On $(t-1)$-colored paths in t-colored complete graphs*

Amir Khamseh

Abstract

Given t distinct colors, we order the t subsets of $t-1$ colors in some arbitrary manner. Let $G_1, G_2, \ldots, G_t$ be graphs. The $(t-1)$-chromatic Ramsey number, denoted by $r_{t-1}^{t}(G_1, G_2, \ldots, G_t)$, is defined to be the least number n such that if the edges of the complete graph K_n are colored in any fashion with t colors, then for some i the subgraph whose edges are colored with the ith subset of colors contains a G_i. In this paper, we find the value of $r_4^5(G_1, \ldots, G_5)$ when each G_i is a path.

1 Introduction

At first, let us fix some notation and introduce some terminology. If G is a graph, V will denote its vertex set and E its edge set. The number of vertices of G is denoted by $|G|$. As usual, P_i is a path on i vertices and C_i is a cycle of length i. Recall that a t-coloring of the edges of G is a partition of E into t classes. Typically we use $1, 2, \ldots, t$ as the set of colors. For every coloring of the edges of G, E_c is the set of edges in color c and for $x \in V$, $d_c(x)$ is the number of edges incident to x in color c. An s-colored graph G is a graph whose edges are colored with a set of s colors. In particular $P_{i(c_1,c_2,\ldots,c_s)}$ and $C_{i(c_1,c_2,\ldots,c_s)}$ respectively denote a path and a cycle with i vertices whose edges are colored in $c_1, c_2, \ldots, c_s$.

Let $G_1, G_2, \ldots, G_t$ be graphs. Then $r(G_1, G_2, \ldots, G_t)$ denotes the classical t-color Ramsey number for these graphs and is defined as the least integer n such

*This research was in part supported by a grant from IPM (No. 94030059).
Received by the editors in January 2016 - In revised form in August 2017.
Communicated by A. Weiermann.
2010 *Mathematics Subject Classification* : 05C55; 05D10.
Key words and phrases : Ramsey numbers, $(t-1)$-chromatic Ramsey numbers, edge coloring.

that, in any coloring of the edges of the complete graph K_n with t colors $1, 2, \dots, t$ for some i the subgraph induced by color i contains a copy of G_i. In particular $r(G_1, G_2)$ is the smallest integer n such that in any two-coloring of the edges of the complete graph K_n there is a monochromatic copy of G_1 in the first color or a monochromatic copy of G_2 in the second color. Note that any re-ordering of the components of $(G_1, G_2, \dots, G_t)$ in the above definition will have no effect on the value of n. The two-color Ramsey number of paths was determined by Gerencsér and Gyárfás.

Theorem 1.1. *([4]) For* $2 \leq i \leq j$, $r(P_i, P_j) = j + [i/2] - 1$.

For three colors, Faudree and Schelp [3] proved that if $k \geq 6(i+j)^2$, then $r(P_i, P_j, P_k) = k + [i/2] + [j/2] - 2$ for $i, j \geq 2$ and conjectured that for all i, $r(P_i, P_i, P_i) = 2i - 2 + (i \bmod 2)$. The conjecture is true for $i \leq 9$ (see [11]) and for i large enough, it was proved in [5]. Although a formula for $r(P_{i_1}, \dots, P_{i_k})$ was presented in [3] for large i_1, the exact value of the Ramsey number of paths is not known even in the case of three colors. For more information we refer the reader to [11].

Let us now consider a special case of generalized Ramsey numbers defined by Chung and Liu [2]. The interested reader can find some results concerning d-chromatic Ramsey numbers in [1], [2], [7], and [9]. Given t distinct colors, we order the t subsets of $t-1$ colors in some arbitrary manner. The $(t-1)$-chromatic Ramsey number, denoted by $r^t_{t-1}(G_1, G_2, \dots, G_t)$, is defined to be the least number n such that if the edges of the complete graph K_n are colored in any fashion with t colors, then for some i the subgraph whose edges are colored with the ith subset of colors contains a G_i. Although in classical Ramsey numbers we are looking for a monochromatic copy of G_i in color i, in this special case of the relaxed version of Chung and Liu, we are looking for a copy of G_i in the subgraph of K_n colored in $t-1$ colors and hence G_i need not be monochromatic. We shall denote the t colors $1, 2, \dots, t$ and order the t subsets of $t-1$ colors as $A_1, A_2, \dots, A_t$, where for $i = 1, 2, \dots, t$, $A_i = \{1, 2, \dots, t\} \setminus \{i\}$. Thus for example, the 2-chromatic Ramsey number $r^3_2(G_1, G_2, G_3)$ is defined to be the least number n such that if the edges of the complete graph K_n are colored with three colors $1, 2, 3$, then there is either a G_1 in colors $2, 3$ or a G_2 in colors $1, 3$ or a G_3 in colors $1, 2$ in the graph K_n. Note that if $t = 2$, $(t-1)$-colored is the same as monochromatic and so $r^2_1(G_1, G_2) = r(G_1, G_2)$.

For graphs G_1, G_2, and G_3 with $|G_1| \leq |G_2| \leq |G_3|$ it is shown in [2] that $r^3_2(G_1, G_2, G_3) \leq r(G_1, G_2)$ and equality holds if $|G_3| \geq r(G_1, G_2)$. Theorem 1.2, is a straightforward generalization of this result. For a proof of this theorem see [9].

