (2.5) imply that

$$||x_i e_i|| < \frac{\varepsilon}{2} \text{ if } i \in I \backslash I_0,$$

hence by (2.3) we have that

$$\left\| x_{\varphi^n(j)} e_{\varphi^n(j)} \right\| < \frac{\varepsilon}{2} \quad \text{for } j \in I_0. \tag{2.7}$$

By the second inequality in (2.6),

$$\left| \alpha \left(\prod_{v=0}^{n-1} b_{\varphi^v(j)} \right) x_{\varphi^n(j)} - 1 \right| \leq \frac{1}{2} \quad \text{for } j \in I_0,$$

which implies $x_{\varphi^n(j)} \neq 0$ and

$$\left| \frac{1}{\alpha \left(\prod_{v=0}^{n-1} b_{\varphi^v(j)} \right) x_{\varphi^n(j)}} \right| \leq 2 \tag{2.8}$$

for every $j \in I_0$.
Now, by (2.7), (2.8) and $|\alpha| \neq 0$ we have

$$\begin{aligned} \left\| \left(\alpha \prod_{v=0}^{n-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^n(j)} \right\| &= \left| \frac{1}{\alpha \left(\prod_{v=0}^{n-1} b_{\varphi^v(j)} \right) x_{\varphi^n(j)}} \right| \left\| x_{\varphi^n(j)} e_{\varphi^n(j)} \right\| \\ &\leq 2 \left\| x_{\varphi^n(j)} e_{\varphi^n(j)} \right\| < \varepsilon \end{aligned} \tag{2.9}$$

for all $j \in I_0$. This implies condition (2.1) because $0 < |\alpha| \leq 1$.

As for (2.2), we deduce from (2.3) and the definition of ψ^n that

$$\psi^n(I_0 \cap \varphi^n(I)) \cap I_0 = \emptyset. \tag{2.10}$$

By (2.4), the second inequality in (2.5) implies that

$$\left\| \alpha \left(\prod_{v=0}^{n-1} b_{\varphi^v(j)} \right) x_{\varphi^n(j)} e_j \right\| < \frac{\varepsilon}{2} \quad \text{if } j \in I \backslash I_0.$$

So by (2.10) and the fact that $e_{\psi^n(i)} = 0$ for all $i \in I \backslash \varphi^n(I)$,

$$\left\| \alpha \left(\prod_{v=1}^{n} b_{\psi^v(i)} \right) x_i e_{\psi^n(i)} \right\| < \frac{\varepsilon}{2} \quad \text{if } i \in I_0. \tag{2.11}$$

By the first inequality in (2.6) we have

$$0 < \frac{1}{|x_i|} \leq 2 \quad \text{for } i \in I_0. \tag{2.12}$$

Now, (2.11) and (2.12) imply that for each $i \in I_0$

$$\begin{aligned} \left\| \alpha \left(\prod_{v=1}^{n} b_{\psi^v(i)} \right) e_{\psi^n(i)} \right\| &= \frac{1}{|x_i|} \left\| \alpha \left(\prod_{v=1}^{n} b_{\psi^v(i)} \right) x_i e_{\psi^n(i)} \right\| \\ &< \varepsilon. \end{aligned} \tag{2.13}$$

Thus from (2.9) and (2.13) we can deduce that

$$\begin{aligned}
&\left\| \left(\prod_{v=0}^{n-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^n(j)} \right\| \left\| \left(\prod_{v=1}^{n} b_{\psi^v(i)} \right) e_{\psi^n(i)} \right\| \\
= &\left\| \left(\alpha \prod_{v=0}^{n-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^n(j)} \right\| \left\| \alpha \left(\prod_{v=1}^{n} b_{\psi^v(i)} \right) e_{\psi^n(i)} \right\| \\
< &\ \varepsilon^2 \leq \varepsilon
\end{aligned}$$

for any $i, j \in I_0$. Therefore (2.2) holds.

