Closed range composition operators for non-injective smooth symbols $\mathbb{R} \to \mathbb{R}^d$

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Abstract

In 2011 Kenessey and Wengenroth gave a full description of closed range composition operators $C_\psi : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, corresponding to smooth injective symbols $\psi : \mathbb{R} \to \mathbb{R}^d$. In 2012 Przestacki gave a sufficient condition for the range of C_ψ to be closed in case if $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth not necessarily injective symbol. Using their ideas we give a sufficient condition ensuring that the range of C_ψ is closed when $\psi : \mathbb{R} \to \mathbb{R}^d$ is a smooth not necessarily injective function.

1 Introduction

Let $C^\infty(\mathbb{R}^d)$ be the space of real valued smooth functions on $\mathbb{R}^d$ equipped with the usual topology of uniform convergence of functions and all (partial) derivatives on compact sets. One of the most important classes of operators acting between spaces of smooth functions are composition operators, i.e., operators of the form $C_\psi : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$, $F \mapsto F \circ \psi$, where $\psi : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function. The aim of this paper is to investigate for which smooth symbols $\psi : \mathbb{R} \to \mathbb{R}^d$ the operator C_ψ has closed range. This is an interesting problem with a long history. In [15] Whitney proved that the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ with the symbol $\psi(x) = x^2$ has closed range. Later on Glaeser [6] showed that if

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$\psi : \mathbb{R}^m \to \mathbb{R}^n$ is a semiproper (i.e., for every compact set $K \subset \mathbb{R}^n$ there exists a compact set $L \subset \mathbb{R}^m$ such that $K \cap \psi(\mathbb{R}^m) \subset \psi(L)$) analytic function with a dense set of points where its Jacobian has rank equal to n, then the composition operator $C_\psi : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$ has closed range. Finally, Bierstone and Milman [3], [4] and Bierstone, Milman and Pawłucki [5] gave a complete characterization of composition operators with closed range for analytic symbols .

In the case when the symbol of the composition operator is only assumed to be smooth relatively little is known. One of the most important things in studying closed range composition operators is investigating the behavior of the closure of $\operatorname{Im} C_\psi := \{F \circ \psi : F \in C^\infty(\mathbb{R}^d)\}$. In [2] Allan, Kakiko, O'Farrell and Watson proved that for smooth and injective symbols $\psi : \mathbb{R} \to \mathbb{R}^d$ we have

$$\overline{\operatorname{Im} C_\psi} = \{f \in C^\infty(\mathbb{R}) : \forall x \in \mathbb{R}\ \exists F \in C^\infty(\mathbb{R}^d)\ \forall n \geq 0\ f^{(n)}(x) = (F \circ \psi)^{(n)}(x)\}.$$

Moreover, as they noted in [1], the proper non-injective analogue of the set from the right side of above equality should be the following set

$$\widehat{\operatorname{Im} C_\psi} = \{f \in C^\infty(\mathbb{R}) : \forall x \in \mathbb{R}^d\ \exists F \in C^\infty(\mathbb{R}^d)\ \forall a \in \psi^{-1}(\{x\})\ \forall n \geq 0 \\ f^{(n)}(a) = (F \circ \psi)^{(n)}(a)\}$$

and they obtained ([1, Cor. 5]) that we always have $\overline{\operatorname{Im} C_\psi} \subset \widehat{\operatorname{Im} C_\psi}$. Let us notice that Tougeron in [14, Thm 1.1 and Thm 1.2] showed that if $\psi : \mathbb{R}^n \to \mathbb{R}^m$ is a real analytic function, then $\overline{\operatorname{Im} C_\psi} = \widehat{\operatorname{Im} C_\psi}$.

