Periodic points on T-fiber bundles over the circle

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Introduction

Let $f : M \to M$ be a map and $x \in M$, where M a compact manifold. The point x is called a periodic point of f if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$, in this case x a periodic point of f of period n. The set of all $\{x \in M |$ x is periodic$\}$ is called the set of periodic points of f and is denoted by $P(f)$.

If M is a compact manifold then the Nielsen theory can be generalized to periodic points. Boju Jiang introduced (Chapter 3 in [1]) a Nielsen-type homotopy invariant $NF_n(f)$ being a lower bound of the number of n-periodic points, for each g homotopic to f; $Fix(g^n) \geq NF_n(f)$. In case $dim(M) \geq 3$, M compact PL- manifold, then any map $f : M \to M$ is homotopic to a map g satisfying $Fix(g^n) = NF_n(f)$, this was proved in [2].

Consider a fiber bundle $F \to M \xrightarrow{p} B$ where F, M, B are closed manifolds and $f : M \to M$ a fiber-preserving map over B. In natural way is to study periodic points of f on M, that is, given $n \in \mathbb{N}$ we want to study the set $\{x \in M \mid f^n(x) = x\}$. The our main question is; when f can be deformed by a fiberwise homotopy to a map $g : M \to M$ such that $Fix(g^n) = \emptyset$?

This paper is organized into three sections besides one. In Section 1 we describe our problem in the general context of fiber bundle with base and fiber closed manifolds.

In section 2, given a positive integer n and a fiber-preserving map $f : M \to M$, in a fiber bundle with base circle and fiber torus, we present necessary and sufficient conditions to deform $f^n : M \to M$ to a fixed point free map over S^1, see Theorem 2.3. In the Theorem 2.4 we described linear models of maps,

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on the universal covering of the torus, which induces fiber-preserving maps on the fiber bundle.

In section 3, in the Theorem 3.1, we used the models of maps of the section 2 to find a map $g : M \to M$, fiberwise homotopic to a given map $f : M \to M$ such that $g^n : M \to M$ is a fixed point free map over S^1.

1 General problem

Let $F \to M \xrightarrow{p} B$ be a fibration and $f : M \to M$ a fiber-preserving map over B, where F, M, B are closed manifolds. Given $n \in \mathbb{N}$, from relation $p \circ f = p$, we obtain $p \circ f^n = p$, thus $f^n : M \to M$ is also a fiber-preserving map for each $n \in \mathbb{N}$. We want to know when f can be deformed by a fiberwise homotopy to a map $g : M \to M$ such that $Fix(g^n) = \emptyset$. The the following lemma give us a necessary condition to a positive answer the question above.

Lemma 1.1. *Let $f : M \to M$ be a fiber-preserving map and n a positive integer. If the map $f^k : M \to M$ can not be deformed to a fixed point free map by a fiberwise homotopy, where k is a positive divisor of n, then there is not map $g : M \to M$ fiberwise homotopic to f such that $g^n : M \to M$ is a fixed point free map.*

Proof. Suppose that exists $g \sim_B f$ such that $Fix(g^n) = \emptyset$. Since $Fix(g^k) \subset Fix(g^n)$ and $Fix(g^k) \neq \emptyset$ then we have a contradiction. ■

Therefore, a necessary condition to deform $f : M \to M$ to a map $g : M \to M$ by a fiberwise homotopy, such that $Fix(g^n) = \emptyset$, is that for all positive integer k, where k divides n, the map $f^k : M \to M$ must be deformed by a fiberwise homotopy to a fixed point free map.

Note that for each n the square of the following diagram is commutative;

$$\begin{array}{ccccccccc}
\dots \longrightarrow & \pi_1(F, x_0) & \xrightarrow{i_\#} & \pi_1(M, x_0) & \xrightarrow{p_\#} & \pi_1(B, p(x_0)) & \longrightarrow 0 \\
& \downarrow{\scriptstyle (f^n|_F)_\#} & & \downarrow{\scriptstyle f^n_\#} & & \downarrow{\scriptstyle Id} & \\
\dots \longrightarrow & \pi_1(F, f^n(x_0)) & \xrightarrow{i_\#} & \pi_1(M, f^n(x_0)) & \xrightarrow{p_\#} & \pi_1(B, p(x_0)) & \longrightarrow 0
\end{array}$$

In our case, all topological spaces are path-connected then we will represent the generators of the groups $\pi_1(M, f^n(x_0))$ for each n, with the same letters. The same thing we will do with $\pi_1(T, f^n(0))$.

Let $M \times_B M$ be the pullback of $p : M \to B$ by $p : M \to B$ and $p_i : M \times_B M \to M$, $i = 1, 2$, the projections to the first and the second coordinates, respectively.

The inclusion $M \times_B M - \Delta \hookrightarrow M \times_B M$, where Δ is the diagonal in $M \times_B M$, is replaced by the fiber bundle $q : E_B(M) \to M \times_B M$, whose fiber is denoted by $\mathcal{F}$. We have $\pi_m(E_B(M)) \approx \pi_m(M \times_B M - \Delta)$ where $E_B(M) = \{(x, \omega) \in B \times A^I | i(x) = \omega(0)\}$, with $A = M \times_B M$, $B = M \times_B M - \Delta$ and q is given by $q(x, \omega) = \omega(1)$.

E. Fadell and S. Husseini in [4] studied the problem to deform the map f^n, for each $n \in \mathbb{N}$, to a fixed point free map. They supposed that $dim(F) \geq 3$ and that F, M, B are closed manifolds. The necessary and sufficient condition to deform f^n is given by the following theorem that the proof can be find in [4].

Theorem 1.1. *Given a positive integer n, the map $f^n : M \to M$ is deformable to a fixed point free map if and only if there exists a lift $\sigma(n)$ in the following diagram;*

$$
\begin{array}{ccccc}
 & & \mathcal{F} & & \mathcal{F} \\
 & & \downarrow & & \downarrow \\
 & & E_B(f^n) & \xrightarrow{\bar{q}_{f^n}} & E_B(M) \\
 & \overset{\sigma(n)}{\nearrow} & \downarrow q_{f^n} & & \downarrow q \\
M & \xrightarrow{1} & M & \xrightarrow{(1,f^n)} & M \times_B M
\end{array}
\tag{1}
$$

where $E_B(f^n) \to M$ is the fiber bundle induced from q by $(1, f^n)$.

In the Theorem 1.1 we have $\pi_{j-1}(\mathcal{F}) \cong \pi_j(M \times_B M, M \times_B M - \Delta) \cong \pi_j(F, F - x)$ where x is a point in F. In this situation, that is, $dim(F) \geq 3$ the classical obstruction was used to find a cross section.

When F is a surface with Euler characteristic ≤ 0 then by Proposition 1.6 from [5] we have necessary e sufficient conditions to deform f^n to a fixed point free map over B. The next proposition gives a relation between a geometric diagram and our problem.