Theorem 1.2. *Let* $G_1, \dots, G_t$ *be graphs and* $|G_1| \leq \dots \leq |G_t|$. *Then we have* $r^t_{t-1}(G_1, \dots, G_t) \leq r^{t-1}_{t-2}(G_1, \dots, G_{t-1})$ *and equality holds if* $|G_t| \geq r^{t-1}_{t-2}(G_1, \dots, G_{t-1})$.

For further reference, we state the following corollary of Theorem 1.2.

Theorem 1.3. *Let* $2 \leq i \leq j \leq k \leq l \leq m$. *Then*

$$r^5_4(P_i, P_j, P_k, P_l, P_m) \leq r^4_3(P_i, P_j, P_k, P_l) \leq r^3_2(P_i, P_j, P_k) \leq r(P_i, P_j).$$

Moreover

- *If* $m \geq r_3^4(P_i, P_j, P_k, P_l)$, *then* $r_4^5(P_i, P_j, P_k, P_l, P_m) = r_3^4(P_i, P_j, P_k, P_l)$.
- *If* $l \geq r_2^3(P_i, P_j, P_k)$, *then* $r_3^4(P_i, P_j, P_k, P_l) = r_2^3(P_i, P_j, P_k)$.
- *If* $k \geq r(P_i, P_j)$, *then* $r_2^3(P_i, P_j, P_k) = r(P_i, P_j)$.

The exact value of the $(t-1)$-chromatic Ramsey number of paths when the number of colors is three or four is known.

Theorem 1.4. *([10]) Let* $2 \leq i \leq j \leq k$. *Then the value of* $r_2^3(P_i, P_j, P_k)$ *is equal to* $[\frac{4k+2j+i-2}{6}]$ *if* $k < r(P_i, P_j)$ *and is equal to* $r(P_i, P_j)$, *otherwise.*

Theorem 1.5. *([8]) Let* $2 \leq i \leq j \leq k \leq l$. *Then the value of* $r_3^4(P_i, P_j, P_k, P_l)$ *is equal to* $[\frac{8l+4k+2j+i-2}{14}]$ *if* $l < r_2^3(P_i, P_j, P_k)$ *and is equal to* $r_2^3(P_i, P_j, P_k)$, *otherwise.*

Following the above pattern, the authors of [8] presented the following conjecture.

Conjecture. For each $t \geq 3$, and for $n_1, n_2, \ldots, n_t$ with $n_1 \leq n_2 \leq \cdots \leq n_t$,

$$r_{t-1}^t(P_{n_1}, P_{n_2}, \ldots, P_{n_t}) = \left[\frac{\sum_{i=0}^{t-1} 2^i n_{i+1} - 2}{\sum_{i=1}^{t-1} 2^i}\right],$$

where $n_t < r_{t-2}^{t-1}(P_{n_1}, P_{n_2}, \ldots, P_{n_{t-1}})$.

Note that the conjecture is consistent with the main result of [6].

Theorem 1.6. *([6]) Every* t*-coloring of* K_n *contains a* $(t-1)$*-colored matching of size* k *provided that* $n \geq 2k + \left[\frac{k-1}{2^{t-1}-1}\right]$.

By Theorems 1.1, 1.4, 1.5, respectively for $t = 2, 3, 4$, not only a $(t-1)$-colored matching of size k can be guaranteed, but a $(t-1)$-colored path on $2k$ vertices. In this paper, we prove the above conjecture for $t = 5$ by showing that for $2 \leq i \leq j \leq k \leq l \leq m$, the value of $r_4^5(P_i, P_j, P_k, P_l, P_m)$ is equal to $[\frac{16m+8l+4k+2j+i-2}{30}]$ if $m < r_3^4(P_i, P_j, P_k, P_l)$ and is equal to $r_3^4(P_i, P_j, P_k, P_l)$, otherwise.

2 Main Result

Lemma 2.1. *Let* $2 \leq i \leq j \leq k \leq l \leq m$ *and* $i \leq 3$. *Then*

$$r_4^5(P_i, P_j, P_k, P_l, P_m) \leq [\frac{16m + 8l + 4k + 2j + i - 2}{30}].$$

Proof. By Theorems 1.3 and 1.1 and the fact that $2 \leq i \leq 3$,

$$r_4^5(P_i, P_j, P_k, P_l, P_m) \leq r(P_i, P_j) = j + [i/2] - 1 = j \leq [\frac{16m + 8l + 4k + 2j + i - 2}{30}]. \quad \blacksquare$$

Lemma 2.2. *Let* $4 \le i \le j \le k \le l \le m < r_3^4(P_i, P_j, P_k, P_l)$, *and* $s = [\frac{16m+8l+4k+2j+i-2}{30}]$. *Suppose that the edges of* K_s *are colored by* 1, 2, 3, 4 *and* 5. *If* K_s *contains either* $C_{[i-1](2,3,4,5)}$, $C_{[j-1](1,3,4,5)}$, $C_{[k-1](1,2,4,5)}$, $C_{[l-1](1,2,3,5)}$ *or* $C_{[m-1](1,2,3,4)}$, *then* K_s *contains either* $P_{i(2,3,4,5)}$, $P_{j(1,3,4,5)}$, $P_{k(1,2,4,5)}$, $P_{l(1,2,3,5)}$ *or* $P_{m(1,2,3,4)}$, *respectively.*

