Laplacian solitons on nilpotent Lie groups

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Abstract

We investigate the existence of closed G_2-structures which are solitons for the Laplacian flow on nilpotent Lie groups. We obtain that seven of the twelve Lie algebras admitting a closed G_2-structure do admit a Laplacian soliton. Moreover, one of them admits a continuous family of Laplacian solitons which are pairwise non-homothetic and the Laplacian flow evolution on four of the Lie groups is not diagonal.

1 Introduction

A closed G_2-structure φ on a 7-manifold M is said to be a *Laplacian soliton* if

$$\Delta_\varphi \varphi = \lambda\varphi + \mathcal{L}_X \varphi, \tag{1.1}$$

for some $c \in \mathbb{R}$ and vector field X on M, where Δ_φ is the Hodge Laplacian on forms defined by φ and $\mathcal{L}_X$ denotes the Lie derivative. Laplacian solitons are also characterized as the G_2-structures that evolves self-similarly under the Laplacian flow $\frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t)$ introduced by Bryant in [B] (see [LoW] for further information).

For left-invariant G_2-structures on a simply connected Lie group G, one has the following 'algebraic' versions of Laplacian solitons (see [L2]): a *semi-algebraic soliton* is a Laplacian soliton for which the field X is defined by the one-parameter subgroup of automorphisms of G associated to some derivation D of the Lie algebra $\mathfrak{g}$ of G. If D^t is also a derivation, then it is called an *algebraic soliton*, which is known to be equivalent to evolve 'diagonally' under the Laplacian flow (see [L2, Theorem 4.10]).

Conti and Fernández proved in [CF] that there are, up to isomorphism, twelve 7-dimensional nilpotent Lie algebras that admit a left-invariant closed G_2-structure.

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On the other hand, Fernández, Fino and Manero studied in [FFM] the existence of left-invariant closed G_2-structures defining a Ricci soliton metric among the Lie algebras given in [CF]. It is also natural to ask which of these twelve Lie algebras admit a closed Laplacian soliton. In this paper, we find a closed Laplacian soliton on each of the first seven Lie algebras. Our main result is summarized as follows.

Theorem 1.1. *For each $i = 1, \ldots, 7$, let $\mathfrak{n}_i$ be the Lie algebra given in Table 1.*

(i) $\mathfrak{n}_2$ *admits an algebraic soliton (see Table 2).*

(ii) $\mathfrak{n}_3$ *admits a pairwise non-homothetic one-parameter family of algebraic solitons (see Table 2).*

(iii) Each of $\mathfrak{n}_4, \mathfrak{n}_5, \mathfrak{n}_6, \mathfrak{n}_7$ *does admit a semi-algebraic soliton which is not algebraic (see Table 3 and Table 4).*

The Laplacian solitons obtained are all expanding (i.e. $\lambda > 0$ in (1.1)). It is not hard to see that in the cases $\mathfrak{n}_1$ and $\mathfrak{n}_2$, the Laplacian soliton is also a Ricci soliton. In cases $\mathfrak{n}_4$ and $\mathfrak{n}_6$, the Laplacian soliton we found is not a Ricci soliton, though $\mathfrak{n}_4$ and $\mathfrak{n}_6$ are known to admit closed G_2-structures with Ricci soliton associated metrics. The remaining algebras $\mathfrak{n}_3$, $\mathfrak{n}_5$ and $\mathfrak{n}_7$ do not admit a closed G_2-structure with Ricci soliton associated metric (see [FFM]).

The family of non-homothetic Laplacian solitons found on $\mathfrak{n}_3$ shows that the uniqueness up to isometry and scaling of Ricci solitons on nilpotent Lie algebras (see [L1]) does not hold in the Laplacian case. This abundance of solitons on the same nilpotent Lie algebra is kind of unexpected, bearing in mind that the uniqueness of the solitons seems to hold even for some other geometric flows like Chern-Ricci flow (see [LR]) and symplectic curvature flow (see[LW]). Another relevant difference between Laplacian and Ricci solitons is the fact that any homogeneous Ricci soliton is isometric to an algebraic soliton (see [J]). On the contrary, we proved that four of the Lie algebras admit semi-algebraic Laplacian solitons that are not equivalent to any algebraic soliton.

It would be desirable to find a Laplacian soliton on every Lie algebra in Table 1, but the computations became very complicated. Indeed, the Ricci soliton on $\mathfrak{n}_{10}$, whose existence was proved in [FC, Example 2], is not known explicitly and the existence of a closed G_2-structure with a Ricci soliton associated metric on $\mathfrak{n}_{10}$ is still open (see [FFM, Remark 3.5]).

2 Preliminaries

Given a 7-dimensional differentiable manifold M, we consider a differentiable 3-form $\varphi \in \Omega^3 M$. For each $p \in M$, φ_p is said to be *positive* if there exists a basis $\{e_1, \ldots, e_7\}$ of T_pM such that

$$\varphi_p = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{2.1}$$

where $e^{ijk} := e^i \wedge e^j \wedge e^k$ and $\{e^1, \ldots, e^7\}$ is the dual basis of $\{e_1, \ldots, e_7\}$. When φ_p is positive for every $p \in M$, we call φ a *G_2-structure* (see [B, LoW, L2] for further information on G_2-structures). Any G_2-structure induces a Riemannian metric g_φ and an orientation, and so a Hodge star operator denoted by $*_\varphi : \Omega M \to \Omega M$. The Hodge star operator in combination with the differential of forms on M define the Hodge Laplacian operator Δ_φ. In particular, on 3-forms, $\Delta_\varphi : \Omega^3 M \to \Omega^3 M$ is given by $\Delta_\varphi = *_\varphi d *_\varphi d - d *_\varphi d *_\varphi$.