Using the result of Allan, Kakiko, O'Farrell and Watson, in 2011 Kenessey and Wengenroth [8] obtained a full description of closed range composition operators corresponding to injective symbols $\psi : \mathbb{R} \to \mathbb{R}^d$. Recall that a point x is called a flat point of a smooth function function $\psi = (\psi_1, \ldots, \psi_d) : \mathbb{R} \to \mathbb{R}^d$ if for every $1 \leq j \leq d$ all derivatives of ψ_j vanish at x. Recall also that a closed set $F \subset \mathbb{R}^d$ is called Whitney regular if for every $x \in F$ there are $d, \delta, C > 0$ such that for any two points $y, z \in F$ with $||y - x|| \leq d$, $||z - x|| \leq d$ there is a rectifiable curve in F connecting y and z of length not greater than $C||y - z||^\delta$ (here $|| \cdot ||$ stands for the euclidean distance in $\mathbb{R}^d$). If $\psi : \mathbb{R} \to \mathbb{R}^d$ is smooth, then the set $\psi(\mathbb{R})$ is a curve and, roughly speaking, it is Whitney regular if it has no sharp cusps. A typical example of a set of that form which is not Whitney regular is the set $\psi(\mathbb{R})$, where $\psi : \mathbb{R} \to \mathbb{R}^2$ is defined by the formula

$$\psi(x) = \begin{cases} (x^2, e^{-\frac{1}{x}}), & x > 0 \\ (x^2, 0), & x \leq 0 \end{cases}.$$

Theorem. *(Kenessey, Wengenroth, 2011) Let $\psi : \mathbb{R} \to \mathbb{R}^d$ be a smooth injective function. The composition operator $C_\psi : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R})$ has closed range if and only if ψ is semiproper, has Whitney regular image and no flat points.*

In 2012 in [11], the author gave the following sufficient condition for the range of C_ψ to be closed in case if $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth not necessarily injective symbol (in some cases this condition is also necessary - see [12, 13]). For every $y \in \mathbb{R}$ the set $\psi^{-1}(\{y\})$ is called the fiber of ψ over y.

Theorem. *(Przestacki, 2012) Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth semiproper function such that every fiber over a boundary point of $\psi(\mathbb{R})$ contains a non-flat point and every fiber over an interior point of $\psi(\mathbb{R})$ contains either a non-flat non extreme point or both a non-flat local minimum and a non-flat local maximum. Then $\operatorname{Im} C_\psi = \overline{\operatorname{Im} C_\psi} = \widehat{\operatorname{Im} C_\psi}$.*

In this paper we prove the following theorem which gives a sufficient condition for the range of C_ψ to be closed in case if $\psi : \mathbb{R} \to \mathbb{R}^d$ is a smooth not necessarily injective function. Note that for two different points $a, b \in \mathbb{R}$ by $[a, b]$ (or (a, b)) we denote the closed (or open) interval with endpoints a and b.

Theorem. *Let $\psi : \mathbb{R} \to \mathbb{R}^d$ be a smooth function satisfying the following conditions:*

1. *For every $b \in \psi(\mathbb{R})$ there are an open neighbourhood U_b of b, non-flat points $a_1, \dots, a_n \in \psi^{-1}(\{b\})$, and nonzero $\delta_1, \dots, \delta_n \in \mathbb{R}$ such that:*
   - *(a) $\psi([a_i, a_i + \delta_i]) \cap \psi([a_j, a_j + \delta_j]) = \{b\}$ for every $1 \le i \ne j \le n$;*
   - *(b) $U_b \cap \psi(\mathbb{R}) \subset \bigcup_{i=1}^{n} \psi([a_i, a_i + \delta_i])$;*
   - *(c) ψ restricted to $[a_i, a_i + \delta_i]$ is injective for every $1 \le i \le n$.*
2. *The set $\psi(\mathbb{R})$ is closed and Whitney regular.*

Then

$$\operatorname{Im} C_\psi = \overline{\operatorname{Im} C_\psi} = \widehat{\operatorname{Im} C_\psi}.$$

It is easy to see that our result extends the sufficiency part of the theorem due to Kenessey and Wengenroth and generalizes the mentioned result of Przestacki. Indeed, the conditions in our main theorem imply that the function ψ is semiproper and that there are no flat points in the injective case and there are sufficiently many non-flat points in the non-injective case.