Coming back to the proof of (2). Since I is a countably infinite set, we fix $I := \{i_1, i_2, \cdots, i_n, \cdots\}$ and set $I_k := \{i_1, i_2, \cdots, i_k\}$ for each $k \in \mathbb{N}, k \geq 1$. Using the above fact, we define inductively an increasing sequence $(n_k)_{k\geq 1}$ of positive integers by letting n_k be a positive integer satisfying (2.1) and (2.2) for $I_0 = I_k$, $\varepsilon = \frac{1}{k}$ and $N = n_{k-1}$, where we set $N = 0$ when $k = 1$. To prove (2) we only need to verify that the sequence $(n_k)_{k\geq 1}$ satisfies both (a) and (b). This is clear, since for any fixed $i, j \in I$ there exists $n_0' \in \mathbb{N}$ such that $i, j \in I_k$ for each $k \geq n_0'$, which means

$$\left\| \left(\prod_{v=0}^{n_k-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^{n_k}(j)} \right\| < \frac{1}{k} \quad \text{if } k \geq n_0',$$

and

$$\left\| \left(\prod_{v=0}^{n_k-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^{n_k}(j)} \right\| \left\| \left(\prod_{v=1}^{n_k} b_{\psi^v(i)} \right) e_{\psi^{n_k}(i)} \right\| < \frac{1}{k} \quad \text{if } k \geq n_0'.$$

So (a) and (b) hold.

(2) $\Rightarrow$ (3). Suppose (2) holds. Set $X_0 = Y_0 = span\{e_i, i \in I\}$ which are dense in X and define a linear mapping: $S : Y_0 \to X$ by

$$S(e_j) = b_j^{-1} e_{\varphi_{(j)}} \qquad \text{for each } j \in I,$$

thus

$$S^n(e_j) = \left(\prod_{v=0}^{n-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^n(j)} \quad (n \in \mathbb{N}, j \in I).$$

Since

$$T^n e_i = \left(\prod_{v=1}^{n} b_{\psi^v(i)} \right) e_{\psi^n(i)} \quad (n \in \mathbb{N}, i \in I),$$

we have $T^n S^n(e_j) = e_j$ for each $n \in \mathbb{N}, j \in I$. Let (n_k) be the sequence given in condition (2). By (a) and (b), it follows that for any $i, j \in I$

$$\lim_{k\to\infty} \|S^{n_k} e_j\| = 0,$$

and

$$\lim_{k\to\infty} \|T^{n_k}e_i\| \, \|S^{n_k}e_j\| = 0.$$

By Proposition 1.4, T satisfies the Disk-Cyclicity Criterion.

(3) $\Rightarrow$ (1). This implication follows from Proposition 1.4. ■

Using a similar argument as in the proof of Theorem 2.3, we obtain equivalent conditions for T to be codisk-cyclic.

Theorem 2.4. *Let $T = T_{b,\varphi} : X \to X$ be a weighted pseudo-shift. If (φ^n) is a run-away sequence, then the following assertions are equivalent:*

(1) T is codisk-cyclic;

(2) There exists an increasing sequence (n_k) of positive integers such that, for every $i, j \in I$,

(a) $\lim_{k\to\infty} \left\| \left(\prod_{v=1}^{n_k} b_{\psi^v(i)} \right) e_{\psi^{n_k}(i)} \right\| = 0;$

(b) $\lim_{k\to\infty} \left\| \left(\prod_{v=0}^{n_k-1} b_{\varphi^v(j)} \right)^{-1} e_{\varphi^{n_k}(j)} \right\| \left\| \left(\prod_{v=1}^{n_k} b_{\psi^v(i)} \right) e_{\psi^{n_k}(i)} \right\| = 0.$

(3) T satisfies the Codisk-Cyclicity Criterion.

3 Disk-cyclic operator weighted shifts on Hilbert space $\ell^2(\mathbb{Z}, \mathcal{K})$

Bilateral operator weighted shifts on space $\ell^2(\mathbb{Z}, \mathcal{K})$ were studied by Hazarika and Arora in [7]. Here we prove that the bilateral operator weighted shifts are special weighted pseudo-shifts. Before stating the main results of this section, we settle some terminologies.

Let $\mathcal{K}$ be a separable complex Hilber space with an orthonormal basis $\{f_k\}_{k=0}^{\infty}$. Define a separable Hilbert space

$$\ell^2(\mathbb{Z}, \mathcal{K}) := \{x = (\ldots, x_{-1}, [x_0], x_1, \ldots) : x_i \in \mathcal{K} \text{ and } \sum_{i\in\mathbb{Z}} ||x_i||^2 < \infty\}$$

under the inner product $\langle x, y\rangle = \sum_{i\in\mathbb{Z}} \langle x_i, y_i\rangle_{\mathcal{K}}$.