Proposition 1.1. *Let $f : M \to M$ be a fiber-preserving map over B. Then there is a map g, $g \sim_B f$, such that $Fix(g^n) = \emptyset$ if and only if there is a map $h_n : M \to M \times_B M - \Delta$ of the form $h_n = (Id, s^n)$, where $s : M \to M$, is fiberwise homotopic to f and makes the diagram below commutative up to homotopy.*

$$
\begin{array}{ccc}
 & & M \times_B M - \Delta \\
 & \overset{h_n}{\nearrow} & \downarrow i \\
M & \xrightarrow[(1,f^n)]{} & M \times_B M
\end{array}
\tag{2}
$$

Proof. ($\Rightarrow$) Suppose that exists $g : M \to M$, $g \sim_B f$, with $Fix(g^n) = \emptyset$. Is enough to define $h_n = (Id, g^n)$, that is, $s = g$.

($\Leftarrow$) If there is $h_n : M \to M \times_B M - \Delta$ such that $h_n = (Id, s^n)$, where $s \sim_B f$, then $s^n(x) \neq x$ for all $x \in M$. Thus, takes $g = s$. ∎

2 Torus fiber-preserving maps

Let T be, the torus, defined as the quotient space $\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{Z}$. We denote by (x, y) the elements of $\mathbb{R} \times \mathbb{R}$ and by $[(x, y)]$ the elements in T.

Let $MA = \frac{T \times [0,1]}{([(x,y)],0) \sim ([A\binom{x}{y}],1)}$ be the quotient space, where A is a homeomorphism of T induced by an operator in $\mathbb{R}^2$ that preserves $\mathbb{Z} \times \mathbb{Z}$. The space MA is a fiber bundle over the circle S^1 where the fiber is the torus. For more details on these bundles see [5].

Given a fiber-preserving map $f : MA \to MA$, i.e. $p \circ f = p$, we want to study the set $Fix(g^n)$ for each map g fiberwise homotopic to f.

Consider the loops in MA given by; $a(t) =< [(t,0)], 0 >$, $b(t) = < [(0,t)], 0 >$ and $c(t) =< [(0,0)], t >$ for $t \in [0,1]$. We denote by B the matrix of the homomorphism induced on the fundamental group by the restriction of f to the fiber T. From [5] we have the following theorem that provides a relationship between the matrices A and B, where

$$A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}.$$

From [5] the induced homomorphism $f_\# : \pi_1(MA) \to \pi_1(MA)$ is given by; $f_\#(a) = a^{b_1}b^{b_2}$, $f_\#(b) = a^{b_3}b^{b_4}$, $f_\#(c) = a^{c_1}b^{c_2}c$. Thus

$$B = \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}.$$

Theorem 2.1. (1) $\pi_1(MA, 0) = \langle a, b, c | [a, b] = 1, cac^{-1} = a^{a_1}b^{a_2}, cbc^{-1} = a^{a_3}b^{a_4} \rangle$

(2) *B commutes with A.*

(3) *If f restricted to the fiber is deformable to a fixed point free map then the determinant of $B - I$ is zero, where I is the identity matrix.*

(4) *If v is an eigenvector of B associated to 1 $(for\ B \neq Id)$ then $A(v)$ is also an eigenvector of B associated to 1.*

(5) *Consider $w = A(v)$ if the pair v, w generators $\mathbb{Z} \times \mathbb{Z}$, otherwise let w be another vector so that v, w span $\mathbb{Z} \times \mathbb{Z}$. Define the linear operator $P : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $P(v) = \binom{1}{0}$ and $P(w) = \binom{0}{1}$. Consider an isomorphism of fiber bundles, also denoted by P, $P : MA \to M(A^1)$ where $A^1 = P \cdot A \cdot P^{-1}$. Then MA is homeomorphic to $M(A^1)$ over S^1. Moreover we have one of the cases of the table below with $B^1 = P \cdot A \cdot P^{-1}$ and $B^1 \neq Id$, except in case I:*

Case I	$A^1 = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}, B^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $a_3 \neq 0$
Case II	$A^1 = \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix}, B^1 = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix}$ $a_3(b_4 - 1) = 0$
Case III	$A^1 = \begin{pmatrix} 1 & a_3 \\ 0 & -1 \end{pmatrix}, B^1 = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix}$ $a_3(b_4 - 1) = -2b_3$
Case IV	$A^1 = \begin{pmatrix} -1 & a_3 \\ 0 & -1 \end{pmatrix}, B^1 = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix}$ $a_3(b_4 - 1) = 0$
Case V	$A^1 = \begin{pmatrix} -1 & a_3 \\ 0 & 1 \end{pmatrix}, B^1 = \begin{pmatrix} 1 & b_3 \\ 0 & b_4 \end{pmatrix}$ $a_3(b_4 - 1) = 2b_3$

From Theorem 4.1 in [5], we have necessary and sufficient conditions to deform f to a fixed point free map over S^1. The next theorem is equivalent to Theorem 4.1 in [5], this equivalence was made in [6].

Theorem 2.2. *A fiber-preserving map $f : MA \to MA$ can be deformed to a fixed point free map by a homotopy over S^1 if and only if one of the cases below holds:*

(1) MA is as in case I and f is arbitrary

(2) MA is as in one of the cases II or III and $c_1(b_4-1)-c_2b_3=0$

(3) MA is as in case IV and $b_4(b_3+1)-1-c_1(b_4-1)+b_3c_2 \equiv 0 \ mod\ 2$ except when:

a_3 is odd and $[(c_1,c_2)] = [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,2),(0,4)\rangle}$ or

a_3 is even and $[(c_1,c_2)] = [(0,0)]$, with $[(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,0),(0,2)\rangle}$.

(4) MA is as in case V and either

a_3 is even and $(b_4-1)(c_1-\frac{a_3}{2}c_2-1) \equiv 0 \ mod\ 2$, except when $c_1-\frac{a_3}{2}c_2-1$ and $\frac{b_4-1}{L}$ are odd, or

a_3 is odd and $\frac{b_4-1}{2}(1+c_2) \equiv 0 \ mod\ 2$ except when $1+c_2$ and $\frac{b_4-1}{L}$ are odd, where $L := gcd(b_4-1,c_2)$.

Given $n \in \mathbb{N}$ we denote the induced homomorphism $f_\#^n : \pi_1(MA) \to \pi_1(MA)$ by $f_\#(a) = a^{b_{1n}}b^{b_{2n}}$, $f_\#(b) = a^{b_{3n}}b^{b_{4n}}$ and $f_\#(c) = a^{c_{1n}}b^{c_{2n}}c$, where $b_{j1} = b_j, j = 1,...,4$ and $c_{j1} = c_j, j = 1,2$. Thus the matrix of the homomorphism induced on the fundamental group by the restriction of f^n to the fiber T is given by:

$$B_n = \begin{pmatrix} b_{1n} & b_{3n} \\ b_{2n} & b_{4n} \end{pmatrix},$$

where $B_1 = B$ is the matrix of $(f_{|T})_\#$ and $B_n = B^n$. From [8] we have

$$N(h^n) = |L(h^n)| = |det([h_\#]^n - I)|,$$

for each map $h : T \to T$ on torus, where $[h_\#]$ is the matrix of induced homomorphism and I is the identity.