For a one-parameter family $\varphi(t)$ of G_2-structures on M, we have a natural geometric flow, introduced by R. Bryant in 1992, given by

$$\frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t), \tag{2.2}$$

so called the *Laplacian flow* (see [B]). A G_2-structure φ on a 7-differentiable manifold flows in a self-similar way along the Laplacian flow, i.e. the solution $\varphi(t)$ with $\varphi(0) = \varphi$ has the form

$$\varphi(t) = c(t) f(t)^* \varphi, \quad \text{for some } c(t) \in \mathbb{R}^* \text{ and } f(t) \in \mathrm{Diff}(M),$$

if and only if

$$\Delta_\varphi \varphi = c\varphi + \mathcal{L}_X \varphi, \qquad \text{for some} \quad c \in \mathbb{R}, \quad X \in \mathfrak{X}(M) \text{ (complete)},$$

where $\mathcal{L}_X$ denotes the Lie derivative. In that case, $c(t) = \left(\frac{2}{3}ct + 1\right)^{3/2}$ and φ is called a *Laplacian soliton*. Furthermore, φ is said to be *expanding*, *steady* or *shrinking*, when $c > 0$, $c = 0$ or $c < 0$, respectively.

A G_2-structure φ on a 7-differentiable manifold is said to be *closed* if $d\varphi = 0$. In the closed case, the intrinsic torsion is only given by the 2-form

$$\tau_\varphi = - *_\varphi d *_\varphi \varphi, \qquad d\tau_\varphi = \Delta_\varphi \varphi.$$

We now consider a 7-dimensional vector space $\mathfrak{g}$. It is known that a 3-form $\psi \in \Lambda^3 \mathfrak{g}^*$ is positive, i.e. ψ can be written as

$$\varphi_0 := e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{2.3}$$

relative to some basis $\{e_1, \dots, e_7\}$ of $\mathfrak{g}$, if and only if ψ is in the orbit $\mathrm{GL}(\mathfrak{g}) \cdot \varphi_0$. Here the action is given by,

$$(h \cdot \phi)(X_1, \dots, X_k) = \phi(h^{-1}X_1, \dots, h^{-1}X_k), \qquad \forall X_1, \dots, X_k \in \mathfrak{g}, \quad \phi \in \Lambda^k \mathfrak{g}^*. \tag{2.4}$$

Also, we know that φ_0 induces an inner product on $\mathfrak{g}$ as follows:

$$\langle X, Y \rangle_{\varphi_0} \mathrm{vol}_0 := \tfrac{1}{6} \iota_X \varphi_0 \wedge \iota_Y \varphi_0 \wedge \varphi_0,$$

where $\mathrm{vol}_0 := e^{1\dots 7}$ and ι_X is defined by $(\iota_X \phi)(\cdot, \cdot) := \phi(X, \cdot, \cdot)$. It is easy to check that the basis $\{e_1, \dots, e_7\}$ is orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle_{\varphi_0}$ and oriented relative to vol_0.

Every positive 3-form $\psi = h \cdot \varphi_0$ with $h \in \mathrm{GL}(\mathfrak{g})$ defines an inner product $\langle \cdot, \cdot \rangle_\psi$ and a volume form vol_ψ by

$$\langle \cdot, \cdot \rangle_\psi := \langle h^{-1} \cdot, h^{-1} \cdot \rangle_{\varphi_0}, \qquad \mathrm{vol}_\psi := h \cdot \mathrm{vol}_0 \,. \tag{2.5}$$

If $\{f_1, \dots, f_7\}$ is an orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_\psi)$, then we also denote by $\langle \cdot, \cdot \rangle_\psi$ the inner product on $\Lambda^k \mathfrak{g}^*$, which makes of $\{f^{i_1 \dots i_k} : i_1 < \cdots < i_k\}$ an orthonormal basis.

The following facts are direct consequences of the above definitions.

Lemma 2.1. *Let $\mathfrak{g}$ be a 7-dimensional vector space. If $X, Y \in \mathfrak{g}$, $h \in \mathrm{GL}(\mathfrak{g})$ and $\psi \in \Lambda^3\mathfrak{g}^*$ is positive, then,*

(i) $\langle X, Y\rangle_\psi \operatorname{vol}_\psi = \frac{1}{6}\iota_X\psi \wedge \iota_Y\psi \wedge \psi$.

(ii) $\langle X, Y\rangle_{h\cdot\psi} = \langle h^{-1}X, h^{-1}Y\rangle_\psi, \quad \forall X, Y \in \mathfrak{g}, \quad$ *(i.e.* $\langle\cdot,\cdot\rangle_{h\cdot\psi} = h\cdot\langle\cdot,\cdot\rangle_\psi$*).*

(iii) $\langle\cdot,\cdot\rangle_{c\psi} = c^{\frac{2}{3}}\langle\cdot,\cdot\rangle_\psi, \quad \forall c \in \mathbb{R}^*$.

For our next lemma, we need to introduce a definition. Let $\mathfrak{g}$ be a Lie algebra and G the corresponding simply connected Lie group. We note that each positive 3-form $\varphi \in \Lambda^3\mathfrak{g}^*$ defines a left-invariant G_2-structure on G. Given $D \in \mathrm{Der}(\mathfrak{g})$ and $t \in \mathbb{R}$, we denote by $f_t \in \mathrm{Aut}(G)$ the automorphism such that $df_t|_e = e^{tD} \in \mathrm{Aut}(\mathfrak{g})$ and by X_D the corresponding vector field on G:

$$X_D(a) := \frac{d}{dt}\Big|_0 f_t(a), \quad \forall a \in G.$$

It is easy to prove that the Lie derivative of a left-invariant form $\psi \in \Lambda^k\mathfrak{g}^*$ with respect to X_D is given by

$$(\mathcal{L}_{X_D}\psi)(X_1,\dots,X_k) := \psi(DX_1, X_2,\dots,X_k) + \cdots + \psi(X_1, X_2,\dots,DX_k), \quad (2.6)$$

for all $X_1,\dots,X_k \in \mathfrak{g}$. The proofs of the following results are all straightforward.

Lemma 2.2. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra and consider $\psi \in \Lambda^k\mathfrak{g}^*$, $h \in \mathrm{Aut}(\mathfrak{g})$.*

(i) $d(h\cdot\psi) = h\cdot d\psi$.