Proof . If $r_2^3(P_i, P_j, P_k) \le l$, then by Theorem 1.3, $m < r_3^4(P_i, P_j, P_k, P_l) \le r_2^3(P_i, P_j, P_k) \le l \le m$, a contradiction. So $l < r_2^3(P_i, P_j, P_k)$ and by Theorem 1.5, $r_3^4(P_i, P_j, P_k, P_l) = [\frac{8l+4k+2j+i-2}{14}]$. Similarly, if $r(P_i, P_j) \le k$, then by Theorem 1.3, $l < r_2^3(P_i, P_j, P_k) \le r(P_i, P_j) \le k \le l$, a contradiction. So $k < r(P_i, P_j)$ and by Theorem 1.4, $r_2^3(P_i, P_j, P_k) = [\frac{4k+2j+i-2}{6}]$. Now we get from $m < [\frac{8l+4k+2j+i-2}{14}]$ that $m \le s$ and from $l < [\frac{4k+2j+i-2}{6}]$ that $l \le r_3^4(P_i, P_j, P_k, P_l)$. Since the arguments for all five possible cases are similar, we only consider that K_s contains $C = C_{[i-1](2,3,4,5)}$ but not a $P_{i(2,3,4,5)}$ and show that K_s contains either $P_{j(1,3,4,5)}$, $P_{k(1,2,4,5)}$, $P_{l(1,2,3,5)}$ or $P_{m(1,2,3,4)}$. Let Q be the graph induced by the $q = s - (i-1) = [\frac{16m+8l+4k+2j-29i+28}{30}]$ vertices in $V(K_s) \setminus V(C)$. Since there exists no $P_{i(2,3,4,5)}$ in K_s all of the edges between C and Q have color 1. We consider two cases as follows.

Case 1. $j - 2q \le 1$. If $q < i-1$, then we have a $P_{[2q+1](1)}$ and hence a $P_{j(1)}$. If $q \ge i-1$, then there is a $P_{[2(i-1)+1](1)}$. So if $j \le 2(i-1)+1$, we have a $P_{j(1)}$. Thus we may assume that $q \ge i-1$ and $j \ge 2(i-1)+2 = 2i$. We now show that

$$r_4^5(P_i, P_{j-2(i-1)}, P_{k-2(i-1)}, P_{l-2(i-1)}, P_{m-2(i-1)}) \le q \tag{1}$$

First suppose that $i \le j - 2(i-1)$. Since $k < j + [i/2] - 1$, $k - 2(i-1) < j - 2(i-1) + [i/2] - 1$ and so

$$\begin{aligned} r_4^5(P_i, P_{j-2(i-1)}, P_{k-2(i-1)}, P_{l-2(i-1)}, P_{m-2(i-1)}) &\le r_2^3(P_i, P_{j-2(i-1)}, P_{k-2(i-1)}) \\ &= [\tfrac{4(k-2(i-1))+2(j-2(i-1))+i-2}{6}] \\ &= [\tfrac{4k+2j-11i+10}{6}] \\ &\le [\tfrac{16m+8l+4k+2j-29i+28}{30}] = q. \end{aligned}$$

Note that for the first inequality we use Theorem 1.3 and then we use Theorem 1.4.

If $j - 2(i-1) < i \le k - 2(i-1)$, then $l \ge k \ge 3i-2$ and so by Theorems 1.3 and 1.1,

$$\begin{aligned} r_4^5(P_i, P_{j-2(i-1)}, P_{k-2(i-1)}, P_{l-2(i-1)}, P_{m-2(i-1)}) &\le r(P_{j-2(i-1)}, P_i) \\ &= i + [(j-2(i-1))/2] - 1 = [j/2] \\ &\le [\tfrac{16m+8l+4k+2j-29i+28}{30}] = q. \end{aligned}$$

If $k - 2(i-1) < i$, then $3i > k+2$ and so by Theorems 1.3 and 1.1,

$$\begin{aligned} r_4^5(P_i, P_{j-2(i-1)}, P_{k-2(i-1)}, P_{l-2(i-1)}, P_{m-2(i-1)}) \\ &\leq r(P_{j-2(i-1)}, P_{k-2(i-1)}) \\ &= k - 2(i-1) + [(j-2(i-1))/2] - 1 \\ &= [\tfrac{2k+j-6i+4}{2}] \\ &\leq [\tfrac{16m+8l+4k+2j-29i+28}{30}] = q. \end{aligned}$$

We have thus proved (1). Hence Q contains either $P_{i(2,3,4,5)}$, $P_{[j-2(i-1)](1,3,4,5)}$, $P_{[k-2(i-1)](1,2,4,5)}$, $P_{[l-2(i-1)](1,2,3,5)}$, or $P_{[m-2(i-1)](1,2,3,4)}$. Denote this path by P. If $P = P_{i(2,3,4,5)}$, we are done. So let $P \neq P_{i(2,3,4,5)}$. Since $s = q + (i-1) \geq m \geq j \geq 2i$, $q \geq m - i + 1$ and so there are at least $i-1$ vertices of Q not in P (see Figure 1). So $i-1$ such vertices together with $i-1$ vertices of C make the monochro-