Since $(B^n - I) = (B-I)(B^{n-1}+ ... + B + I)$ then $det(B^n - I) = det(B-I)det(B^{n-1}+...+B+I)$. Therefore, if $f_{|T}$ is deformable to a fixed point free map then $f_{|T}^n$ is deformable to a fixed point free map.

Remark 2.1. *C.Y.You in [10] proved that if $h : X \to X$ is a map, where X is a torus, then there exist g homotopic to h such that $NF_n(h) = \#Fix(g^n)$. Note that we do not have yet the Nielsen Jiang number defined for a map $f : M \to M$ in a fiber bundle over B. This work investigates when there exist a such map g, fiberwise homotopic to f, with $Fix(g^n) = \emptyset$, with $n > 1$.*

In the Theorems 2.1 and 2.2, putting f^n in the place of f we will get conditions to f^n. The conditions in Theorem 2.1 to f^n is the same of f but the conditions to f^n in the Theorem 2.2 are different of f and are in the Theorem 2.3.

Given a fiber-preserving map $f : MA \to MA$, if $f \sim_{S^1} g$ then $f^n \sim_{S^1} g^n$. Therefore, if $Fix(g^n) = \emptyset$ then the homomorphism $f_\#^n : \pi_1(M) \to \pi(M)$ satisfies the condition of deformability gives in [5].

Proposition 2.1. *Let $f : MA \to MA$ be a fiber-preserving map, where MA is a T-bundle over S^1. Suppose that f restricted to the fiber can be deformed to a fixed point free map. This implies $L(f|_T) = 0$. From Theorem 2.1 we can suppose that the induced* *homomorphism* $f_\# : \pi_1(MA) \to \pi_1(MA)$ *is given by;* $f_\#(a) = a$, $f_\#(b) = a^{b_3}b^{b_4}$, $f_\#(c) = a^{c_1}b^{c_2}c$. *Given* $n \in \mathbb{N}$ *then from relation* $(f_\#)^n = f_\#^n$ *we obtain;*

$$f_\#^n(a) = a,$$

$$f_\#^n(b) = a^{b_3 \sum_{i=0}^{n-1} b_4^i} b^{b_4^n},$$

$$f_\#^n(c) = a^{nc_1 + b_3c_2 \sum_{i=0}^{n-1}(n-1-i)b_4^i} b^{c_2 \sum_{i=0}^{n-1} b_4^i} c.$$

Proof. In fact, $f_\#^2(b) = f_\#(a^{b_3}b^{b_4}) = a^{b_3}(a^{b_3}b^{b_4})^{b_4} = a^{b_3+b_3b_4}b^{b_4^2}$ and $f_\#^2(c) = f_\#(a^{c_1}b^{c_2}c) = a^{c_1}(a^{b_3}b^{b_4})^{c_2}(a^{c_1}b^{c_2}c) = a^{2c_1+b_3c_2}b^{c_2+c_2b_4}c$. Suppose $f_\#^n(b) = a^{b_3 \sum_{i=0}^{n-1} b_4^i} b^{b_4^n}$ and $f_\#^n(c) = a^{nc_1 + b_3c_2 \sum_{i=0}^{n-1}(n-1-i)b_4^i} b^{c_2 \sum_{i=0}^{n-1} b_4^i} c$. Then,

$$\begin{aligned} f_\#^{n+1}(b) &= f_\#(a^{b_3 \sum_{i=0}^{n-1} b_4^i} b^{b_4^n}) &&= a^{b_3 \sum_{i=0}^{n-1} b_4^i}(a^{b_3}b^{b_4})^{b_4^n} \\ &= a^{b_3 \sum_{i=0}^{n-1} b_4^i}(a^{b_3 b_4^n} b^{b_4^{n+1}}) &&= a^{b_3 \sum_{i=0}^{n} b_4^i} b^{b_4^{n+1}}; \end{aligned}$$

$$\begin{aligned} f_\#^{n+1}(c) &= f_\#(a^{nc_1 + b_3c_2 \sum_{i=0}^{n-1}(n-1-i)b_4^i} b^{c_2 \sum_{i=0}^{n-1} b_4^i} c) \\ &= a^{nc_1 + b_3c_2 \sum_{i=0}^{n-1}(n-1-i)b_4^i} (a^{b_3}b^{b_4})^{c_2 \sum_{i=0}^{n-1} b_4^i} (a^{c_1}b^{c_2}c) \\ &= a^{(nc_1 + b_3c_2 \sum_{i=0}^{n-1}(n-1-i)b_4^i) + (b_3c_2 \sum_{i=0}^{n-1} b_4^i) + (c_1)} b^{(c_2 \sum_{i=1}^{n} b_4^i) + (c_2)} c \\ &= a^{(n+1)c_1 + b_3c_2 \sum_{i=0}^{n}(n-i)b_4^i} b^{c_2 \sum_{i=0}^{n} b_4^i} c. \end{aligned}$$

We will denote; $f_\#^n(b) = a^{b_{3n}}b^{b_{4n}}$ and $f_\#^n(c) = a^{c_{1n}}b^{c_{2n}}c$. ∎

Theorem 2.3. *Let* $f : MA \to MA$ *be a fiber-preserving map, where* MA *is a T-bundle over* S^1. *Suppose that* f *restricted to the fiber can be deformed to a fixed point free map and that the induced homomorphism* $f_\# : \pi_1(MA) \to \pi_1(MA)$ *is given by;* $f_\#(a) = a$, $f_\#(b) = a^{b_3}b^{b_4}$, $f_\#(c) = a^{c_1}b^{c_2}c$ *as in cases of the Theorem 2.2. If* n *is a positive integer, then* $f^n : MA \to MA$ *can be deformed to a fixed point free map over* S^1 *if and only if the following conditions are satisfies;*

1) MA *is as in case* I *and* f *is arbitrary.*

2) MA *is as in cases* II, III *and* $(c_1(b_4-1) - c_2b_3)\left(\sum_{i=0}^{n-1} b_4^i\right) = 0$

3) MA *is as in case* IV *and* $n(b_4(b_3+1) - 1 - c_1(b_4-1) + b_3c_2) - (n-1)(b_4-1) \equiv 0 \bmod 2$ *except when:*
a_3 *is odd and* $[(nc_1 + \frac{n(n-1)}{2}b_3c_2, nc_2)] = [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,2),(0,4)\rangle}$ *or*
a_3 *is even and* $[(nc_1 + \frac{n(n-1)}{2}b_3b_4c_2, c_2 + (n-1)b_4c_2)] = [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,0),(0,2)\rangle}$.