Let $\{A_n\}_{n=-\infty}^{\infty}$ be a uniformly bounded sequence of invertible positive diagonal operators on $\mathcal{K}$. The bilateral forward and backward operator weighted shifts on $\ell^2(\mathbb{Z}, \mathcal{K})$ are defined as follows:

(i) The bilateral forward operator weighted shift T on $\ell^2(\mathbb{Z}, \mathcal{K})$ is defined by

$$T(\ldots, x_{-1}, [x_0], x_1, \ldots) = (\ldots, A_{-2}x_{-2}, [A_{-1}x_{-1}], A_0x_0, \ldots).$$

Since $\{A_n\}_{n=-\infty}^{\infty}$ is uniformly bounded, T is bounded and $||T|| = \sup_{i\in\mathbb{Z}} ||A_i|| < \infty$. For $n > 0$,

$$T^n(\ldots, x_{-1}, [x_0], x_1, \ldots) = (\ldots, y_{-1}, [y_0], y_1, \ldots),$$

where $y_j = \prod_{s=0}^{n-1} A_{j+s-n}x_{j-n}$.

(ii) The bilateral backward operator weighted shift T on $\ell^2(\mathbb{Z},\mathcal{K})$ is defined by

$$T(\ldots,x_{-1},[x_0],x_1,\ldots)=(\ldots,A_0x_0,[A_1x_1],A_2x_2,\ldots).$$

Then

$$T^n(\ldots,x_{-1},[x_0],x_1,\ldots)=(\ldots,y_{-1},[y_0],y_1,\ldots),$$

where $y_j=\prod_{s=1}^{n}A_{j+s}x_{j+n}$.

Since each A_n is an invertible diagonal operator on $\mathcal{K}$, we conclude that

$$||A_n||=\sup_k ||A_nf_k|| \text{ and } ||A_n^{-1}||=\sup_k ||A_n^{-1}f_k||.$$

Our main goal in this section is to prove the theorem stated below, which is a special case of Theorem 2.3.

Theorem 3.1. *Let T be a bilateral forward operator weighted shift on $\ell^2(\mathbb{Z},\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$. Then the following statements are equivalent:*

(1) T is disk-cyclic;

(2) There exists an increasing sequence (n_k) of positive integers such that, for every $i_1,i_2\in\mathbb{N}$ and $j_1,j_2\in\mathbb{Z}$,

(a) $\lim_{k\to\infty}\left\|\prod_{v=j_1-n_k}^{j_1-1}A_v^{-1}f_{i_1}\right\|=0;$

(b) $\lim_{k\to\infty}\left\|\prod_{v=j_1-n_k}^{j_1-1}A_v^{-1}f_{i_1}\right\|\left\|\prod_{s=j_2}^{j_2+n_k-1}A_sf_{i_2}\right\|=0.$

(3) T satisfies the Disk-Cyclicity Criterion.

Proof. We start by proving that T is a weighted pseudo-shift on the Hilbert sequence space $\ell^2(\mathbb{Z},\mathcal{K})$. For any $x=(x_j)_{j\in\mathbb{Z}}\in\ell^2(\mathbb{Z},\mathcal{K})$, since each x_j is in $\mathcal{K}$, there exist scalars $\{x_{i,j}\}_{i\in\mathbb{N}}$ such that $x_j=\sum_{i=0}^{\infty}x_{i,j}f_i$. If we identify the tuple

$$(\ldots,x_{-1},[x_0],x_1,\ldots)=(\ldots,(x_{i,(-1)})_{i\in\mathbb{N}},[(x_{i,0})_{i\in\mathbb{N}}],(x_{i,1})_{i\in\mathbb{N}},\ldots)$$

with $(x_{i,j})_{i\in\mathbb{N},j\in\mathbb{Z}}$, the space $\ell^2(\mathbb{Z},\mathcal{K})$ can be regarded as a Hilbert sequence space over $I:=\mathbb{N}\times\mathbb{Z}$.