(ii) If $k = 3$ and ψ is positive, then

(a) $\Delta_{h\cdot\psi} h\cdot\psi = h\cdot\Delta_\psi\psi$.

(b) $\Delta_{c\psi} c\psi = c^{\frac{1}{3}}\Delta_\psi\psi, \quad \forall c \in \mathbb{R}^*$.

(iii) $\mathcal{L}_{X_{hDh^{-1}}}(h\cdot\psi) = h\cdot\mathcal{L}_{X_D}\psi$, *for any* $D \in \mathrm{Der}(\mathfrak{g})$.

Laplacian solitons on Lie groups have been deeply studied in [L2]. The following definition will be used from now on along the paper.

Definition 2.3. Given $\mathfrak{g}$ a 7-dimensional Lie algebra and ψ a positive 3-form on $\mathfrak{g}$, we call $(\mathfrak{g}, \psi)$ a *semi-algebraic soliton* if there exist $D \in \mathrm{Der}(\mathfrak{g})$ and $\lambda \in \mathbb{R}$ such that

$$\Delta_\psi\psi = \mathcal{L}_{X_D}\psi + \lambda\psi. \quad (2.7)$$

In the case when $D^t \in \mathrm{Der}(\mathfrak{g})$, we say that $(\mathfrak{g}, \psi)$ is an *algebraic soliton*.

Let $\theta : \mathfrak{gl}(\mathfrak{g}) \to \mathrm{End}(\Lambda^3\mathfrak{g}^*)$ be the derivative of the action given by (2.4), i.e.

$$\theta(A)\psi(\cdot,\cdot,\cdot) = -\psi(A\cdot,\cdot,\cdot) - \psi(\cdot,A\cdot,\cdot) - \psi(\cdot,\cdot,A\cdot), \qquad \forall A \in \mathfrak{gl}(\mathfrak{g}), \quad \psi \in \Lambda^3\mathfrak{g}^*.$$

It is shown in [L2, (11)] that for any closed G_2-structure ψ on $\mathfrak{g}$, there exists a unique symmetric operator $Q_\psi \in \mathfrak{gl}(\mathfrak{g})$ such that $\theta(Q_\psi)\psi = \Delta_\psi\psi$. The following useful formula for Q_ψ was given in [L2, Proposition 2.2]: for any closed G_2-structure ψ,

$$Q_\psi = \mathrm{Ric}_\psi - \frac{1}{12}\operatorname{tr}(\tau_\psi^2)I + \frac{1}{2}\tau_\psi^2, \tag{2.8}$$

where Ric_ψ is the Ricci operator of (G, g_ψ) and $\tau_\psi \in \mathfrak{so}(TG)$ also denotes the skew-symmetric operator determined by the 2-form τ_ψ (i.e. $\tau_\psi = \langle \tau_\psi \cdot, \cdot \rangle_\psi$).

According to [L2, Proposition 4.5], $(\mathfrak{g}, \psi)$ is a semi-algebraic soliton with $\Delta_\psi\psi = \mathcal{L}_{X_D}\psi + \lambda\psi$, if and only if $Q_\psi = -\frac{1}{3}\lambda I - \frac{D+D^t}{2}$. Recall that ψ is an algebraic soliton if and only if $\frac{D+D^t}{2} \in \mathrm{Der}(\mathfrak{g})$.

Definition 2.4. We say that two G_2-structures $(\mathfrak{g}_1, \psi_1)$ and $(\mathfrak{g}_2, \psi_2)$ are *equivalent* if there exists a Lie algebra isomorphism $h : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $h \cdot \psi_1 = \psi_2$. We denote it briefly by $(\mathfrak{g}_1, \psi_1) \simeq (\mathfrak{g}_2, \psi_2)$. Also, we say that $(\mathfrak{g}_1, \psi_1)$ and $(\mathfrak{g}_2, \psi_2)$ are *homothetic* if there exists $c \in \mathbb{R}^*$ such that $(\mathfrak{g}_1, \psi_1) \simeq (\mathfrak{g}_2, c\psi_2)$.

Proposition 2.5. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra, $\psi_1, \psi_2 \in \Lambda^2\mathfrak{g}^*$ positive such that $(\mathfrak{g}, \psi_1)$ and $(\mathfrak{g}, \psi_2)$ are homothetic. Then $(\mathfrak{g}, \psi_1)$ is a semi-algebraic soliton if and only if $(\mathfrak{g}, \psi_2)$ is so.*

Proof. Recall that $(\mathfrak{g}, \psi_1)$ is semi-algebraic soliton if and only if there exist $D \in \mathrm{Der}(\mathfrak{g})$ and $\lambda \in \mathbb{R}$ such that $\Delta_{\psi_1}\psi_1 = \mathcal{L}_{X_D}\psi_1 + \lambda\psi_1$. Therefore, by Lemma 2.2, we have that

$$\begin{aligned} c^{\frac{1}{3}}\Delta_{\psi_2}\psi_2 &= \Delta_{c\psi_2}(c\psi_2) = \Delta_{h\cdot\psi_1}(h\cdot\psi_1) = h\cdot\Delta_{\psi_1}\psi_1 = h\cdot(\mathcal{L}_{X_D}\psi_1 + \lambda\psi_1) \\ &= \mathcal{L}_{X_{hDh^{-1}}}(h\cdot\psi_1) + \lambda(h\cdot\psi_1) = \mathcal{L}_{X_{hDh^{-1}}}(c\psi_2) + \lambda(c\psi_2) \\ &= c\mathcal{L}_{X_{hDh^{-1}}}\psi_2 + c\lambda\psi_2. \end{aligned}$$

So, $\Delta_{\psi_2}\psi_2 = c^{\frac{2}{3}}\mathcal{L}_{X_{hDh^{-1}}}\psi_2 + c^{\frac{2}{3}}\lambda\psi_2 = \mathcal{L}_{X_{c^{\frac{2}{3}}hDh^{-1}}}\psi_2 + c^{\frac{2}{3}}\lambda\psi_2$. Since $c^{\frac{2}{3}}hDh^{-1} \in \mathrm{Der}(\mathfrak{g})$, we conclude that $(\mathfrak{g}, \psi_2)$ is a semi-algebraic soliton. ∎

3 Closed Laplacian solitons

In [CF], Conti and Fernández studied the existence of closed G_2-structures on a 7-dimensional nilpotent Lie algebra. They obtained that, up to isomorphism, there are 12 nilpotent Lie algebras with that property, which are shown in Table 1. It is of interest to know whether these Lie algebras admit closed Laplacian solitons.