4) MA *is as in case* V *and either*
a_3 *is even and* $n(b_4-1)(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1)(b_4-1) \equiv 0 \bmod 2$, *except when* $n(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1)$ *and* $\frac{b_4-1}{L}$ *are odd, or*
a_3 *is odd and* $\frac{b_4-1}{2}((1+c_2)(1+(n-1)b_4)) \equiv 0 \bmod 2$ *except when* $(1+c_2)(1+(n-1)b_4)$ *and* $\frac{b_4-1}{L}$ *are odd, where* $L := gcd(b_4-1, c_2)$.

Proof. By Proposition 2.1 we know $f^n_\#(a) = a$, $f^n_\#(b) = a^{b_{3n}}b^{b_{4n}}$ and $f^n_\#(c) = a^{c_{1n}}b^{c_{2n}}c$.

(1) From Theorem 2.2 each map $f : MA \to MA$ is fiberwise homotopic to a fixed point free map over S^1 in particular that happens with $f^n : MA \to MA$ for each $n \in \mathbb{N}$.

(2) If $b_4 = 1$ then the assumption of the Theorem means $c_2b_3 = 0$. Moreover $b_{3n} = nb_3$, $b_{4n} = 1$, $c_{1n} = nc_1 + b_3c_2\frac{n(n-1)}{2}$ and $c_{2n} = nc_2$. In this sense, following Theorem 2.2, in cases II and III, f^n can be deformed, by a fiberwise homotopy, to a fixed point free map if and only if $c_{1n}(b_{4n} - 1) - c_{2n}b_{3n} = 0$. However, $c_{1n}(b_{4n} - 1) - c_{2n}b_{3n} = -n^2c_2b_3$, and $-n^2c_2b_3 = 0$ if and only if $c_2b_3 = 0$.

For $b_4 \neq 1$ we have $b_{3n} = b_3\sum_{i=0}^{n-1} b_4^i = b_3\frac{b_4^n-1}{b_4-1}$, $b_{4n} = b_4^n$, $c_{1n} = nc_1 + b_3c_2 \sum_{i=0}^{n-1}(n-1-i)b_4^i$ and $c_{2n} = c_2\sum_{i=0}^{n-1} b_4^i = c_2\frac{b_4^n-1}{b_4-1}$.

Note that; $\sum_{i=0}^{n-1}(n-1-i)b_4^i = \sum_{i=0}^{n-1}\frac{(n-1-i)b_4^i(b_4-1)^2}{(b_4-1)^2} =$

$$= \frac{\sum_{i=0}^{n-1}(n-1-i)b_4^{i+2} - 2\sum_{i=0}^{n-1}(n-1-i)b_4^{i+1} + \sum_{i=0}^{n-1}(n-1-i)b_4^i}{(b_4-1)^2}$$
$$= \frac{\sum_{i=2}^{n+1}(n+1-i)b_4^i - 2\sum_{i=1}^{n}(n-i)b_4^i + \sum_{i=0}^{n-1}(n-1-i)b_4^i}{(b_4-1)^2} =$$
$$\frac{\sum_{i=2}^{n-1}[(n+1-i) - 2(n-i) + (n-1-i)]b_4^i + b_4^n + (-2(n-1) + n - 2)b_4 + n - 1}{(b_4-1)^2}$$
$$= \frac{b_4^n - nb_4 + n - 1}{(b_4-1)^2}.$$

Therefore, $c_{1n}(b_{4n} - 1) - c_{2n}b_{3n} = \frac{n(b_4^n-1).(c_1(b_4-1)-c_2b_3)}{b_4-1}$. In fact,

$$\begin{aligned} c_{1n}(b_{4n}-1) &= \left(nc_1 + c_2b_3\frac{b_4^n - nb_4 + n - 1}{(b_4-1)^2}\right)(b_4^n - 1) \\ &= nc_1(b_4^n - 1) + c_2b_3\left(\frac{(b_4^n-1) - n(b_4-1)}{(b_4-1)^2}\right)(b_4^n - 1) \\ &= nc_1(b_4^n - 1) + c_2b_3\left(\frac{b_4^n-1}{b_4-1}\right)^2 - nc_2b_3\left(\frac{b_4^n-1}{b_4-1}\right); \\ c_{2n}b_{3n} &= \left(c_2\frac{b_4^n-1}{b_4-1}\right)\left(b_3\frac{b_4^n-1}{b_4-1}\right) = c_2b_3\left(\frac{b_4^n-1}{b_4-1}\right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} c_{1n}(b_{4n}-1) - c_{2n}b_{3n} &= nc_1(b_4^n - 1) - nc_2b_3\left(\frac{b_4^n-1}{b_4-1}\right) \\ &= n(b_4^n - 1)\left(c_1 - \frac{c_2b_3}{b_4-1}\right) \\ &= n\left(\frac{b_4^n-1}{b_4-1}\right)(c_1(b_4-1) - c_2b_3) \\ &= n(c_1(b_4-1) - c_2b_3)\left(\sum_{i=0}^{n-1} b_4^i\right). \end{aligned}$$

(3) Following Theorem 2.2, in cases IV, f^n can be deformed, by a fiberwise homotopy, to a fixed point free map iff $b_{4n}(b_{3n} + 1) - 1 - c_{1n}(b_{4n} - 1) + c_{2n}b_{3n} \equiv$

$0 \ mod \ 2$ except when a_3 even and $[(c_{1n}, c_{2n})] = [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,0),(0,2)\rangle}$, or a_3 odd and $[(c_{1n}, c_{2n})] = [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,2),(0,4)\rangle}$.

As in (2), we have $-c_{1n}(b_{4n}-1)+c_{2n}b_{3n} = -n(c_1(b_4-1)-c_2b_3)\left(\sum_{i=0}^{n-1} b_4^i\right)$ and $b_{4n}(b_{3n}+1)-1 = b_4^n\left(1+b_3\sum_{i=0}^{n-1} b_4^i\right)-1 = (b_4^n-1)+b_4^nb_3\left(\sum_{i=0}^{n-1} b_4^i\right) = (b_4-1)\left(\sum_{i=0}^{n-1} b_4^i\right)+b_4^nb_3\left(\sum_{i=0}^{n-1} b_4^i\right)$. Thus,

$$
\begin{aligned}
& b_{4n}(b_{3n}+1)-1-c_{1n}(b_{4n}-1)+c_{2n}b_{3n} = \\
& (b_4-1)\left(\sum_{i=0}^{n-1} b_4^i\right)+b_3b_4^n\left(\sum_{i=0}^{n-1} b_4^i\right)-n(c_1(b_4-1)-c_2b_3)\left(\sum_{i=0}^{n-1} b_4^i\right) = \\
& (b_4-1+b_3b_4^n-n(c_1(b_4-1)-c_2b_3))\left(\sum_{i=0}^{n-1} b_4^i\right) \equiv mod\ 2 \\
& (b_4-1+b_3b_4-n(c_1(b_4-1)-c_2b_3))(1+(n-1)b_4) \equiv mod 2 \\
& n(b_3b_4-c_1(b_4-1)+b_3c_2)+b_4-1 = \\
& n(b_4(b_3+1)-1-c_1(b_4-1)+b_3c_2)-(n-1)(b_4-1).
\end{aligned}
$$

The exceptions holds for a_3 even and $[(c_{1n}, c_{2n})] = [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,0),(0,2)\rangle}$, or a_3 odd and $[(c_{1n}, c_{2n})] = [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,2),(0,4)\rangle}$.