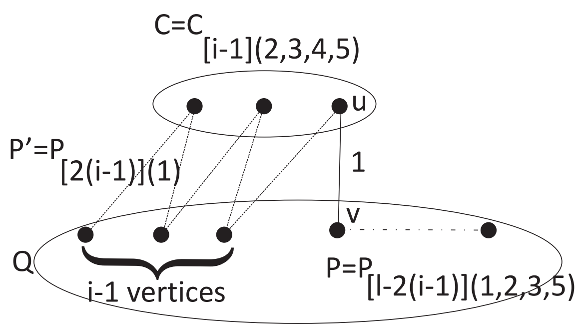

Figure 1: Graph K_s

matic path $P' = P_{[2(i-1)](1)}$. Remembering that the vertices of C are joined to the vertices of Q by edges of color 1, note that the constructed path P' visits alternatingly a vertex of C and a vertex among the extra $i-1$ vertices of Q. Let u be the end-vertex of P' that does lie on C and v be an end-vertex of P. Clearly u is joined to v by an edge of color 1. We now add P' to P to obtain either $P_{j(1,3,4,5)}$, $P_{k(1,2,4,5)}$, $P_{l(1,2,3,5)}$ or $P_{m(1,2,3,4)}$.

Case 2. $j - 2q \geq 2$. We shall show that

$$r_4^5(P_i, P_{j-2q}, P_{k-2q}, P_{l-2q}, P_{m-2q}) \leq i - 1. \tag{2}$$

Since $30q > 16m + 8l + 4k + 2j - 29i - 2$, we have $30q - 15m + 15i > m + 8l + 4k + 2j - 14i - 2 > 0$, which implies $m - 2q < i$ and so by Theorem 1.3, $r_4^5(P_i, P_{j-2q}, P_{k-2q}, P_{l-2q}, P_{m-2q}) \leq r_3^4(P_{j-2q}, P_{k-2q}, P_{l-2q}, P_{m-2q})$. By definition of q, $30q > 16m + 8l + 4k + 2j - 29i - 2$ so $30q - 8m - 4l - 2k - j + 14i + 2 > 0$, which implies $[\frac{8m+4l+2k+j-2-30q}{14}] \leq i - 1$.
Hence if $m - 2q < r_2^3(P_{j-2q}, P_{k-2q}, P_{l-2q})$,

$$\begin{aligned} r_4^5(P_i, P_{j-2q}, P_{k-2q}, P_{l-2q}, P_{m-2q}) &\leq r_3^4(P_{j-2q}, P_{k-2q}, P_{l-2q}, P_{m-2q}) \\ &= [\tfrac{8(m-2q)+4(l-2q)+2(k-2q)+j-2q-2}{14}] \\ &= [\tfrac{8m+4l+2k+j-2-30q}{14}] \\ &\leq i - 1, \end{aligned}$$

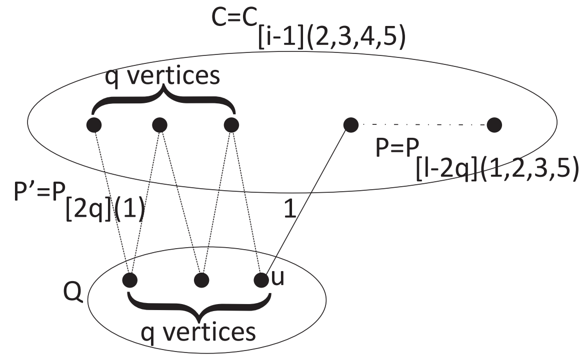

Figure 2: Graph K_s

Note that for the first inequality we use Theorem 1.3 and then we use Theorem 1.5.
On the other hand, if $r_2^3(P_{j-2q}, P_{k-2q}, P_{l-2q}) \leq m-2q$, remembering that $m-2q < i$, by Theorems 1.3 and 1.5,

$$\begin{aligned} r_4^5(P_i, P_{j-2q}, P_{k-2q}, P_{l-2q}), P_{m-2q}) &\leq r_3^4(P_{j-2q}, P_{k-2q}, P_{l-2q}, P_{m-2q}) \\ &= r_2^3(P_{j-2q}, P_{k-2q}, P_{l-2q}) \\ &\leq m-2q \\ &\leq i-1. \end{aligned}$$

We have thus proved (2). Hence in the subgraph induced by the $i-1$ vertices of C, there exist either $P_{i(2,3,4,5)}$, $P_{[j-2q](1,3,4,5)}$, $P_{[k-2q](1,2,4,5)}$, $P_{[l-2q](1,2,3,5)}$ or $P_{[m-2q](1,2,3,4)}$. Denote this path by P. If $P = P_{i(2,3,4,5)}$, we are done. So let $P \neq P_{i(2,3,4,5)}$. Since $s = i-1+q \geq m$, C contains at least q vertices not in P (see Figure 2). So q such vertices together with q vertices of Q make the monochromatic path $P' = P_{[2q](1)}$. Remembering that the vertices of Q are joined to the vertices of C by edges of color 1, note that the constructed path P' visits alternatingly a vertex of Q and a vertex among the extra q vertices of C. Let u be the end-vertex of P' that does lie on Q. Clearly u is joined to the end-vertices of P by edges of color 1. We now add P' to P to obtain either $P_{j(1,3,4,5)}$, $P_{k(1,2,4,5)}$, $P_{l(1,2,3,5)}$, or $P_{m(1,2,3,4)}$. □