We prove that for the first seven Lie algebras of the table, there exists at least one closed Laplacian soliton.

Theorem 3.1. *For each $i = 1, \dots, 7$, let $\mathfrak{n}_i$ be the Lie algebra given in Table 1.*

(i) *$\mathfrak{n}_2$ admits an algebraic soliton (see Table 2).*

(ii) *$\mathfrak{n}_3$ admits a pairwise non-homothetic one-parameter family of algebraic solitons (see Table 2).*

$\mathfrak{g}$	Lie bracket
$\mathfrak{n}_1$	$[\cdot,\cdot]=0$
$\mathfrak{n}_2$	$[e_1,e_2]=-e_5,[e_1,e_3]=-e_6$
$\mathfrak{n}_3$	$[e_1,e_2]=-e_4,[e_1,e_3]=-e_5,[e_2,e_3]=-e_6$
$\mathfrak{n}_4$	$[e_1,e_2]=-e_3,[e_1,e_3]=-e_6,[e_2,e_4]=-e_6,[e_1,e_5]=-e_7$
$\mathfrak{n}_5$	$[e_1,e_2]=-e_3,[e_1,e_3]=-e_6,[e_1,e_4]=-e_7,[e_2,e_5]=-e_7$
$\mathfrak{n}_6$	$[e_1,e_2]=-e_4,[e_1,e_3]=-e_5,[e_1,e_4]=-e_6,[e_1,e_5]=-e_7$
$\mathfrak{n}_7$	$[e_1,e_2]=-e_4,[e_1,e_3]=-e_5,[e_1,e_4]=-e_6,[e_2,e_3]=-e_6,[e_1,e_5]=-e_7$
$\mathfrak{n}_8$	$[e_1,e_2]=-e_3,[e_1,e_3]=-e_4,[e_2,e_3]=-e_5,[e_1,e_5]=-e_6,$ $[e_2,e_4]=-e_6,[e_1,e_6]=-e_7,[e_3,e_4]=-e_7$
$\mathfrak{n}_9$	$[e_1,e_2]=-e_3,[e_1,e_3]=-e_4,[e_2,e_3]=-e_5,[e_1,e_5]=-e_6,$ $[e_2,e_4]=-e_6,[e_1,e_6]=-e_7,[e_3,e_4]=-e_7,[e_2,e_5]=-e_7$
$\mathfrak{n}_{10}$	$[e_1,e_2]=-e_3,[e_1,e_3]=-e_5,[e_2,e_4]=-e_5,[e_1,e_4]=-e_6,$ $[e_4,e_6]=-e_7,[e_3,e_4]=-e_7,[e_1,e_5]=-e_7,[e_2,e_3]=-e_7$
$\mathfrak{n}_{11}$	$[e_1,e_2]=-e_3,[e_1,e_3]=-e_5,[e_2,e_4]=-e_6,[e_2,e_3]=-e_6,$ $[e_2,e_5]=-e_7,[e_3,e_4]=-e_7,[e_1,e_5]=-e_7,[e_1,e_6]=-e_7,[e_2,e_6]=3e_7$
$\mathfrak{n}_{12}$	$[e_1,e_2]=-e_4,[e_2,e_3]=-e_5,[e_1,e_3]=e_6,[e_2,e_6]=-2e_7,$ $[e_3,e_4]=2e_7,[e_1,e_6]=2e_7,[e_2,e_5]=-2e_7$

Table 1: Nilpotent Lie algebras that admit a closed G_2-structure (see [CF]).

(iii) Each of $\mathfrak{n}_4, \mathfrak{n}_5, \mathfrak{n}_6, \mathfrak{n}_7$ *does admit a semi-algebraic soliton which is not algebraic (see Table 3 and Table 4).*

Proof. We only give a proof for the cases $\mathfrak{n}_3$ and $\mathfrak{n}_4$, the other cases follow in much the same way.

To prove that $\mathfrak{n}_3$ admits a family of algebraic solitons up to isomorphism and scaling, we consider $\mathfrak{n}_3(a,b,c)$ to be the 7-dimensional Lie algebra with basis $\{e_1,\dots,e_7\}$ and Lie bracket defined by

$$[e_1,e_2] = -ae_4, \quad [e_1,e_3] = -be_5, \quad [e_2,e_3] = -ce_6, \quad a,b,c \in \mathbb{R}^*,$$

or equivalently,

$$de^{12} = ae^4, \quad de^{13} = be^5, \quad de^{23} = ce^6, \qquad a,b,c \in \mathbb{R}^*. \tag{3.1}$$

We have a linear isomorphism that carries $\mathfrak{n}_3(1,1,1)$ into $\mathfrak{n}_3(a,b,c)$, whose matrix is $\mathrm{Diag}(1,1,a,ab/c,1,ab,d)$. From now on $\mathfrak{n}_3$ denotes $\mathfrak{n}_3(a,b,c)$. We consider the 3-form

$$\varphi_3 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3\mathfrak{n}_3^*.$$

If $h_3 \in \mathrm{GL}_7(\mathbb{R})$ is the permutation $(1,6,4,3,5,2,7)$, then $h_3 \cdot \varphi_3 = \varphi_0$, which implies that φ_3 is positive. It is easy to check by using (3.1) that $d\varphi_3 = (a-b-c)e^{1237}$, so φ_3 is closed if and only if $a = b + c$. If we assume φ_3 to be closed, then the Laplacian can be computed as follows:

$$\begin{aligned}
*\varphi_3 &= -e^{1247} - e^{1256} - e^{1346} + e^{1357} + e^{2345} + e^{2367} + e^{4567},\\
d * \varphi_3 &= ae^{12567} - be^{13467} + ce^{23457},\\
*d * \varphi_3 &= ce^{16} - be^{25} + ae^{34},\\
d * d * \varphi_3 &= -(a^2+b^2+c^2)e^{123}.
\end{aligned}$$

By replacing in the condition $a = b + c$, we obtain $\Delta_{\varphi_3}\varphi_3 = 2(b^2+c^2+bc)e^{123}$.