In this sense, we have $(c_{1n}, c_{2n}) = \left(nc_1 + b_3c_2\sum_{i=0}^{n-1}(n-1-i)b_4^i, c_2\sum_{i=0}^{n-1} b_4^i\right)$. If a_3 is odd then $b_4 = 1$, $c_2\sum_{i=0}^{n-1} 1^i = nc_2$ and $nc_1 + b_3c_2\sum_{i=0}^{n-1}(n-1-i)1^i = nc_1 + b_3c_2\dfrac{n(n-1)}{2}$. If a_3 is even then $c_2\sum_{i=0}^{n-1} b_4^i \equiv c_2(1+(n-1)b_4)\ mod\ 2$ and $nc_1 + b_3c_2\sum_{i=0}^{n-1}(n-1-i)b_4^i \equiv nc_1 + \dfrac{n(n-1)}{2}b_3b_4c_2\ mod\ 2$.

(4) From Theorem 2.2 the map f^n can be deformed, over S^1, to a fixed point free map if and only if the following condition holds:
a_3 is even and $(b_{4n}-1)(c_{1n}-\frac{a_3}{2}c_{2n}-1) \equiv 0\ mod\ 2$, except when $c_{1n}-\frac{a_3}{2}c_{2n}-1$ and $\frac{b_{4n}-1}{L}$ are odd, or
a_3 is odd and $\frac{b_{4n}-1}{2}(1+c_{2n}) \equiv 0\ mod\ 2$ except when $1+c_{2n}$ and $\frac{b_{4n}-1}{L}$ are odd, where $L := gcd(b_{4n}-1, c_{2n})$.

Note that if $b_4 = 1$ then from Theorem 2.1 we must have $b_3 = 0$ and this situation return in the case I. Therefore let us suppose $b_4 \neq 1$.

From previous calculation we have; $b_{4n} = b_4^n$, $b_{3n} = b_3\frac{b_4^n-1}{b_4-1}$, $c_{2n} = c_2\frac{b_4^n-1}{b_4-1}$ and $c_{1n} = nc_1 + b_3c_2\frac{b_4^n-nb_4+n-1}{(b_4-1)^2}$. From Theorem 2.1 we have $a_3(b_4-1) = 2b_3$.

Suppose a_3 even. Since $c_{1n}(b_{4n}-1)-c_{2n}b_{3n} = \frac{n(b_4^n-1)(c_1(b_4-1)-c_2b_3)}{b_4-1}$. Then

$(b_{4n}-1)(c_{1n}-\frac{a_3}{2}c_{2n}-1)=n(b_4^n-1)(c_1-\frac{a_3}{2}c_2-1)+(n-1)(b_4^n-1)$. In fact,

$$\begin{aligned} c_{1n}-\tfrac{a_3}{2}c_{2n} &= nc_1+b_3c_2\tfrac{b_4^n-nb_4+n-1}{(b_4-1)^2}-\tfrac{a_3}{2}c_2\tfrac{b_4^n-1}{b_4-1} \\ &= nc_1+b_3c_2\tfrac{(b_4^n-1)-n(b_4-1)}{(b_4-1)^2}-b_3c_2\tfrac{b_4^n-1}{(b_4-1)^2} \\ &= nc_1-\tfrac{b_3c_2n}{b_4-1} \\ &= n(c_1-\tfrac{a_3}{2}c_2). \end{aligned}$$

We have defined $L := gcd(b_4-1,c_2)$. Therefore, $kL = gcd(k(b_4-1),kc_2)$. We also define $L' := gcd(b_{4n}-1,c_{2n})$. Now $L' = \frac{b_4^n-1}{(b_4-1)}L$, since $b_{4n}-1 = \frac{b_4^n-1}{(b_4-1)}(b_4-1)$ and $c_{2n}=c_2\frac{b_4^n-1}{(b_4-1)}$. Furthermore, $\frac{b_{4n}-1}{L'} = \frac{b_{4n}-1}{L}\frac{b_4-1}{(b_4^n-1)} = \frac{b_4-1}{L}$. With these calculations we obtain the conditions statements on the theorem.

In the case a_3 odd we must have: $\frac{b_4^n-1}{2}(1+c_2\frac{b_4^n-1}{b_4-1}) \equiv 0 \ mod\ 2$ except when $1+c_2\frac{b_4^n-1}{b_4-1}$ and $\frac{b_4-1}{L}$ are odd, where $L := gcd(b_4-1,c_2)$.

Note that $\frac{b_4^n-1}{b_4-1}$ is even if and only if $1+(n-1)b_4$ is even, and b_4^n-1 is even if and only if b_4-1 is even, for all $n\in\mathbb{N}$. With this we obtain the enunciate of the theorem. ∎

Corollary 2.1. *From Theorem 2.3, if $f : MA \to MA$ can be deformed to a fixed point free map over S^1 and n is a odd positive integer, then $f^n : MA \to MA$ can be deformed to a fixed point free map over S^1.*

Proof. If $f : MA \to MA$ is deformed to a fixed point free map over S^1 then the conditions of the Theorem 2.2 are satisfied. Suppose n odd then the conditions of the Theorem 2.3 also are satisfied. Thus $f^n : MA \to MA$ can be deformed to a fixed point free map over S^1. ∎

In the corollary above if n is even the above statement may not holds, for example in the case V of the Theorem 2.3 if n, b_4, a_3 and $c_1-\frac{a_3}{2}c_2-1$ are even then $f : MA \to MA$ is deformed to a fixed point free map over S^1 but f^n is not.

Proposition 2.2. *Let $f : MA \to MA$ be a fiber-preserving map. Suppose that for some n, odd positive integer, $f^n : MA \to MA$ can be deformed to a fixed point free map over S^1, as in Theorem 2.3. If k is a positive divisor of n then the map $f^k : MA \to MA$ can be deformed, by a fiberwise homotopy, to a fixed point free map over S^1.*

Proof. It is enough to verify that if the conditions of the Theorem 2.3 are satisfied for some $n > 1$ odd then those conditions are also satisfied for $n = 1$. The validity of the conditions for any k which divides n follows of the Corollary 2.1. We will analyze each case of the Theorem 2.3.