Lemma 2.3. *Let* $4 \leq i \leq j \leq k \leq l \leq m < r_3^4(P_i, P_j, P_k, P_l)$, $s = [\frac{16m+8l+4k+2j+i-2}{30}]$, *and suppose that the edges of* $G = K_s$ *are colored by* 1, 2, 3, 4 *and* 5. *Let* $f(1) = i$, $f(2) = j$, $f(3) = k$, $f(4) = l$, *and* $f(5) = m$. *Fix* $\alpha \in \{1, \dots, 5\}$. *Suppose that there exists a vertex* x_1 *of* G *such that* $d_\alpha(x_1) \geq d_\gamma(x)$ *for each* γ, $1 \leq \gamma \leq 5$, *and for each vertex* x *of* G. *If* $G - x_1$ *contains either of* $P_{[f(\beta)-2+g(\beta)](\{1,\dots,5\} \setminus \{\beta\})}$, *where* β *ranges over all elements of* $\{1, \dots, 5\} \setminus \{\alpha\}$ *and* $g(\beta) = 0$ *if* $\beta < \alpha$ *and* $g(\beta) = 1$ *if* $\beta > \alpha$, *then* G *contains either of* $P_{[f(\beta)](\{1,\dots,5\} \setminus \{\beta\})}$, *respectively, where again* $\beta \in \{1, \dots, 5\} \setminus \{\alpha\}$.

Proof . We prove the lemma for $\alpha = 5$ and leave the other cases to the reader. Hence there exists a vertex x_1 of G such that $n = d_5(x_1) \geq d_\gamma(x)$ for each γ, $1 \leq \gamma \leq 5$, and for each vertex x of G. We must show that if $G - x_1$ contains either $P_{[i-2](2,3,4,5)}$, $P_{[j-2](1,3,4,5)}$, $P_{[k-2](1,2,4,5)}$, or $P_{[l-2](1,2,3,5)}$, then G contains either

$P_{i(2,3,4,5)}$, $P_{j(1,3,4,5)}$, $P_{k(1,2,4,5)}$, or $P_{l(1,2,3,5)}$, respectively. Since the arguments for all four possible cases are similar, we consider that $G - x_1$ contains $P_{[k-2](1,2,4,5)}$ but G does not contain $P_{k(1,2,4,5)}$ and show that G contains either $P_{i(2,3,4,5)}$, $P_{j(1,3,4,5)}$, or $P_{l(1,2,3,5)}$. Let y and z be the end-vertices of $P_{[k-2](1,2,4,5)}$ and denote this path by $P = (y, \ldots, z)$. If both of x_1y and x_1z are in $E_1 \cup E_2 \cup E_4 \cup E_5$, then the assertion follows from Lemma 2.2. Otherwise, we consider two cases as follows.

Case 1. $x_1y \in E_1 \cup E_2 \cup E_4 \cup E_5$ and $x_1z \in E_3$. First note that since there exists no $P_{k(1,2,4,5)}$, $x_1w \notin E_5$ for each $w \in V \setminus (V(P) \cup \{x_1\})$. This means that all of the n vertices joined to x_1 by edges of color 5 are in $V(P)$. Let $v \neq y$ be a vertex of P with $x_1v \in E_5$. Then v splits P into a yv-path P' and a vz-path. Let $u \in N(v)$ such that $uv \in E(P')$. Then $zu \in E_3$, since otherwise existence of the cycle $(x_1v \ldots zu \ldots y)$ implies existence of the desired path by Lemma 2.2. Note that $v \ldots z$ refers to the subpath of P between v and z. Summarizing, for each edge $x_1v \in E_5$ we get an edge $zu \in E_3$, plus the edge $zx_1 \in E_3$, we see that $d_3(z) \geq d_5(x_1)$ (and it is possible that $x_1y \in E_5$). Let $w \in V \setminus (V(P) \cup \{x_1\})$. If $zw \in E_1 \cup E_2 \cup E_4 \cup E_5$, then $(x_1y \ldots zw)$ is a $P_{k(1,2,4,5)}$, which is impossible. Hence $zw \in E_3$, where $w \in V \setminus (V(P) \cup \{x_1\})$. So $d_3(z) \geq d_5(x_1) + 1$. This contradicts our assumption that $d_5(x_1) \geq d_\gamma(x)$ for each γ, $1 \leq \gamma \leq 5$, and for each vertex x of G.

Case 2. $x_1y, x_1z \in E_3$. Let x_1 be adjacent to n_1 vertices of $V \setminus V(P)$ in color 5. First suppose that $n_1 > 0$. Let w be any vertex of $V \setminus (V(P) \cup \{x_1\})$ with $x_1w \in E_5$. If $zw \in E_1 \cup E_2 \cup E_4 \cup E_5$, then $(y \ldots zwx_1)$ is a $P_{k(1,2,4,5)}$. This contradiction shows that $zw \in E_3$. That is, for each edge $x_1w \in E_5$ we get an edge $zw \in E_3$. Hence if $n = n_1$, then $d_3(z) \geq n_1$, plus the edge $zx_1 \in E_3$, we see that $d_3(z) \geq d_5(x_1) + 1$. This contradicts our assumption that $d_5(x_1) \geq d_\gamma(x)$ for each γ, $1 \leq \gamma \leq 5$, and for each vertex x of G. Thus we may suppose that $n > n_1$. Let v be a vertex of P with $x_1v \in E_5$. Then v splits P into a yv-path P' and a vz-path. Let $u \in N(v)$ such that $uv \in E(P')$. Then $zu \in E_3$, since otherwise $(wx_1v \ldots zu \ldots y)$ is a $P_{k(1,2,4,5)}$. Summarizing, for each edge $x_1v \in E_5$ we get an edge $zu \in E_3$, plus the edge $zx_1 \in E_3$, we see that $d_3(z) \geq d_5(x_1) + 1$. This contradicts our assumption that $d_5(x_1) \geq d_\gamma(x)$ for each γ, $1 \leq \gamma \leq 5$, and for each vertex x of G.