The following examples illustrate our result. The function $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by the formula $\varphi(x) = e^{-1/x^2}$ for $x \neq 0$ and $\varphi(0) = 0$.

Example 1.1. *Let $\psi_1, \psi_2, \psi_3 : \mathbb{R} \to \mathbb{R}^2$ be functions defined by the formulas: $\psi_1(x) = (x, x^2)$,*

$$\psi_2(x) = \left(x \cdot \varphi(x), x^2 \cdot \varphi^2(x)\right),$$

$$\psi_3(x) = \begin{cases} \left(x \cdot \varphi(x), x^2 \cdot \varphi^2(x)\right), & x < 0 \\ \left((x^2 - x) \cdot \varphi(x), (x^2 - x)^2 \cdot \varphi^2(x)\right), & x > 0 \end{cases}.$$

Those functions are smooth, semiproper, and $\psi_1(\mathbb{R}) = \psi_2(\mathbb{R}) = \psi_3(\mathbb{R})$. Moreover, one can easily check that those sets are Whitney regular. Since ψ_1 is injective and has no flat points, the result of Kenessey and Wengenroth implies that the range of C_{ψ_1} is closed. The same result implies that the range of C_{ψ_2} is not closed because ψ_2 is injective and 0 is a flat point of ψ_2. The function ψ_3 is not injective and one can easily check that the conditions in our theorem are satisfied (observe that $\psi_3^{-1}(\{(0,0)\}) = \{0, 1\}$) and, therefore, C_{ψ_3} has closed range.

Example 1.2. *Let $\psi : \mathbb{R} \to \mathbb{R}^2$ be defined by the formula*

$$\psi(x) = \begin{cases} \left(-(x^2+x)\cdot\varphi(x), 0\right), & x < 0 \\ \left(\sin\left(x\cdot\varphi(x)\right), \cos\left(x\cdot\varphi(x)\right) - 1\right), & x > 0 \end{cases}.$$

This function is smooth, semiproper and one can check that the image of ψ is equal to the union of the set $\{(x,0) : x \leq c\}$, where c is a constant bigger than 0 and the circle of radius 1 with center at $(0,-1)$. The image of ψ is Whitney regular and it is easy to check that ψ has sufficiently many non-flat points and, therefore, by our theorem the range of C_ψ is closed.

Unfortunately, we do not know what happens if a function ψ does not satisfy the conditions of our theorem. In particular, we do not know what happens if ψ has a fiber containing only flat points (the answer is not known even in the one-dimensional case) or if the image of ψ has too many self-intersections.

2 Tools

In order to prove our main theorem we will need the following lemmas which are all contained in [8]. The first one is an easy consequence of Rolle's theorem. Recall that a point $x_0 \in \mathbb{R}$ is called a critical point of a smooth function $\psi = (\psi_1, \ldots, \psi_d) : \mathbb{R} \to \mathbb{R}^d$ if $\psi_j'(x_0) = 0$ for $1 \leq j \leq d$.

Lemma 2.1. *Let $\psi : \mathbb{R} \to \mathbb{R}^d$ be smooth. If x_0 is a non-flat point of ψ, then there exists $\varepsilon > 0$ such that the only possible critical point of ψ in $[x_0 - \varepsilon, x_0 + \varepsilon]$ is x_0.*

By $\mathcal{D}([a,b])$ we denote the space of all functions on $[a,b]$ which are restrictions of smooth functions on $\mathbb{R}$ with support contained in $[a,b]$, with the topology inherited from $C^\infty(\mathbb{R})$.

Lemma 2.2. *Let $\psi : [a,b] \to [c,d]$ be a smooth bijection without critical points or with one non-flat critical point at a. Then the composition operator*

$$C_{\psi^{-1}} : \mathcal{D}([a,b]) \to \mathcal{D}([c,d]), F \mapsto F \circ \psi^{-1}$$

is well-defined and continuous.