Case I. In this case for each $n \in \mathbb{N}$ the fiber-preserving map can be deformed over S^1 to a fixed point free map.

Cases II and III. In these cases if for some n odd the fiber-preserving map $f^n : MA \to MA$ is deformed to a fixed point free map over S^1 then we must have; $c_1(b_4-1)-c_2b_3 = 0$. Thus, for all $k \le n$, f^k can be deformed to a fixed point free map over S^1, in particular when k divides n.

Case IV. Suppose that for some odd positive integer n the fiber-preserving map $f^n : MA \to MA$ is deformed to a fixed point free map over S^1, then $n(b_4(b_3+1) - 1 - c_1(b_4-1) + b_3c_2) - (n-1)(b_4-1) \equiv 0 \ mod\ 2$ and if a_3 is odd then $[(nc_1 + \frac{n(n-1)}{2} b_3c_2, nc_2)] \neq [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,2),(0,4)\rangle}$ or if a_3 is even then $[(nc_1 + \frac{n(n-1)}{2} b_3b_4c_2, c_2 + (n-1)b_4c_2)] \neq [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,0),(0,2)\rangle}$.

Suppose a_3 is odd. If $f : MA \to MA$ can not be deformed to a fixed point free map over S^1, then we must have $[(c_1,c_2)] = [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,2),(0,4)\rangle}$ or $(b_4(b_3+1) - 1 - c_1(b_4-1) + b_3c_2)$ odd, that is, $c_2 - 2c_1 \equiv 0 \ mod\ 4$ or $(b_4(b_3+1) - 1 - c_1(b_4-1) + b_3c_2)$ odd. Note that $(b_4(b_3+1) - 1 - c_1(b_4-1) + b_3c_2)$ odd iff $n(b_4(b_3+1) - 1 - c_1(b_4-1) + b_3c_2) - (n-1)(b_4-1)$ odd for any n odd. Now, if $c_2 - 2c_1 \equiv 0 \ mod\ 4$ then we have c_2 even and therefore $c_2 - 2c_1 - (n-1)b_3c_2 \equiv 0 \ mod\ 4$. Thus, we have $c_2 - 2c_1 - (n-1)b_3c_2 \equiv 0 \ mod\ 4$ or $n(b_4(b_3+1) - 1 - c_1(b_4-1) + b_3c_2) - (n-1)(b_4-1)$ odd. These two conditions guarantee that f^n can not be deformed to a fixed point free map over S^1, which is a contradiction by hypothesis.

If a_3 is even then

$$\begin{aligned}[(c_1,c_2)] &= [(c_1 + \tfrac{(n-1)}{2} b_3b_4c_2, c_2)] \\ &\neq [(0,0)] \in \tfrac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,0),(0,2)\rangle}.\end{aligned}$$

Then, $f : MA \to MA$ can be deformed to a fixed point free map over S^1.

Case V. Suppose that for some n odd, $n \in \mathbb{N}$ the fiber-preserving map $f^n : MA \to MA$ can be deformed to a fixed point free map over S^1.

If a_3 is even then f^n can be deformed if $n(b_4-1)(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1)(b_4-1) \equiv 0 \ mod\ 2$, except when $n(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1)$ and $\frac{b_4-1}{L}$ are odd, where $L := gcd(b_4-1, c_2)$. But $n(b_4-1)(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1)(b_4-1)$ even implies $(b_4-1)(c_1 - \frac{a_3}{2}c_2 - 1)$ even, and $n(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1)$ odd implies $(c_1 - \frac{a_3}{2}c_2 - 1)$ odd. Therefore, $f : MA \to MA$ can be deformed to a fixed point free map over S^1. The case a_3 odd is analogous. ∎

Proposition 2.3. *Let $f : MA \to MA$ be a fiber-preserving map. If m, n are odd positive integers, then f^m is deformable to a fixed point free map over S^1 if and only if f^n is deformable to a fixed point free map over S^1.*

Proof. If m, n are odd and f^m is deformable to a fixed point free map over S^1 then by Proposition 2.2 f is deformable to a fixed point free map over S^1. From Corollary 2.1 f^n is deformable to a fixed point free map over S^1. ∎

We have a analogous result for even numbers;

Proposition 2.4. *Let $f : MA \to MA$ be a fiber-preserving map, where MA is a T-bundle over S^1. Suppose that the induced homomorphism $f_\# : \pi_1(MA) \to \pi_1(MA)$ is given by; $f_\#(a) = a$, $f_\#(b) = a^{b_3}b^{b_4}$, $f_\#(c) = a^{c_1}b^{c_2}c$ as in cases of the Theorem 2.2. Given an even positive integer n such that f^n is deformable to a fixed point free map over S^1, as in Theorem 2.3, then f^k is deformable to a fixed point free map over S^1, for all even positive integer k divisor of n.*

Proof. Is enough to verify that if the conditions of the Theorem 2.3 are satisfied for some n even then those conditions are also satisfied by every even k. We will analyze each case of the Theorem 2.3.

Case I. In this case for each $n \in \mathbb{N}$ the fiber-preserving map can be deformed over S^1 to a fixed point free map.

Cases II and III. In these cases if for some n even the fiber-preserving map $f^n : MA \to MA$ is deformed to a fixed point free map over S^1 then we must have; $c_1(b_4 - 1) - c_2 b_3 = 0$ or $b_4 = -1$. Thus, for all even k, f^k can be deformed to a fixed point free map over S^1.

Case IV. If n is an even positive integer and $f^n : MA \to MA$ is deformed to a fixed point free map over S^1, then $n(b_4(b_3+1) - 1 - c_1(b_4 - 1) + b_3 c_2) - (n-1)(b_4 - 1) \equiv 0 \; mod\ 2$ and
if a_3 is odd then $[(nc_1 + \frac{n(n-1)}{2} b_3 c_2, nc_2)] \neq [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,2),(0,4)\rangle}$ or
if a_3 is even then $[(nc_1 + \frac{n(n-1)}{2} b_3 b_4 c_2, c_2 + (n-1) b_4 c_2)] \neq [(0,0)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,0),(0,2)\rangle}$.

Note that b_4 is odd when n is even. If a_3 is odd then $b_4 = 1$ and

$$\begin{array}{rcl} [(nc_1 + \frac{n(n-1)}{2} b_3 c_2, nc_2)] & = & [(0, nc_2 - 2(nc_1 + \frac{n(n-1)}{2} b_3 c_2))] \\ & = & [(0, n(c_2 - 2c_1 - (n-1) b_3 c_2)))] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,2),(0,4)\rangle}; \\ & \Rightarrow & n(c_2 - 2c_1 - (n-1)b_3 c_2) \not\equiv 0 \; mod\ 4 \\ & \Rightarrow & \left\{ \begin{array}{rcl} c_2 - 2c_1 - (n-1)b_3 c_2 & \equiv & 1 \; mod\ 2; \\ n & \equiv & 2 \; mod\ 4. \end{array} \right. \end{array}$$

If a_3 is even we have

$$\left[\left(nc_1 + \frac{n(n-1)}{2} b_3 b_4 c_2, c_2 + (n-1) b_4 c_2\right)\right] = \left[\left(\frac{n}{2} b_3 c_2, 0\right)\right] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,0),(0,2)\rangle}$$

$\Rightarrow \frac{n}{2} b_3 c_2 \equiv 1 \; mod\ 2 \;\Rightarrow n \equiv 2 \; mod\ 4$ and $b_3 c_2 \equiv 1 \; mod\ 2$.