For each $(i_0,j_0)\in I$, we define $e_{i_0,j_0}:=(\ldots,z_{-1},[z_0],z_1,\ldots)\in\ell^2(\mathbb{Z},\mathcal{K})$, by letting $z_{j_0}=f_{i_0}$ and $z_j=0$ for $j\neq j_0$. It is easy to see that $(e_{i,j})_{(i,j)\in I}$ is an OP-basis of $\ell^2(\mathbb{Z},\mathcal{K})$.

As by the hypothesis that $\{A_n\}_{n\in\mathbb{Z}}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$, there exist uniformly bounded positive sequences $\{(a_{i,n})_{i\in\mathbb{N}}\}_{n\in\mathbb{Z}}$, such that for each $n\in\mathbb{Z}$

$$A_nf_i=a_{i,n}f_i \;\text{ and }\; A_n^{-1}f_i=a_{i,n}^{-1}f_i \quad \text{for every } i\in\mathbb{N}.$$

In this interpretation, T is the operator given by

$$T(x_{i,j})_{(i,j)\in I}=(y_{i,j})_{(i,j)\in I} \quad\text{where}\quad y_{i,j}=a_{i,(j-1)}x_{i,(j-1)}.$$

Hence T is a weighted pseudo-shift $T_{b,\varphi}$ with

$$b_{i,j} = a_{i,j-1} \text{ and } \varphi(i,j) = (i,j-1) \quad \text{for } (i,j) \in I.$$

It follows from Theorem 2.3 that (1) and (3) are equivalent to the statement: There exists an increasing sequence (n_k) of positive integers such that, for every $(i_1, j_1), (i_2, j_2) \in I$

$$\begin{aligned}
\lim_{k\to\infty}\left\|\left(\prod_{v=0}^{n_k-1} b_{\varphi^v(i_1,j_1)}\right)^{-1} e_{\varphi^{n_k}(i_1,j_1)}\right\| &= \lim_{k\to\infty}\left\|\left(\prod_{v=0}^{n_k-1} b_{(i_1,j_1-v)}\right)^{-1} e_{(i_1,j_1-n_k)}\right\| \\
&= \lim_{k\to\infty}\left\|\left(\prod_{v=0}^{n_k-1} a_{(i_1,j_1-v-1)}\right)^{-1} e_{(i_1,j_1-n_k)}\right\| \\
&= \lim_{k\to\infty}\left\|\left(\prod_{v=1}^{n_k} a_{(i_1,j_1-v)}\right)^{-1} e_{(i_1,j_1-n_k)}\right\| \\
&= \lim_{k\to\infty}\left\|\prod_{v=j_1-n_k}^{j_1-1} A_v^{-1} f_{i_1}\right\| = 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{k\to\infty}\left\|\left(\prod_{v=0}^{n_k-1} b_{\varphi^v(i_1,j_1)}\right)^{-1} e_{\varphi^{n_k}(i_1,j_1)}\right\| \left\|\left(\prod_{v=1}^{n_k} b_{\psi^v(i_2,j_2)}\right) e_{\psi^{n_k}(i_2,j_2)}\right\| \\
&= \lim_{k\to\infty}\left\|\left(\prod_{v=0}^{n_k-1} a_{(i_1,j_1-v-1)}\right)^{-1} e_{(i_1,j_1-n_k)}\right\| \left\|\left(\prod_{v=1}^{n_k} a_{(i_2,j_2+v-1)}\right) e_{(i_2,j_2+n_k)}\right\| \\
&= \lim_{k\to\infty}\left\|\prod_{v=j_1-n_k}^{j_1-1} A_v^{-1} f_{i_1}\right\| \left\|\prod_{s=j_2}^{j_2+n_k-1} A_s f_{i_2}\right\| = 0,
\end{aligned}$$

which concludes the proof. ∎

By Theorem 2.1 and the same proof as for Theorem 3.1 we get the following result.

Theorem 3.2. *Let T be a bilateral forward operator weighted shift on $\ell^2(\mathbb{Z},\mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$. Then the following statements are equivalent:*

(1) T is hypercyclic;

(2) There exists an increasing sequence (n_k) of positive integers such that, for every $i \in \mathbb{N}$ and $j \in \mathbb{Z}$,

$$\lim_{k\to\infty}\left\|\prod_{v=j-n_k}^{j-1} A_v^{-1} f_i\right\| = 0 \text{ and } \lim_{k\to\infty}\left\|\prod_{v=j}^{j+n_k-1} A_v f_i\right\| = 0.$$

In [13], Liang and Zhou also provided a sufficient and necessary condition for disk-cyclic forward bilateral operator weighted shifts on $\ell^2(\mathbb{Z},\mathcal{K})$.