We now turn to the case $n_1 = 0$. That is, x_1 is adjacent to n vertices of $V(P)$ by edges of color 5. Let v be a vertex of P with $x_1v \in E_5$. Then v splits P into a yv-path P' and a vz-path P''. Let $u \in N(v)$ such that $uv \in E(P')$.

Claim. $zu \in E_3$.
Proof of claim. Suppose, contrary to our claim, that $zu \in E_1 \cup E_2 \cup E_4 \cup E_5$. We aim to get the contradiction $d_3(y) > d_5(x_1)$. Let us first outline the proof. For each vertex a with $x_1a \in E_5$ we get a vertex b with $yb \in E_3$ to conclude that $d_3(y) \geq d_5(x_1)$, and then we find an extra vertex w with $yw \in E_3$. First note that, if $x_1u \in E_5$, then $yz \in E_3$, since otherwise existence of the cycle $(v \ldots zy \ldots ux_1)$ implies existence of the desired path by Lemma 2.2. Let $v' \neq u$ be a vertex of the subpath $(y \ldots u)$ of P' with $x_1v' \in E_5$. Then v' splits P into a yv'-path P'_1 and a $v'z$-path P''_1. Let $u' \in N(v')$ such that $u'v' \in E(P''_1)$. Then $yu' \in E_3$, since otherwise existence of the cycle $(u' \ldots uz \ldots vx_1v' \ldots y)$ implies existence of the

desired path by Lemma 2.2. Similarly, let v'' be a vertex of the subpath $(v \dots z)$ of P'' with $x_1v'' \in E_5$. Then v'' splits P into a yv''-path P_2' and a $v''z$-path P_2''. Let $u'' \in N(v'')$ such that $u''v'' \in E(P_2')$. Then $yu'' \in E_3$, since otherwise existence of the cycle $(u'' \dots vx_1v'' \dots zu \dots y)$ (also existence of the cycle $(vx_1v'' \dots zu \dots y)$, when $v = u''$) implies existence of the desired path by Lemma 2.2. Summarizing, for each edge $x_1v' \in E_5$ and $x_1v'' \in E_5$ we get an edge $yu' \in E_3$ and $yu'' \in E_3$, respectively, plus the edges $x_1v \in E_5$ and $yx_1 \in E_3$, we see that $d_3(y) \geq d_5(x_1)$. Let $w \in V \setminus (V(P) \cup \{x_1\})$. If $yw \in E_1 \cup E_2 \cup E_4 \cup E_5$, then $(x_1v \dots zu \dots yw)$ is a $P_{k(1,2,4,5)}$. So $yw \in E_3$, where $w \in V \setminus (V(P) \cup \{x_1\})$. Therefore $d_3(y) \geq d_5(x_1) + 1$. This contradiction completes the proof of our claim. ⊣

Hence by the claim, for each edge $x_1v \in E_5$ we get an edge $zu \in E_3$, plus the edge $zx_1 \in E_3$, we have $d_3(z) \geq d_5(x_1) + 1$, which is impossible. ∎

Theorem 2.4. *Let* $2 \leq i \leq j \leq k \leq l \leq m < r_3^4(P_i, P_j, P_k, P_l)$. *Then*

$$r_4^5(P_i, P_j, P_k, P_l, P_m) \leq [\frac{16m + 8l + 4k + 2j + i - 2}{30}].$$

Proof . The assertion holds for $i \leq 3$ by Lemma 2.1. Let $i \geq 4$ and the edges of $G = K_s$ be colored by 1, 2, 3, 4, and 5, where $s = [\frac{16m+8l+4k+2j+i-2}{30}]$. We saw in Lemma 2.2 that $s \geq m$. The proof is by induction. First suppose that $i = j$. Since all the eight possible cases use completely similar arguments, we only consider that $i = j < k = l < m$ and leave the other cases to the reader. Moreover, without loss of generality we can consider three cases as follows.