Note that, if f^n can be deformed to a fixed point free map over S^1 then $n \equiv 2 \; mod\ 4$. Let k be an even positive integer, then

$$k(b_4(b_3+1) - 1 - c_1(b_4 - 1) + b_3 c_2) - (k-1)(b_4 - 1) \equiv 0 \; mod\ 2.$$

Hence, $f^k : MA \to MA$ can be deformed to a fixed point free map over S^1 except when $k \equiv 0 \; mod\ 4$ since;
if a_3 is odd then

$$\begin{array}{rcl} [(kc_1 + \frac{k(k-1)}{2} b_3 c_2, kc_2)] & = & [(0, k(c_2 - 2c_1 - (k-1) b_3 c_2))] \\ & = & [(0,k)] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(1,2),(0,4)\rangle}, \end{array}$$

because $c_2 - 2c_1 - (k-1) b_3 c_2 \equiv 1 \; mod\ 2$
if a_3 is even then

$$\left[\left(kc_1 + \frac{k(k-1)}{2} b_3 b_4 c_2, c_2 + (k-1) b_4 c_2\right)\right] = \left[\left(\frac{k}{2}, 0\right)\right] \in \frac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,0),(0,2)\rangle}.$$

Case V. If n is an even positive integer and $f^n : MA \to MA$ is deformed to a fixed point free map over S^1, then

if a_3 is odd then $\frac{b_4-1}{2}((1+c_2)(1+(n-1)b_4)) \equiv 0 \ mod\ 2$ and at least one of $(1+c_2)(1+(n-1)b_4)$ and $\frac{b_4-1}{L}$ is even, where $L := gcd(b_4-1, c_2)$, or
if a_3 is even then $n(b_4-1)(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1)(b_4-1) \equiv 0 \ mod\ 2$ and at least one of $n(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1)$ and $\frac{b_4-1}{L}$ is even, where $L := gcd(b_4-1, c_2)$.

Let a_3 odd and k an even positive integer then

$$\begin{array}{rcl} (1+(k-1)b_4)) & \equiv & (1+(n-1)b_4) \ mod\ 2 \\ \Rightarrow \quad \frac{b_4-1}{2}((1+c_2)(1+(k-1)b_4)) & \equiv & \frac{b_4-1}{2}((1+c_2)(1+(n-1)b_4)) \ mod\ 2 \\ & \equiv & 0 \ mod\ 2; \\ (1+c_2)(1+(k-1)b_4) & \equiv & (1+c_2)(1+(n-1)b_4) \ mod\ 2. \end{array}$$

Then, $f^k : MA \to MA$ can be deformed to a fixed point free map over S^1 for a_3 odd. Let a_3 even and k an even positive integer then

$$\begin{array}{rcl} n(b_4-1)(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1)(b_4-1) & \equiv & b_4 - 1 \ mod\ 2; \\ n(c_1 - \frac{a_3}{2}c_2 - 1) + (n-1) & \equiv & 1 \ mod\ 2; \\ \Rightarrow \quad k(b_4-1)(c_1 - \frac{a_3}{2}c_2 - 1) + (k-1)(b_4-1) & \equiv & 0 \ mod\ 2. \end{array}$$

Then, $f^k : MA \to MA$ can be deformed to a fixed point free map over S^1 for a_3 even. ■

Given $n \in \mathbb{N}$ and $f : MA \to MA$ a fiber-preserving. If $f^n : MA \to MA$ can be deformed to a fixed point free map over S^1, then from Propositions 2.3 and 2.4 the conditions to deform f and f^n to a fixed point free map over S^1 are enough to deform f^k to a fixed point free map over S^1 for all k divisor of n.

Theorem 2.4. *Let $f : T \times I \to T \times I$ be the map defined by;*

$$f(x,y,t) = (x + b_3 y + c_1 t + \varepsilon, b_4 y + c_2 t + \delta, t).$$

Denoting $f^n : T \times I \to T \times I$ by $f^n(x,y,t) = (x_n, y_n, t)$, then x_n and y_n are given by

$$\begin{array}{rcl} x_n & = & x + b_3 y \sum_{i=0}^{n-1} b_4^i + (nc_1 + b_3 c_2 \sum_{i=0}^{n-1} i b_4^{n-1-i})t + b_3 \delta \sum_{i=0}^{n-1} i b_4^{n-1-i} + n\varepsilon \\ y_n & = & b_4^n y + c_2 t \sum_{i=0}^{n-1} b_4^i + \delta \sum_{i=0}^{n-1} b_4^i. \end{array}$$

If for each positive integer n and ε, δ satisfying the following conditions, in each case of the Theorem 2.1,

Case I) $a_1\epsilon + a_3\delta = \epsilon + k$ *and* $a_2\epsilon + a_4\delta = \delta + l$ *where* $k, l \in \mathbb{Z}$
Case II) $a_3\delta \in \mathbb{Z}$
Case III) $a_3\delta \in \mathbb{Z}$ *and* $\delta = \frac{k}{2}, k \in \mathbb{Z}$
Case IV) $\epsilon = \frac{a_3 m + 2k}{4}$ *and* $\delta = \frac{m}{2}$ *where* $m, k \in \mathbb{Z}$
Case V) $\epsilon = \frac{a_3\delta + k}{2}$ *where* $k \in \mathbb{Z}$

then the map $f : T \times I \to T \times I$ induces a fiber-preserving map in the fiber bundle MA, as in Theorem 2.1, such that the induce homomorphism $f_\#$ is given by; $f_\#(a) = a$, $f_\#(b) = a^{b_3}b^{b_4}$, $f_\#(c) = a^{c_1}b^{c_2}c$. Moreover, the map $f^n : T \times I \to T \times I$ induces a fiber-preserving map in the fiber bundle MA, which we will represent by $f^n(< x, y, t >) =< x_n, y_n, t >$, such that the induces homomorphism $(f^n)_\#$ is as in the Proposition 2.1.

Proof. Denote $f^n(x,y,t) = (x_n, y_n, t)$ for each positive integer n. We have $x_2 = x_1 + b_3y_1 + c_1t + \varepsilon = (x + b_3y + c_1t + \varepsilon) + b_3(b_4y + c_2t + \delta) + c_1t + \varepsilon = x + b_3y(b_4 + 1) + (2c_1 + b_3c_2)t + b_3\delta + 2\varepsilon$. Also, $y_2 = b_4y_1 + c_2t + \delta = b_4(b_4y + c_2t + \delta) + c_2t + \delta = b_4^2y + c_2(b_4 + 1)t + (b_4 + 1)\delta$.