The following lemma will play a crucial role in the proof of our main theorem. This lemma and its proof is contained in Step 2 of the proof of the main result in [8] which deals with composition operators corresponding to injective symbols. However, an easy examination of that proof allows us to transfer it to the non-injective case. We include the proof for the convenience of the reader. Recall that a smooth function $G \in C^\infty(\mathbb{R}^d)$ is called flat at a point x_0 if all (partial) derivatives of this function vanish at x_0. For $f \in C^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$, and $K \subset \mathbb{R}^d$

$$\|f\|_{n,K} := \max_{x \in K} \max_{0 \leq |\alpha| \leq n} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right|.$$

Lemma 2.3. *Let $\psi = (\psi_1, \ldots, \psi_d) : \mathbb{R} \to \mathbb{R}^d$ be a smooth function and let $a, b \in \mathbb{R}$. Assume that:*

1. *There is an index $1 \leq j \leq d$ such that the only possible critical point of ψ_j in $[a,b]$ is a at which ψ_j is non-flat.*
2. *The set $\psi(\mathbb{R})$ is closed and Whitney regular.*
3. *There exists an open set $W \subset \mathbb{R}^d$ such that $W \cap \psi(\mathbb{R}) = \psi((a,b))$.*

Let

$$X = \{G \in C^\infty(\mathbb{R}^d) : G \text{ is equal to zero and is flat on } \psi(\mathbb{R}) \backslash \psi((a,b))\}.$$

Then the operator

$$T : X \to \mathcal{D}([a,b]),\ G \mapsto G \circ \psi|_{[a,b]}$$

is surjective.

Proof. Let $\pi_j : \mathbb{R}^d \to \mathbb{R}$ be the projection on the j-th coordinate, i.e, $\pi_j(x_1, \ldots, x_d) = x_j$. From the first assumption of the lemma it follows that the set

$$\{x \in \mathbb{R}^d : \pi_j(x) \in \psi_j((a,b))\}$$

is open. Thus the set

$$U = W \cap \{x \in \mathbb{R}^d : \pi_j(x) \in \psi_j((a,b))\}$$

is also open and moreover

$$U \cap \psi(\mathbb{R}) = \psi((a,b)).$$

Note that the function

$$\psi_j : (a,b) \to \psi_j((a,b))$$

has a smooth inverse.

Step 1.

First we will show that the space $\mathcal{D}((a,b))$ of functions from $\mathcal{D}([a,b])$ with support contained in (a,b) belongs to the image of T. Let $g \in \mathcal{D}((a,b))$ be arbitrary. We can find a smooth function Φ_g such that

$$\operatorname{supp}(\Phi_g) \subset U,\ \ \Phi_g = 1 \text{ near } \psi(\operatorname{supp}(g)).$$

Consider now the function G defined by the formula

$$G(x) = \Phi_g(x) \cdot (g \circ \psi_j^{-1} \circ \pi_j)(x)$$

for every $x \in U$ and as zero outside the set U. It is clear that G is smooth and that $G \in X$. Of course we have $G \circ \psi|_{[a,b]} = g$.

Step 2. Note that $\mathcal{D}((a,b))$ is dense in $\mathcal{D}([a,b])$. Hence, from Step 1, to prove the lemma it is enough to show that the range of T is closed. To prove this it suffices to show (see [9, Thm. 26.3]) that

$$\operatorname{Ker}(T)^\circ \subset \operatorname{Range}(T^t),$$

where $T^t : \mathcal{D}([a,b])' \to X'$ is the transposed operator and $\operatorname{Ker}(T)^\circ$ is the polar set of the kernel of the operator T, i.e.,

$$\operatorname{Ker}(T)^\circ = \{u \in X' : u(G) = 0 \text{ for all } G \in X \text{ with } G \circ \psi|_{[a,b]} = 0\}.$$

Take $u \in \operatorname{Ker}(T)^\circ$. By Hahn-Banach theorem, we can extend u to a distribution with compact support $\widehat{u} \in C^\infty(\mathbb{R}^d)'$. It is easy to see that the support of $\widehat{u}$ is contained in $\psi(\mathbb{R})$. Our aim is to find v in $\mathcal{D}([a,b])'$ such that

$$v(G \circ \psi) = u(G)$$

for all $G \in X$.