What is left to show is that $\Delta_{\varphi_3}\varphi_3 = \mathcal{L}_{X_D}\varphi_3 + \lambda\varphi_3$ for some $D \in \mathrm{Der}(\mathfrak{n}_3)$ and $\lambda \in \mathbb{R}$. We propose $D := d\,\mathrm{Diag}(1,1,1,2,2,2,2)$ with $d \in \mathbb{R}^*$, so the resulting Lie derivative of φ_3 with respect to the field X_D is

$$\mathcal{L}_{X_D}\varphi_3 = 3de^{123} + 5de^{145} + 5de^{167} + 5de^{246} - 5de^{257} - 5de^{347} - 5de^{356}.$$

It follows that $\Delta_{\varphi_3}\varphi_3 = \mathcal{L}_{X_D}\varphi_3 + \lambda\varphi_3$ if and only if $\lambda = -5d$ and $d = -(b^2+c^2+bc)$. Since $D = D^t$, one obtains that $(\mathfrak{n}_3, \varphi_3)$ is an algebraic soliton.

Lemma 3.2. *If* $a,b,c \in \mathbb{R}^*$ *and* $\mathfrak{n}_3(a,b,c)$ *are as above, then*

(i) φ_3 *is closed if and only if* $a = b + c$.

(ii) $(\mathfrak{n}_3(b+c,b,c), \varphi_3)$ *is an algebraic soliton.*

Remark 3.3. For all $b,c \in \mathbb{R}^*$, the algebraic soliton $(\mathfrak{n}_3(b+c,b,c), \varphi_3)$ is expanding since $\lambda > 0$.

As we have two free parameters, it is natural to ask whether there are two non-equivalent algebraic solitons on $\mathfrak{n}_3$.

Proposition 3.4. *There exists a pairwise non-homothetic continuous family of algebraic solitons on* $\mathfrak{n}_3$.

Remark 3.5. This is in contrast to the known uniqueness up to isometry and scaling of Ricci solitons on nilpotent Lie algebras (see [L1]).

Proof. By using *e.g.* the formula for the Ricci operator given in [L1, (8)], it is easy to see that

$$\mathrm{Ric}_{b,c} = \frac{1}{2}\,\mathrm{Diag}(-a^2-b^2, -a^2-c^2, -b^2-c^2, a^2, b^2, c^2, 0),$$

where $a = b + c$. Clearly, $\mathrm{Ric}_{b,c}$ has three positives eigenvalues, one equal to zero and three negatives for each $b, c \in \mathbb{R}^*$. If we set $b := 1-t$ and $c := t$ with $t \in (0, \frac{1}{2})$, then for every $t \in (0, \frac{1}{2})$ the positive eigenvalues are ordered in the following way:

$$\frac{t^2}{2} < \frac{1-2t+t^2}{2} < \frac{1}{2}.$$

Now, if $(\mathfrak{n}_3(b_1+c_1, b_1, c_1), \varphi_3)$ and $(\mathfrak{n}_3(kb_2+kc_2, kb_2, kc_2), \varphi_3)$ are equivalent for some $k \in \mathbb{R}^*$ (where $b_i = 1-t_i$, $c_i = t_i$), then there are in particular isometric, hence

$$\frac{1}{2} = k^2\frac{1}{2}, \quad \frac{1-2t_1+t_1^2}{2} = k^2\frac{1-2t_2+t_2^2}{2}, \quad \frac{t_1^2}{2} = k^2\frac{t_2^2}{2},$$

which implies that $k^2 = 1$ and $t_1 = t_2$.

In the case when $t = \frac{1}{2}$, the Ricci operator results $\mathrm{Ric} = \mathrm{Diag}(-\frac{5}{8}, -\frac{5}{8}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}, 0)$, which has two of the three positive eigenvalues equal. Thus $\mathfrak{n}_3(1, \frac{1}{2}, \frac{1}{2})$ is non-homothetic to $\mathfrak{n}_3(1, 1-t, t)$ for any $t \in (0, \frac{1}{2})$; concluding the proof of the proposition. ∎

Remark 3.6. Let R_φ denote the scalar curvature of φ, i.e. $R_\varphi = \mathrm{tr}\,\mathrm{Ric}_\varphi$. The number $\frac{R_\varphi^2}{|\,\mathrm{Ric}_\varphi\,|^2}$ is therefore an invariant up to isometry or scaling. For $(\mathfrak{n}_3(b+c, b, c), \varphi_3)$, $\frac{R_{b,c}^2}{|\,\mathrm{Ric}_{b,c}\,|^2} = \frac{1}{2}$ for all $b, c \in \mathbb{R}^*$, so it can not be used to prove non-homothety.

It follows from (2.8) that $Q_{\varphi_3} = \frac{a^2+b^2+c^2}{6}\,\mathrm{Diag}(-2, -2, -2, 1, 1, 1, 1)$. Note that this coincides with $-\frac{1}{3}\lambda I - D$ above.

We can now proceed to the proof of part (iii) for $\mathfrak{n}_4$. Let $\mathfrak{n}_4 = \mathfrak{n}_4(a, b, c, d)$ be the 7-dimensional nilpotent Lie algebra with basis $\{e_1, \ldots, e_7\}$ and Lie bracket given by

$$[e_1, e_2] = -ae_3, \quad [e_1, e_3] = -be_6, \quad [e_2, e_4] = -ce_6, \quad [e_1, e_5] = -de_7 \quad a, b, c, d \in \mathbb{R}^*,$$

or equivalently,

$$de^3 = ae^{12}, \quad de^6 = be^{13} + ce^{24}, \quad de^7 = de^{15} \qquad a, b, c, d \in \mathbb{R}^*.$$

We have a linear isomorphism that carries $\mathfrak{n}_4(1,1,1,1)$ into $\mathfrak{n}_4(a,b,c,d)$, whose matrix is $\mathrm{Diag}(1,1,a,ab/c,1,ab,d)$. From now on $\mathfrak{n}_4$ denotes $\mathfrak{n}_4(a,b,c,d)$.