Case 1. There exists a vertex x_1 such that $n = d_5(x_1) \geq d_\gamma(x)$ for each γ, $1 \leq \gamma \leq 5$, and for each vertex x of G. If $m \geq r_3^4(P_{i-2}, P_{j-2}, P_{k-2}, P_{l-2})$, by Theorems 1.3 and 1.5 we obtain

$$\begin{aligned} &r_4^5(P_{i-2}, P_{j-2}, P_{k-2}, P_{l-2}, P_m) \\ &\quad = r_3^4(P_{i-2}, P_{j-2}, P_{k-2}, P_{l-2}) \\ &\quad = \begin{cases} [\frac{8(l-2)+4(k-2)+2(j-2)+i-4}{14}] & \text{if } l-2 < r_2^3(P_{i-2}, P_{j-2}, P_{k-2}), \\ r_2^3(P_{i-2}, P_{j-2}, P_{k-2}) & \text{if } r_2^3(P_{i-2}, P_{j-2}, P_{k-2}) \leq l-2 \end{cases} \\ &\quad \leq s-1, \end{aligned}$$

and if $m < r_3^4(P_{i-2}, P_{j-2}, P_{k-2}, P_{l-2})$, then by the induction hypothesis,

$$\begin{aligned} &r_4^5(P_{i-2}, P_{j-2}, P_{k-2}, P_{l-2}, P_m) \leq \\ &\qquad [\frac{16m + 8(l-2) + 4(k-2) + 2(j-2) + (i-2) - 2}{30}] = s - 1. \end{aligned}$$

So $G - x_1$ contains either $P_{[i-2](2,3,4,5)}$, $P_{[j-2](1,3,4,5)}$, $P_{[k-2](1,2,4,5)}$, $P_{[l-2](1,2,3,5)}$, or $P_{m(1,2,3,4)}$. If $P_{m(1,2,3,4)}$ is present, there is nothing to prove. If $P_{[i-2](2,3,4,5)}$, $P_{[j-2](1,3,4,5)}$, $P_{[k-2](1,2,4,5)}$, or $P_{[l-2](1,2,3,5)}$ is present, G contains the desired path by Lemma 2.3.

Case 2. There exists a vertex x_1 such that $n = d_4(x_1) \geq d_\gamma(x)$ for each γ, $1 \leq \gamma \leq 5$, and for each vertex x of G. We shall show that $r_4^5(P_{i-2}, P_{j-2}, P_{k-2}, P_l, P_{m-1}) \leq$

$s-1$. If $r_3^4(P_{i-2}, P_{j-2}, P_{k-2}, P_l) \leq m-1$, then

$$r_4^5(P_{i-2}, P_{j-2}, P_{k-2}, P_l, P_{m-1}) = r_3^4(P_{i-2}, P_{j-2}, P_{k-2}, P_l) \leq m-1 \leq s-1,$$

and if $m-1 < r_3^4(P_{i-2}, P_{j-2}, P_{k-2}, P_l)$, by induction hypothesis,

$$r_4^5(P_{i-2}, P_{j-2}, P_{k-2}, P_l, P_{m-1}) \leq \left[\frac{16(m-1)+8l+4(k-2)+2(j-2)+(i-2)-2}{30}\right] = s-1.$$

So $G - x_1$ contains either $P_{[i-2](2,3,4,5)}$, $P_{[j-2](1,3,4,5)}$, $P_{[k-2](1,2,4,5)}$, $P_{l(1,2,3,5)}$, or $P_{[m-1](1,2,3,4)}$. If $P_{l(1,2,3,5)}$ is present, there is nothing to prove. If $P_{[i-2](2,3,4,5)}$, $P_{[j-2](1,3,4,5)}$, $P_{[k-2](1,2,4,5)}$, or $P_{[m-1](1,2,3,4)}$ is present, G contains the desired path by Lemma 2.3.

Case 3. There exists a vertex x_1 such that $n = d_2(x_1) \geq d_\gamma(x)$ for each γ, $1 \leq \gamma \leq 5$, and for each vertex x of G. We leave it to the reader to verify that $r_4^5(P_{i-2}, P_j, P_{k-1}, P_{l-1}, P_{m-1}) \leq s-1$ and so $G - x_1$ contains either $P_{[i-2](2,3,4,5)}$, $P_{j(1,3,4,5)}$, $P_{[k-1](1,2,4,5)}$, $P_{[l-1](1,2,3,5)}$, or $P_{[m-1](1,2,3,4)}$. If $P_{j(1,3,4,5)}$ is present, there is nothing to prove. If $P_{[i-2](2,3,4,5)}$, $P_{[k-1](1,2,4,5)}$, $P_{[l-1](1,2,3,5)}$, or $P_{[m-1](1,2,3,4)}$ is present, G contains the desired path by Lemma 2.3.

Now suppose that $i < j$. Let x_1 be a vertex with $\Sigma_{n=2}^5 d_n(x_1) \leq \Sigma_{n=2}^5 d_n(x)$, for each vertex x. That is, among the vertices of G, x_1 has the minimum value in the sum of the degrees in colors $2, \ldots, 5$ and hence the maximum degree in color 1. If $\Sigma_{n=2}^5 d_n(x_1) \geq \lceil j/2 \rceil$ and the subgraph induced by $\cup_{n=2}^5 E_n$ is connected then G, by the standard result stating that every connected graph G has a path of length at least $\min\{2\delta(G), |G|-1\}$, contains a $P_{[2\lceil j/2 \rceil](2,3,4,5)}$ and hence a $P_{i(2,3,4,5)}$. Otherwise, if the subgraph induced by $\cup_{n=2}^5 E_n$ is disconnected, then all of its components are of order at least $\lceil j/2 \rceil$ and so G contains $P_{j(1)}$. Thus we may suppose that $\Sigma_{n=2}^5 d_n(x_1) \leq [j/2]$. It is obvious that $G - x_1$ contains either $P_{i(2,3,4,5)}$, $P_{[j-1](1,3,4,5)}$, $P_{[k-1](1,2,4,5)}$, $P_{[l-1](1,2,3,5)}$, or $P_{[m-1](1,2,3,4)}$. Suppose that one of the latter four paths is present and denote it by P. Since $d_1(x_1) \geq s-1-[j/2]$, x_1 is adjacent to two successive vertices of P by edges of color 1, which implies the desired path. □