First we will define v on $\mathcal{D}((a,b))$. From Step 1 we know that every $g \in \mathcal{D}((a,b))$ can be written as $G \circ \psi$, where $G \in X$. We define

$$v(g) = u(G).$$

This definition makes sense since $u \in \operatorname{Ker}(T)^\circ$. We will show that v is continuous on $\mathcal{D}((a,b))$ with the topology inherited from $\mathcal{D}([a,b])$. Since the set $\psi(\mathbb{R})$ is Whitney regular, there is a natural number n and $C > 0$ such that

$$|\widehat{u}(F)| \leq C\|F\|_{n,\psi(\mathbb{R})}$$

for every $F \in C^\infty(\mathbb{R}^d)$ (see [7, Thm. 2.3.11]). This gives us that

$$|v(g)| = |u(G)| = |\widehat{u}(G)| \leq C\|G\|_{n,\psi(\mathbb{R})}$$

for every smooth function $G \in X$ such that $g = G \circ \psi$. With the specific choice of the function G from Step 1 we obtain that

$$|v(g)| \leq C\|G\|_{n,\psi(\mathbb{R})} = C\|\Phi_g \cdot (g \circ \psi_j^{-1} \circ \pi_j)\|_{n,\psi(\mathbb{R})\cap U}.$$

The properties of the functions g and Φ_g imply that

$$\|\Phi_g \cdot (g \circ \psi_j^{-1} \circ \pi_j)\|_{n,\psi(\mathbb{R})\cap U} = \|g \circ \psi_j^{-1} \circ \pi_j\|_{n,\psi([a,b])}.$$

Now the projection π_j and the function ψ_j^{-1} induce continuous composition operators (see Lemma 2.2) and therefore, altogether, we can find a natural number k and $D > 0$ such that

$$|v(g)| \leq D\|g\|_{k,[a,b]}.$$

This shows that v is continuous. Hence, by continuity, we can extend it to $\widehat{v} \in \mathcal{D}([a,b])'$. Now, we have to verify that

$$\widehat{v}(G \circ \psi) = u(G)$$

for every $G \in X$. If $G \in X$ is such that $G \circ \psi|_{[a,b]} \in \mathcal{D}((a,b))$, then from the construction it is clear that

$$\widehat{v}(G \circ \psi) = u(G).$$

To finish the proof it is enough to show that the linear subspace

$$Y = \{G \in X : G \circ \psi|_{[a,b]} \in \mathcal{D}((a,b))\}$$

is dense in X. To do this we prove that every $w \in X'$ which vanishes on Y also vanishes on X. Let $w \in X'$ be a linear functional which vanishes on Y. We can extend w using Hahn-Banach theorem to a distribution with compact support $\widehat{w}$. It is easy to see that the support of $\widehat{w}$ is contained in $\psi(\mathbb{R})\backslash\psi((a,b))$. Using this, the definition of X, and [7, Thm. 2.3.3] we conclude that $\widehat{w}$ vanishes on X. ∎

3 Proof of the main result

Throughout this section we assume that $\psi : \mathbb{R} \to \mathbb{R}^d$ is a smooth function satisfying the conditions of the main theorem. Our goal is to show that $\operatorname{Im} C_\psi$ is closed and equal to its formal closure $\widehat{\operatorname{Im} C_\psi}$. As noted in the Introduction, it is well-known (see [1, Cor. 5]) that

$$\operatorname{Im} C_\psi \subset \overline{\operatorname{Im} C_\psi} \subset \widehat{\operatorname{Im} C_\psi}$$