We consider the 3-form

$$\varphi_4 = -e^{124} - e^{456} + e^{347} + e^{135} + e^{167} + e^{257} - e^{236} \in \Lambda^3\mathfrak{n}_4^*.$$

Let $h_4 \in \mathrm{GL}_7(\mathbb{R})$ be the permutation $(1,-6,3,4,5,2,7)$, then $h_4 \cdot \varphi_4 = \varphi_0$, which implies that φ_4 is positive.

Lemma 3.7. *If $a,b,c,d \in \mathbb{R}^*$ and $\mathfrak{n}_4(a,b,c,d)$ is as above, then*

(i) φ_4 is closed if and only if $a=c$ and $b=d$.

(ii) If $a^2 = 2b^2$, then $(\mathfrak{n}_4(a,b,a,b),\varphi_3)$ is a semi-algebraic soliton.

Proof. It is easy to see that $d\varphi_4 = (a-c)e^{1247} + (d-b)e^{1345}$, so φ_4 is closed if and only if $a=c$ and $b=d$. Assuming φ_4 to be closed we proceed to compute the Laplacian $\Delta_{\varphi_4}\varphi_4$:

$$\begin{aligned} *\varphi_4 &= e^{3567} + e^{1237} + e^{1256} - e^{2467} + e^{2345} + e^{1346} + e^{1457}, \\ d*\varphi_4 &= ae^{12567} - ce^{23457} + be^{12347} + de^{12456}, \\ *d*\varphi_4 &= ae^{34} - ce^{16} + be^{56} - de^{37}, \\ d*d*\varphi_4 &= (a^2+c^2)e^{124} - (b^2+d^2)e^{135} - bce^{245} - ade^{127}. \end{aligned}$$

Replacing in the condition $a=c$ and $b=d$, we obtain $\Delta_{\varphi_4}\varphi_4 = -2a^2e^{124} + 2b^2e^{135} + abe^{245} + abe^{127}$.

To prove that $(\mathfrak{n}_4,\varphi_4)$ is a semi-algebraic soliton, we have to find some $\lambda \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{n}_4)$ such that $\Delta_{\varphi_4}\varphi_4 = \lambda\varphi_4 + \mathcal{L}_{X_D}\varphi_4$. We propose

$$D := \begin{bmatrix} -b^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2b^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3b^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2b^2 & 0 & 0 & 0 \\ -ab & 0 & 0 & 0 & -3b^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4b^2 & 0 \\ 0 & 0 & 0 & -ab & 0 & 0 & -4b^2 \end{bmatrix}$$

and $\lambda = 9b^2$. Then the Lie derivative equals

$$\begin{aligned} \mathcal{L}_{X_D}\varphi_4 =& 5b^2e^{124} + 9b^2e^{456} - abe^{146} - 9b^2e^{347} - 7b^2e^{135} - 9b^2e^{167} \\ &+ abe^{146} - 9b^2e^{257} + abe^{127} + abe^{245} + 9b^2e^{236}. \end{aligned}$$

The soliton equation holds if $a^2 = 2b^2$, i.e. if $a^2 = 2b^2$ then

$$\mathcal{L}_{X_D}\varphi_4 + 9b^2\varphi_4 = -4b^2e^{124} + 2b^2e^{135} + abe^{127} + abe^{245} = \Delta_{\varphi_4}\varphi_4.$$

Note that $(\mathfrak{n}_4(a,b,a,b),\varphi_4)$ is not an algebraic soliton. Indeed, $D^t \notin \mathrm{Der}(\mathfrak{n}_4)$ since $[D^te_2,e_7] + [e_2,D^te_7] = -ab[e_2,e_4] = abce_6 \neq 0 = D^t[e_2,e_7]$. ∎

Remark 3.8. For every $a,b \in \mathbb{R}^*$ such that $a^2 = 2b^2$, $(\mathfrak{n}_4(a,b,a,b),\varphi_4)$ is an expanding semi-algebraic soliton since $\lambda > 0$.

On the other hand, we are interested in computing Q_{φ_4}. It is not hard to see that $\mathrm{Ric}_{\varphi_4} = \mathrm{Diag}\left(-\frac{a^2+b^2+d^2}{2}, -\frac{a^2+c^2}{2}, \frac{a^2-b^2}{2}, -\frac{c^2}{2}, -\frac{d^2}{2}, \frac{b^2+c^2}{2}, \frac{d^2}{2}\right)$ and $\tau_{\varphi_4} = -ae^{34} + ce^{16} - be^{56} + de^{37}$. It follows from (2.8) that

$$Q_{\varphi_4} = \begin{bmatrix} -\frac{\alpha}{3} & 0 & 0 & 0 & \frac{bc}{2} & 0 & 0 \\ 0 & \frac{\alpha-3a^2-3c^2}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha-3b^2-3d^2}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha-3a^2-3c^2}{6} & 0 & 0 & \frac{ad}{2} \\ \frac{bc}{2} & 0 & 0 & 0 & \frac{\alpha-3b^2-3d^2}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha}{6} & 0 \\ 0 & 0 & 0 & \frac{ad}{2} & 0 & 0 & \frac{\alpha}{6} \end{bmatrix},$$

where $\alpha = a^2 + b^2 + c^2 + d^2$. Thus, $Q_{\varphi_4} = -\frac{1}{3}\lambda I - D$, where λ and D are as above.

The remaining cases are analogous and the following lemmas provide information about them.

Lemma 3.9. *If $a, b \in \mathbb{R}^*$, $\mathfrak{n}_2(a,b)$ is the Lie algebra with Lie bracket $[e_1, e_2] = -ae_5$, $[e_1, e_3] = -be_6$ and $\varphi_2 := e^{147} + e^{267} + e^{357} + e^{123} + e^{156} + e^{245} - e^{346}$, then*

(i) φ_2 is closed if and only if $a = b$.