Claim 1. [13, Theorem 2.2] Let T be a forward bilateral operator weighted shift on $\ell^2(\mathbb{Z}, \mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$. Then the following statements are equivalent:

(1) T is disk-cyclic;

(2) For all $q \in \mathbb{N}$,

(a) $\liminf_{n\to\infty} \max \left\{ \left\| \prod_{k=j-n}^{j-1} A_k^{-1} \right\|, \ |j| \leq q \right\} = 0,$

(b) $\liminf_{n\to\infty} \max \left\{ \left\| \prod_{k=j}^{j+n-1} A_k \right\| \left\| \prod_{s=h-n}^{h-1} A_s^{-1} \right\|, \ |h|, |j| \leq q \right\} = 0;$

(3) T satisfies the Disk-Cyclicity Criterion.

However, we discover that there is a gap in the proof of "(1) $\Rightarrow$ (2)" in the above claim: in paper [13], line 21 of page 836 does not imply line 23 of page 836, since the selection of the integer n in line 21 depends on f_i.

The following counter-example demonstrates that condition (2) of Claim 1 is not necessary for disk-cyclicity.

Example 3.3. Let $\{A_n\}_{n=-\infty}^{\infty}$ be the uniformly bounded sequence of positive invertible diagonal operators on $\mathcal{K}$, defined as follows:

$$\text{if} \quad n \geq 0 : A_n(f_k) = \begin{cases} 2f_k, & 0 \leq k \leq n, \\ 3f_k, & k > n. \end{cases}$$

$$\text{if} \quad n < 0 : A_n(f_k) = 3f_k, \quad \text{for all } k \geq 0.$$

Let T be the bilateral forward operator weighted shift on $\ell^2(\mathbb{Z}, \mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$. Then

(1) T is disk-cyclic;

(2) T is not hypercyclic;

(3) T does not satisfy condition (2) of Claim 1.

Proof. To prove (1), we apply Theorem 3.1 with $(n_k) = (1, 2, 3, \cdots)$. For any fixed integers $i_1, i_2 \in \mathbb{N}$ and $j_1, j_2 \in \mathbb{Z}$, by the definition of $\{A_n\}_n$ we have

$$\left\| \prod_{v=j_1-n}^{j_1-1} A_v^{-1} f_{i_1} \right\| \leq \frac{1}{2^{|j_1|} \cdot 3^{n-|j_1|}}, \tag{3.1}$$

and

$$\left\| \prod_{v=j_1-n}^{j_1-1} A_v^{-1} f_{i_1} \right\| \left\| \prod_{s=j_2}^{j_2+n-1} A_s f_{i_2} \right\| \leq \frac{1}{2^{|j_1|} \cdot 3^{n-|j_1|}} \cdot 3^{|j_2|+i_2} \cdot 2^{n-|j_2|-i_2}, \tag{3.2}$$

when $n > |j_1| + |j_2| + i_2 + 1$.

It is obvious that condition (2) of Theorem 3.1 is satisfied, so T is disk-cyclic.

But for each integer $n \geq 1$ and any integers $i \in \mathbb{N}, j \in \mathbb{Z}$, we have

$$\left\| \prod_{v=j}^{j+n-1} A_v f_i \right\| \geq 2,$$

By Theorem 3.2, T is not hypercyclic.