and hence to prove the theorem it is enough to show

$$\widehat{\operatorname{Im} C_\psi} \subset \operatorname{Im} C_\psi.$$

Let $f \in \widehat{\operatorname{Im} C_\psi}$ be an arbitrary function. We will show that for every $b \in \psi(\mathbb{R})$ we can find an open set $W_b \subset \mathbb{R}^d$ containing b and a function $F_b \in C^\infty(\mathbb{R}^d)$ satisfying

$$F_b \circ \psi = f \text{ on } \psi^{-1}(W_b).$$

This is enough. Indeed, the family $\{W_b : b \in \psi(\mathbb{R})\}$ forms an open covering of the closed set $\psi(\mathbb{R})$. Let $\{\varphi_b : b \in \psi(\mathbb{R})\}$ be a smooth partition of unity associated to this covering, i.e., let $\{\varphi_b : b \in \psi(\mathbb{R})\}$ be a family of smooth function on $\mathbb{R}^d$ such that

$$0 \leq \varphi_b \leq 1, \ \operatorname{supp}(\varphi_b) \subset W_b, \ \{\operatorname{supp}(\varphi_b) : b \in \psi(\mathbb{R})\} \text{ is locally finite, and } \sum_{b \in \psi(\mathbb{R})} \varphi_b = 1 \text{ on } \psi(\mathbb{R}).$$

Such a partition of unity exists by [10, Thm. 1.2.3 and Cor. 1.2.6]. Consider now the function

$$F = \sum_{b \in \psi(\mathbb{R})} \varphi_b \cdot F_b.$$

Clearly F is a smooth function on $\mathbb{R}^d$ and it is easy to verify that

$$F \circ \psi = f.$$

Let us now fix $b \in \psi(\mathbb{R})$. We will proceed now with constructing an open set W_b containing b and a function $F_b \in C^\infty(\mathbb{R}^d)$ such that $F_b \circ \psi = f$ on $\psi^{-1}(W_b)$.

Step 1.

From the assumptions of the theorem we can find an open set $U_b \subset \mathbb{R}^d$ containing b, non-flat points $a_1, \dots, a_n \in \psi^{-1}(\{b\})$ and nonzero $\delta_1, \dots, \delta_n \in \mathbb{R}$ such that the conditions (a), (b), (c) in the theorem are satisfied. Using Lemma 2.1 and the continuity of ψ we can find nonzero numbers $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{R}$ such that

- $|\varepsilon_i| \leq |\delta_i|$ for $1 \leq i \leq n$;
- $\operatorname{sign}(\varepsilon_i) = \operatorname{sign}(\delta_i)$ for $1 \leq i \leq n$;
- $\psi([a_i, a_i + \varepsilon_i]) \subset U_b$ for $1 \leq i \leq n$;

- for every $1 \leq i \leq n$ there is $1 \leq j \leq d$ such that the only possible critical point of ψ_j in $[a_i, a_i + \varepsilon_i]$ is a_i.