(ii) $(\mathfrak{n}_2(a,a), \varphi_2)$ is an algebraic soliton.

Lemma 3.10. *If $\mathfrak{n}_5(a,b,c,d)$ is the Lie algebra with Lie bracket given by*

$$[e_1, e_2] = -ae_3, \quad [e_1, e_3] = -be_6, \quad [e_1, e_4] = -ce_7, \quad [e_2, e_5] = -de_7,$$

where $a, b, c, d \in \mathbb{R}^*$ and $\varphi_5 := e^{134} + e^{457} - e^{246} - e^{125} - e^{356} + e^{167} - e^{237}$, then

(i) φ_5 is closed if and only if $a = d$ and $b = c$.

(ii) If $a^2 = 2b^2$, then $(\mathfrak{n}_5(a,b,b,a), \varphi_5)$ is a semi-algebraic soliton.

Lemma 3.11. *If $a, b, c, d \in \mathbb{R}^*$, $\mathfrak{n}_6(a,b,c,d)$ is the Lie algebra with Lie bracket given by*

$$[e_1, e_2] = -ae_4, \quad [e_1, e_3] = -be_5, \quad [e_1, e_4] = -ce_6, \quad [e_1, e_5] = -de_7$$

and $\varphi_6 := e^{123} + e^{347} + e^{356} + e^{145} - e^{246} + e^{167} + e^{257}$, then

(i) φ_6 is closed if and only if $a = b$ and $c = d$.

(ii) If $a^2 = 2c^2$, then $(\mathfrak{n}_6(a,a,c,c), \varphi_6)$ is a semi-algebraic soliton.

Lemma 3.12. *If $\mathfrak{n}_7(a,b,c,d,e)$ is the Lie algebra with Lie bracket given by*

$$[e_1, e_2] = -ae_4, \quad [e_1, e_7] = -be_6, \quad [e_2, e_7] = -ce_5, [e_5, e_7] = -de_3, \quad [e_6, e_7] = -ee_4,$$

where $a, b, c, d, e \in \mathbb{R}^*$ and $\varphi_7 := e^{127} + e^{135} - e^{146} - e^{236} - e^{245} + e^{347} + e^{567}$, then

(i) φ_7 is closed if and only if $a = -b - c$ and $d = e$.

(ii) If $e^2 = \frac{b^2+c^2+bc}{2}$, then $(\mathfrak{g}_7(b+c,b,c,e,e), \varphi_7)$ is a semi-algebraic soliton.

	$\mathfrak{n}_2$	$\mathfrak{n}_3$
$[\cdot,\cdot]$	$[e_1,e_2]=-e_5,\ [e_1,e_3]=-e_6.$	$[e_1,e_2]=-e_4,\ [e_1,e_3]=(c-1)e_5,$ $[e_2,e_3]=-ce_6,\quad 0<c\leq 1/2.$
φ	$e^{147}+e^{267}+e^{357}$ $+e^{123}+e^{156}+e^{245}-e^{346}$	$e^{123}+e^{145}+e^{167}$ $+e^{246}-e^{257}-e^{347}-e^{356}$
τ_φ	$-e^{35}+e^{26}$	$-ce^{16}+(1-c)e^{25}-e^{34}$
$\Delta_\varphi\varphi$	$2e^{123}$	$2(1-c+c^2)e^{123}$
Ric_φ	$-\operatorname{Diag}\left(1,\frac{1}{2},\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2},0\right)$	$\frac{1}{2}\operatorname{Diag}(-2+2c-c^2,-1-c^2,$ $-1+2c-2c^2,1,(-1+c)^2,c^2,0)$
R_φ	-1	$-1+c-c^2$
$\frac{R_\varphi^2}{tr\,\mathrm{Ric}_\varphi^2}$	$\frac{1}{2}$	$\frac{1}{2}$
λ	5	$5(1-c+c^2)$
D	$-\operatorname{Diag}(1,1,1,2,2,2,2)$	$-(1-c+c^2)\operatorname{Diag}(1,1,1,2,2,2,2)$

Table 2:

The Lie bracket given in the above lemma is isomorphic to the one given by Table 1, the isomorphism is given by:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & be/a & 0 & 0 & 0 & 0 \\ 0 & bcde/a & 0 & 0 & 0 & 0 & bcde/a \\ 0 & 0 & -be & 0 & 0 & be & 0 \\ 0 & 0 & 0 & 0 & -bce/a & 0 & 0 \\ 0 & 0 & 0 & -b & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This concludes the proof of the theorem. ■

The following tables provide information about the solitons found for the Lie algebras $\mathfrak{n}_2,\ldots,\mathfrak{n}_7$. For any Lie algebra $\mathfrak{n}_i$, the 3-form φ, the number λ and the derivation D given in the tables are such that

$$\Delta_\varphi\varphi=\mathcal{L}_{X_D}\varphi+\lambda\varphi.$$

Note that for each i, the given λ is always positive. This implies that all the Laplacian solitons are expanding.