For the proof of (3), letting $q = 0$ in (2) of Claim 1 we can obtain

$$\liminf_{n\to\infty} \max \left\{ \left\| \prod_{k=j}^{j+n-1} A_k \right\| \left\| \prod_{s=h-n}^{h-1} A_s^{-1} \right\|, \ |h|, |j| \leq 0 \right\}$$

$$= \liminf_{n\to\infty} \left\{ \left\| \prod_{k=0}^{n-1} A_k \right\| \left\| \prod_{s=-n}^{-1} A_s^{-1} \right\| \right\}$$
$$= \liminf_{n\to\infty} 3^n \frac{1}{3^n} = 1 \neq 0,$$

which means that T does not satisfy condition (2) of Claim 1. ∎

Remark 3.4. We note that Theorem 2.2 in paper [13] was motivated by Theorem 3.1 in [7] by Hazarika and Arora. In paper [7] Theorem 3.1 and its proof contain the same mistake as [13]. Theorem 3.2 is the correct version of it. Indeed, we have the following counter-example: Let T be the bilateral forward operator weighted shift on $\ell^2(\mathbb{Z}, \mathcal{K})$ with weight sequence defined by

$$A_n(f_k) = \begin{cases} \frac{1}{2} f_k & \text{if } n \geq k, \\ f_k & \text{if } -k < n < k, \\ 2 f_k & \text{if } n \leq -k, \end{cases}$$

Then T is hypercyclic by Theorem 3.2, but it does not satisfy condition (3.1) of Theorem 3.1 in [7].

4 Invertible shifts

In [3], Feldman showed that for bilateral weighted shifts on $\ell^2(\mathbb{Z})$ that are invertible, the characterizing conditions for hypercyclicity simplify. It is clear that if T is a bilateral operator weighted shift on $\ell^2(\mathbb{Z}, \mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, then T is invertible if and only if there exists $m > 0$ such that $||A_n^{-1}|| \leq m$ for all $n \in \mathbb{Z}$. For such shifts, the characterizing conditions of Theorem 3.1 simplify. Following Feldman [3] we notice that for this simplification it suffices to demand that there is some $m > 0$ such that $||A_n^{-1}|| \leq m$ for all $n < 0$ (or for all $n > 0$). Thus we have the following.

Theorem 4.1. *Let T be a bilateral forward operator weighted shift on $\ell^2(\mathbb{Z}, \mathcal{K})$ with weight sequence $\{A_n\}_{n=-\infty}^{\infty}$, where $\{A_n\}$ is a uniformly bounded sequence of positive invertible operators on $\mathcal{K}$ and there exists $m > 0$ such that $||A_n^{-1}|| \leq m$ for all $n < 0$*

(or for all $n > 0$). Then T is disk-cyclic if and only if there exists an increasing sequence (n_k) of positive integers such that, for every $i_1, i_2 \in \mathbb{N}$,

(a) $\lim_{k\to\infty} \left\| \prod_{v=1}^{n_k} A_{-v}^{-1} f_{i_1} \right\| = 0;$

(b) $\lim_{k\to\infty} \left\| \prod_{v=1}^{n_k} A_{-v}^{-1} f_{i_1} \right\| \left\| \prod_{s=1}^{n_k} A_s f_{i_2} \right\| = 0.$

Proof. If T is disk-cyclic the result follows from Theorem 3.1. For the converse, it is sufficient to show that for any $\varepsilon > 0, K \in \mathbb{N}$ with $K > 1$ and every $N \in \mathbb{N}$, there exists an integer $n > N$ such that for any $|j_1|, |j_2| \le K$ and $i_1, i_2 \le K$

$$\left\| \prod_{v=j_1-n}^{j_1-1} A_v^{-1} f_{i_1} \right\| < \varepsilon, \tag{4.1}$$

and

$$\left\| \prod_{v=j_1-n}^{j_1-1} A_v^{-1} f_{i_1} \right\| \left\| \prod_{s=j_2}^{j_2+n-1} A_s f_{i_2} \right\| < \varepsilon. \tag{4.2}$$

To see this, we fix $m_1 = 1$ and for $k = 2, 3, 4, \cdots$ let m_k be a number n satisfying (4.1) and (4.2) for $\varepsilon = \frac{1}{k}, K = k$ and $N = m_{k-1}$. It is clear that the increasing sequence $(m_k)_{k\ge1}$ satisfies condition (2) of Theorem 3.1, so that T is disk-cyclic.