Theorem 2.5. *Let* $2 \leq i \leq j \leq k \leq l \leq m < r_3^4(P_i, P_j, P_k, P_l)$. *Then*

$$r_4^5(P_i, P_j, P_k, P_l, P_m) > \left[\frac{16m+8l+4k+2j+i-2}{30}\right] - 1.$$

Proof . Let $s = \left[\frac{16m+8l+4k+2j+i-2}{30}\right]$, $x_1 = \left\lceil\frac{8m+4l+2k+j-i-1-12s}{3}\right\rceil$, $x_2 = \left[\frac{4m+2l+k-j+i-2-6s}{3}\right]$, $x_3 = 4s-2m-l-k$, $x_4 = 2s-l-m$, and $x_5 = s-m$. First note that $x_1 + x_2 = 4m+2l+k-1-6s$. Moreover by $m < r_3^4(P_i, P_j, P_k, P_l)$ and the definition of s and x_i's, it is straightforward to check that

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = s - 1, \\ x_1 + x_2 + x_3 + x_4 = m - 1, \\ x_1 + x_2 + x_3 + 2x_5 = l - 1, \\ x_1 + x_2 + 2x_4 + 2x_5 = k - 1, \\ x_1 + 2x_3 + 2x_4 + 2x_5 \leq j - 1, \\ 2x_2 + 2x_3 + 2x_4 + 2x_5 + 1 \leq i - 1, \\ x_1 > 0, \\ x_i \geq 0, 2 \leq i \leq 5. \end{cases} \tag{3}$$

Now partition the vertices of K_{s-1} into five sets X_i, $1 \leq i \leq 5$ with $|X_i| = x_i$. Paint with 1 all edges which are incident with two vertices of X_1. For $i = 2, 3, 4, 5$, paint with i the edges having two vertices in X_i or one vertex in X_i and one vertex in X_j where $j < i$. The conditions in (3) guarantee that K_{s-1} does not contain $P_{i(2,3,4,5)}$, $P_{j(1,3,4,5)}$, $P_{k(1,2,4,5)}$, $P_{l(1,2,3,5)}$, and $P_{m(1,2,3,4)}$. □

Corollary 2.6. *Let* $2 \leq i \leq j \leq k \leq l \leq m$. *Then* $r_4^5(P_i, P_j, P_k, P_l, P_m)$ *is equal to* $[\frac{16m+8l+4k+2j+i-2}{30}]$ *if* $m < r_3^4(P_i, P_j, P_k, P_l)$ *and is equal to* $r_3^4(P_i, P_j, P_k, P_l)$, *otherwise.*

Acknowledgements

The author would like to thank the anonymous referee for several valuable comments and suggestions which significantly improved the paper. In particular, the present form of the statement of Lemma 2.3 and some other explanations are due to the referee.

References

[1] K.M. Chung, M.L. Chung and C.L. Liu, A generalization of Ramsey theory for graphs-with stars and complete graphs as forbidden subgraphs, *Congr. Numer.* 19 (1977) 155-161.

[2] K.M. Chung and C.L. Liu, A generalization of Ramsey theory for graphs, *Discrete Math.* 2 (1978) 117-127.

[3] R.J. Faudree and R.H. Schelp, Path Ramsey Numbers in Multicolorings, *Journal of Combinatorial Theory, Series B* 19 (1975) 150-160.

[4] L. Gerencsér and A. Gyárfás, On Ramsey-type problems, *Ann. Univ. Sci. Budapest Eötvös.* 10 (1967) 167-170.

[5] A. Gyárfás, M. Ruszinkó, G.N. Sárközy and E. Szemerédi, Three-color Ramsey Numbers for Paths, *Combinatorica* 27 (2007) 35-69. Corrigendum in 28 (2008) 499-502.

[6] A. Gyárfás, G.N. Sárközy and S. Selkow, Coverings by few monochromatic pieces: a transition between two Ramsey problems, *Graphs Combin.*, 31(1) (2015) 131-140.

[7] H. Harborth and M. Möller, Weakened Ramsey numbers, *Discrete Applied Math.* 95 (1999) 279-284.

[8] A. Khamseh and R. Omidi, A generalization of Ramsey theory for linear forests, *Int. J. Comput. Math.*, 89(10) (2012) 1303-1310.

[9] A. Khamseh and R. Omidi, A generalization of Ramsey theory for stars and one matching, *Math. Reports*, 19(69), 1 (2017), 85-92.

[10] R. Meenakshi and P.S. Sundararaghavan, Generalized Ramsey numbers for paths in 2-chromatic graphs, *Internat. J. Math. Sci.* 9 (1986) 273-276.

[11] S.P. Radziszowski, Small Ramsey numbers, *Electronic J. Combin.* 1 (1994) Dynamic Surveys, DS1.15 (March 3, 2017).

Department of Mathematics, Kharazmi University,
15719-14911 Tehran Iran
and
School of Mathematics,
Institute for Research in Fundamental Sciences (IPM),
PO Box 19395-5746 Tehran Iran
email :khamseh@khu.ac.ir