Using (2.8), we computed Q_φ for any $\mathfrak{n}_i$ with $i=2,\ldots,7$:

	$\mathfrak{n}_4$	$\mathfrak{n}_5$
$[\cdot,\cdot]$	$[e_1,e_2]=-\sqrt{2}e_3,\ [e_1,e_3]=-e_6,$ $[e_2,e_4]=-\sqrt{2}e_6,\ [e_1,e_5]=-e_7.$	$[e_1,e_2]=-\sqrt{2}e_3,\ [e_1,e_3]=-e_6,$ $[e_1,e_4]=-e_7,\ [e_2,e_5]=-\sqrt{2}e_7.$
φ	$-e^{124}-e^{456}+e^{347}$ $+e^{135}+e^{167}+e^{257}-e^{236}$	$e^{134}+e^{457}-e^{246}$ $-e^{125}-e^{356}+e^{167}-e^{237}$
τ_φ	$-\sqrt{2}e^{34}+\sqrt{2}e^{16}-e^{56}+e^{37}$	$-e^{46}+e^{37}-\sqrt{2}e^{35}+\sqrt{2}e^{17}$
$\Delta_\varphi\varphi$	$-4e^{124}+2e^{135}+\sqrt{2}e^{245}+\sqrt{2}e^{127}$	$2e^{134}+\sqrt{2}e^{127}+\sqrt{2}e^{235}-4e^{125}$
Ric_φ	$\mathrm{Diag}\left(-2,-2,\frac{1}{2},-1,-\frac{1}{2},\frac{3}{2},\frac{1}{2}\right)$	$\mathrm{Diag}\left(-2,-2,\frac{1}{2},-\frac{1}{2},-1,\frac{1}{2},\frac{3}{2}\right)$
R_φ	-3	-3
$\frac{{R_\varphi}^2}{tr\,\mathrm{Ric}_\varphi^2}$	$\frac{3}{4}$	$\frac{3}{4}$
λ	9	9
D	$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & -4 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & -4 \end{bmatrix}$

Table 3:

	$\mathfrak{n}_6$	$\mathfrak{n}_7$
$[\cdot,\cdot]$	$[e_1,e_2]=-\sqrt{2}e_4,\ [e_1,e_3]=-\sqrt{2}e_5$ $[e_1,e_4]=-e_6,\ [e_1,e_5]=-e_7.$	$[e_1,e_2]=4e_4,\ [e_1,e_7]=-2e_6,$ $[e_2,e_7]=-2e_5,\ [e_5,e_7]=-\sqrt{6}e_5,$ $[e_6,e_7]=-\sqrt{6}e_4.$
φ	$e^{123}+e^{347}+e^{356}$ $+e^{145}-e^{246}+e^{167}+e^{257}$	$e^{127}+e^{135}-e^{146}$ $-e^{236}-e^{245}+e^{347}+e^{567}$
τ_φ	$-\sqrt{2}e^{34}+\sqrt{2}e^{25}-e^{56}+e^{47}$	$-2e^{15}+2e^{26}-\sqrt{6}e^{36}+\sqrt{6}e^{45}-4e^{47}$
$\Delta_\varphi\varphi$	$4e^{123}-\sqrt{2}e^{136}+\sqrt{2}e^{127}+2e^{145}$	$24e^{127}-4\sqrt{6}e^{125}-2\sqrt{6}e^{137}$ $+2\sqrt{6}e^{247}+12e^{567}$
Ric_φ	$\mathrm{Diag}\left(-3,-1,-1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$	$\mathrm{Diag}\left(-10,-10,3,11,-1,-1,-10\right)$
R_φ	-3	-18
$\frac{{R_\varphi}^2}{tr\,\mathrm{Ric}_\varphi^2}$	$\frac{3}{4}$	$\frac{3}{4}$
λ	9	54
D	$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & -4 \end{bmatrix}$	$\begin{bmatrix} -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\sqrt{6} & -24 & 0 & 0 & 0 & 0 \\ -2\sqrt{6} & 0 & 0 & -24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -18 & 0 & -4\sqrt{6} \\ 0 & 0 & 0 & 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{bmatrix}$

Table 4:

$$Q_{\varphi_2} = \frac{1}{3}\,\mathrm{Diag}(-2,-2,-2,1,1,1,1),$$

$$Q_{\varphi_3} = \frac{1-c+c^2}{3}\,\mathrm{Diag}(-2,-2,-2,1,1,1,1),$$

$$Q_{\varphi_4} = \begin{bmatrix} -2 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 1 \end{bmatrix}, \qquad Q_{\varphi_5} = \begin{bmatrix} -2 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 1 \end{bmatrix},$$

$$Q_{\varphi_6} = \begin{bmatrix} -2 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Q_{\varphi_7} = \begin{bmatrix} -4 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & 0 & -\sqrt{6}/2 & 0 \\ 0 & 0 & 0 & 17 & \sqrt{6}/2 & 0 & -2 \\ 1 & 0 & 0 & -\sqrt{6}/2 & 5 & 0 & 0 \\ 0 & -1 & \sqrt{6}/2 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -4 \end{bmatrix}.$$

Note that in every case, we have that $Q_\varphi = -\frac{1}{3}\lambda I - D$, where λ and D are given in the tables above.

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References

[B] R. BRYANT, Some remarks on G_2-structures, *Proc. Gökova Geometry-Topology Conference* (2005), 75-109.

[CF] D. CONTI, M. FERNÁNDEZ, Nilmanifolds with a calibrated G_2-structure, *Diff. Geom. Appl.* **29** (2011), 493-506.

[FFM] M. FERNÁNDEZ, A. FINO, V. MANERO, Laplacian flow of closed G_2-structures inducing nilsolitons, *J. Geom. Anal.* **26** (2016), 1808-1837.

[FC] E. FERNÁNDEZ-CULMA, Classification of 7-dimensional Einstein nilradicals, *Transform. Groups*, **17**(2012), 639-656.

[J] M. JABLONSKI, Homogeneous Ricci solitons are algebraic, *Geom. Topol.* **18** (2014), 2477-2486.

[L1] J. LAURET, Ricci soliton homogeneous nilmanifolds, *Math. Ann.* **319** (2001), 715-733.

[L2] J. LAURET, Laplacian flow of homogeneous G_2-structures and its solitons, *Proc. London Math. Soc* (2017)Volume 114, Issue 3 527-560.

[LR] J. LAURET, E. A. RODRÍGUEZ VALENCIA, On the Chern-Ricci flow and its solitons for Lie groups, *Math. Nachr.* **288** (2015), 1512-1526.

[LW] J. LAURET, C. WILL, On the symplectic curvature flow for locally homogeneous manifolds, *J. Symp. Geom.* **15** (2017), 1-49.

[LoW] J. LOTAY, Y. WEI, Laplacian flow for closed G_2-structures: Shi-type estimates, uniqueness and compactness, *Geom. Funct. Anal.* **27** (2017), 165-233.

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