We have to prove (4.1) and (4.2) under the assumption of (a) and (b). Firstly, we assume $||A_n^{-1}|| \le m$ for all $n < 0$. Let $\varepsilon > 0$, $K \in \mathbb{N}$ $(K > 1)$ and $N \in \mathbb{N}$ be given. Let (n_k) be a sequence satisfying (a) and (b). Then we define a sequence $(\widetilde{n_k})$ by letting $\widetilde{n_k} := n_k + K + 2$ (this choice of $\widetilde{n_k}$ guarantees that $\widetilde{n_k} + j - 1 \ge n_k + 1$ and $\widetilde{n_k} - j \ge n_k + 1$ for all j with $|j| \le K$). Then for any $j \in \mathbb{Z}$ with $|j| \le K$ and for all $i \in \mathbb{N}$ we can deduce

$$\left\| \prod_{s=j}^{j+\widetilde{n_k}-1} A_s f_i \right\| \le C_j \left\| \prod_{s=1}^{n_k} A_s f_i \right\| \left\| \prod_{s=n_k+1}^{\widetilde{n_k}+j-1} A_s \right\|$$

where $C_j = \left\| \prod_{s=1}^{j-1} A_s^{-1} \right\|$ if $1 < j \le K$, $C_j = 1$ if $j = 1$, $C_j = \left\| \prod_{s=j}^{0} A_s \right\|$ if $-K \le j < 1$. And

$$\begin{aligned} \left\| \prod_{v=j-\widetilde{n_k}}^{j-1} A_v^{-1} f_i \right\| &= \left\| \prod_{v=1-j}^{\widetilde{n_k}-j} A_{-v}^{-1} f_i \right\| \\ &\le C_j' \left\| \prod_{v=1}^{n_k} A_{-v}^{-1} f_i \right\| \left\| \prod_{v=n_k+1}^{\widetilde{n_k}-j} A_{-v}^{-1} \right\| \end{aligned}$$

where $C_j' = \left\| \prod_{v=1-j}^{0} A_{-v}^{-1} \right\|$ if $0 < j \le K$, $C_j' = 1$ if $j = 0$, $C_j' = \left\| \prod_{v=1}^{-j} A_{-v} \right\|$ if $-K \le j < 0$.

Since $\{A_n\}_{n=-\infty}^{\infty}$ is uniformly bounded, there exists $M_1 > 1$ such that

$||A_n|| < M_1$ for all $n \in \mathbb{Z}$.
By setting $C_1 := \max\{C_j : |j| \leq K\}$, $C_2 := \max\{C'_j : |j| \leq K\}$, $C := \max\{M_1,\ m\}$ we can easily obtain that for all $i \in \mathbb{N}$

$$\left\| \prod_{s=j}^{j+\widetilde{n_k}-1} A_s f_i \right\| \leq C_1 C^{2K+1} \left\| \prod_{s=1}^{n_k} A_s f_i \right\| \quad \text{for all } |j| \leq K, \tag{4.3}$$

and

$$\left\| \prod_{v=j-\widetilde{n_k}}^{j-1} A_v^{-1} f_i \right\| \leq C_2 C^{2K+2} \left\| \prod_{v=1}^{n_k} A_{-v}^{-1} f_i \right\| \quad \text{for all } |j| \leq K. \tag{4.4}$$

Combining (4.3) and (4.4) we can get that for any $|j_1|, |j_2| \leq K$ and $i_1, i_2 \in \mathbb{N}$

$$\left\| \prod_{v=j_1-\widetilde{n_k}}^{j_1-1} A_v^{-1} f_{i_1} \right\| \leq C_2 C^{2K+2} \left\| \prod_{v=1}^{n_k} A_{-v}^{-1} f_{i_1} \right\| \tag{4.5}$$

and

$$\left\| \prod_{v=j_1-\widetilde{n_k}}^{j_1-1} A_v^{-1} f_{i_1} \right\| \left\| \prod_{s=j_2}^{j_2+\widetilde{n_k}-1} A_s f_{i_2} \right\| \leq C_1 C_2 C^{4K+3} \left\| \prod_{v=1}^{n_k} A_{-v}^{-1} f_{i_1} \right\| \left\| \prod_{s=1}^{n_k} A_s f_{i_2} \right\|. \tag{4.6}$$

By (a) and (b) we can find an integer $n \in \{\widetilde{n_k}\}_k, n > N$, such that (4.1) and (4.2) hold for $|j_1|, |j_2| \leq K$ and $i_1, i_2 \leq K$.

The proof is similar when $||A_n^{-1}|| \leq m$ for all $n > 0$, in which case we just need to let $\widetilde{n_k} = n_k - K - 1$. ∎

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