The set
$$W_b = U_b \setminus \bigcup_{i=1}^{n} \psi([a_i + \varepsilon_i/2, a_i + \delta_i])$$
is open and it is clear that $b \in W_b$.
Step 2.
For every $1 \leq i \leq n$ the set
$$W_i = U_b \setminus \left(\psi([a_i + \varepsilon_i, a_i + \delta_i]) \cup \bigcup_{j \neq i} \psi([a_j, a_j + \delta_j]) \right)$$
is open and satisfies
$$W_i \cap \psi(\mathbb{R}) = \psi((a_i, a_i + \varepsilon_i)).$$
For every $1 \leq i \leq n$ let
$$X_i = \{G \in C^\infty(\mathbb{R}^d) : G \text{ is equal to zero and is flat on } \psi(\mathbb{R}) \setminus \psi((a_i, a_i + \varepsilon_i))\}$$
and let
$$T_i : X_i \to \mathcal{D}([a_i, a_i + \varepsilon_i]),\ G \mapsto G \circ \psi|_{[a_i, a_i + \varepsilon_i]}.$$
The above considerations show that the assumptions of Lemma 2.3 are satisfied and thus the operator T_i is surjective for $1 \leq i \leq n$.
Step 3.
Let $\Omega_1, \ldots, \Omega_n$ and Ω_b be open, disjoint neighborhoods of the points $\psi(a_1 + \varepsilon_1), \ldots, \psi(a_n + \varepsilon_n)$ and b, respectively. We choose smooth functions $\varphi_1, \ldots, \varphi_n$ and φ_b such that
$$\varphi_i = 1 \text{ near } \psi(a_i + \varepsilon_i),\ \operatorname{supp}(\varphi_i) \subset \Omega_i \text{ for } 1 \leq i \leq n$$
and
$$\varphi_b = 1 \text{ near } b,\ \operatorname{supp}(\varphi_b) \subset \Omega_b.$$
From the definition of $\widehat{\operatorname{Im} C_\psi}$ we can find smooth functions $F_1, \ldots, F_n$ and F_b such that
$$f - (\varphi_i \circ \psi) \cdot (F_i \circ \psi) \text{ is equal to zero and is flat on } \psi^{-1}(\{\psi(a_i + \varepsilon_i)\}) \text{ for } 1 \leq i \leq n$$
and
$$f - (\varphi_b \circ \psi) \cdot (F_b \circ \psi) \text{ is equal to zero and is flat on } \psi^{-1}(\{b\}).$$
From the properties of the functions $\varphi_1, \ldots, \varphi_n$ and φ_b it follows that the function
$$g = f - (\varphi_b \circ \psi) \cdot (F_b \circ \psi) - \sum_{i=1}^{n} (\varphi_i \circ \psi) \cdot (F_i \circ \psi)$$
is equal to zero and is flat on the set
$$\psi^{-1}(\{b\}) \cup \bigcup_{i=1}^{n} \psi^{-1}(\{\psi(a_i + \varepsilon_i)\}).$$

Thus, for $1 \leq i \leq n$ the restriction of g to the interval $[a_i, a_i + \varepsilon_i]$ belongs to $\mathcal{D}([a_i, a_i + \varepsilon_i])$. From Step 2 (the surjectivity of the operators T_i) we can find smooth functions $G_i \in X_i$ such that

$$G_i \circ \psi|_{[a_i,a_i+\varepsilon_i]} = g|_{[a_i,a_i+\varepsilon_i]} \text{ for } 1 \leq i \leq n.$$

Since $g \in \widehat{\operatorname{Im} C_\psi}$, it is constant on fibers of ψ and therefore

$$G_i \circ \psi = g \text{ on } \psi^{-1}(\psi([a_i, a_i + \varepsilon_i])) \text{ for } 1 \leq i \leq n.$$

Moreover, the definition X_i implies that $G_i \circ \psi$ vanishes on $\psi^{-1}(\psi([a_j, a_j + \varepsilon_j]))$ for $j \neq i$ and thus

$$\sum_{i=1}^{n} G_i \circ \psi = g \text{ on } \bigcup_{i=1}^{n} \psi^{-1}(\psi([a_i, a_i + \varepsilon_i]).$$

Therefore

$$f = (\varphi_b \circ \psi) \cdot (F_b \circ \psi) + \sum_{i=1}^{n} G_i \circ \psi + \sum_{i=1}^{n} (\varphi_i \circ \psi) \cdot (F_i \circ \psi) \text{ on } \bigcup_{i=1}^{n} \psi^{-1}(\psi([a_i, a_i + \varepsilon_i]).$$

This shows that we can find a smooth function F such that

$$f = F \circ \psi \text{ on } \bigcup_{i=1}^{n} \psi^{-1}(\psi([a_i, a_i + \varepsilon_i]).$$

To finish the proof it is enough to observe that

$$\psi^{-1}(W_b) \subset \bigcup_{i=1}^{n} \psi^{-1}(\psi([a_i, a_i + \varepsilon_i]). \qquad \blacksquare$$

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