Suppose that $f^n(x,y,t) = (x_n, y_n, t)$ as in hypothesis, then

$$f^{n+1}(x,y,t) = (x_n + b_3y_n + c_1t + \varepsilon, b_4y_n + c_2t + \delta, t) = (x_{n+1}, y_{n+1}, t),$$

where; $x_{n+1} = x_n + b_3y_n + c_1t + \varepsilon$

$$\begin{aligned}
&= (x + b_3y\sum_{i=0}^{n-1} b_4^i + (nc_1 + b_3c_2\sum_{i=0}^{n-1} ib_4^{n-1-i})t + b_3\delta\sum_{i=0}^{n-1} ib_4^{n-1-i} \\
&\quad +n\varepsilon) + b_3(b_4^ny + c_2t\sum_{i=0}^{n-1} b_4^i + \delta\sum_{i=0}^{n-1} b_4^i) + c_1t + \varepsilon \\
&= x + (b_3y\sum_{i=0}^{n-1} b_4^i + b_3yb_4^n) + ((nc_1 + b_3c_2\sum_{i=0}^{n-1} ib_4^{n-1-i})t + c_1t + \\
&\quad b_3c_2t\sum_{i=0}^{n-1} b_4^i) + (b_3\delta\sum_{i=0}^{n-1} ib_4^{n-1-i} + b_3\delta\sum_{i=0}^{n-1} b_4^i) + (n\varepsilon + \varepsilon) \\
&= x + b_3y\sum_{i=0}^{n} b_4^i + ((n+1)c_1 + b_3c_2\sum_{i=0}^{n} ib_4^{n-i})t + b_3\delta\sum_{i=0}^{n} ib_4^{n-i} \\
&\quad +(n+1)\varepsilon;
\end{aligned}$$

$$\begin{aligned}
y_{n+1} &= b_4y_n + c_2t + \delta \\
&= b_4(b_4^ny + c_2t\sum_{i=0}^{n-1} b_4^i + \delta\sum_{i=0}^{n-1} b_4^i) + c_2t + \delta \\
&= b_4^{n+1}y + (c_2t\sum_{i=1}^{n} b_4^i + c_2t) + (\delta\sum_{i=1}^{n} b_4^i + \delta) \\
&= b_4^{n+1}y + c_2t\sum_{i=0}^{n} b_4^i + \delta\sum_{i=0}^{n} b_4^i,
\end{aligned}$$

as we wish. Now, we will verify that $f(< x,y,0 >) = f(< A\binom{x}{y}, 1 >)$.

We have; $< x,y,0 >=< A\binom{x}{y}, 1 >=< a_1x + a_3y, a_2x + a_4y, 1 >$,

$f(< x,y,0 >) =< x + b_3y + \varepsilon, b_4y + \delta, 0 >$ and $f(< A\binom{x}{y}, 1 >) = < (a_1 + a_2b_3)x + (a_3 + b_3a_4)y + c_1 + \varepsilon, b_4a_2x + b_4a_4y + c_2 + \delta, 1 >$.

Now, we will analyze each case of the Theorem 2.1.

Case I. In this case we need consider $b_3 = 0$ and $b_4 = 1$. Thus, in MA we have $f(< x,y,0 >) =< x + \epsilon, y + \delta, 0 > =< a_1x + a_3y + a_1\epsilon + a_3\delta, a_2x + a_4y + a_2\epsilon + a_4\delta, 1 >$ and $f(< A\binom{x}{y}, 1 >) =< a_1x + a_3y + c_1 + \epsilon, a_2x + a_4y + c_2 + \delta, 1 >$. Therefore, $f(< x,y,0 >) = f(< A\binom{x}{y}, 1 >)$ if $a_1\epsilon + a_3\delta = \epsilon + k$ and $a_2\epsilon + a_4\delta = \delta + l$ where $k, l \in \mathbb{Z}$.

Case II. In this case we have $a_1 = a_4 = 1$, $a_2 = 0$ and $a_3(b_4 - 1) = 0$. Therefore, $f(< x,y,0 >) =< x + b_3y + \epsilon, b_4y + \delta, 0 >=< x + (a_3 + b_3)y + \epsilon +$

$a_3\delta, b_4y+\delta, 1> = <x+(a_3+b_3)y+\epsilon+a_3\delta, b_4y+\delta, 1>$, and $f(<A\binom{x}{y}, 1>) = <x+(a_3+b_3)y+c_1+\epsilon, b_4y+c_2+\delta, 1>$. Thus, $f(<x,y,0>) = f(<A\binom{x}{y}, 1>)$ if $a_3\delta \in \mathbb{Z}$.

Case III. In this case we have $a_1 = 1$, $a_4 = -1$, $a_2 = 0$ and $a_3(b_4-1) = -2b_3$. Therefore, $f(<x,y,0>) = <x+b_3y+\epsilon, b_4y+\delta, 0> = <x+(a_3b_4+b_3)y+\epsilon+a_3\delta, -b_4y-\delta, 1> = <x+(a_3-b_3)y+\epsilon+a_3\delta, -b_4y-\delta, 1>$, and $f(<A\binom{x}{y}, 1>) = <x+(a_3-b_3)y+c_1+\epsilon, -b_4y+c_2+\delta, 1>$. Then, $f(<x,y,0>) = f(<A\binom{x}{y}, 1>)$ if $a_3\delta \in \mathbb{Z}$ and $\delta = \frac{k}{2}, k \in \mathbb{Z}$.

Case IV. In this case we have $a_1 = -1$, $a_4 = -1$, $a_2 = 0$ and $a_3(b_4-1) = 0$. Thus, $f(<x,y,0>) = <x+b_3y+\epsilon, b_4y+\delta, 0> = <-x+(a_3b_4-b_3)y-\epsilon+a_3\delta, -b_4y-\delta, 1> = <-x+(a_3-b_3)y-\epsilon+a_3\delta, -b_4y-\delta, 1>$, and $f(<A\binom{x}{y}, 1>) = <-x+(a_3-b_3)y+c_1+\epsilon, -b_4y+c_2+\delta, 1>$. Therefore, $f(<x,y,0>) = f(<A\binom{x}{y}, 1>)$ if $\epsilon = \frac{a_3m+2k}{4}$ and $\delta = \frac{m}{2}$ where $m, k \in \mathbb{Z}$.

Case V. In this case we have $a_1 = -1$, $a_4 = 1$, $a_2 = 0$ and $a_3(b_4-1) = 2b_3$. Therefore, $f(<x,y,0>) = <x+b_3y+\epsilon, b_4y+\delta, 0> = <-x+(a_3b_4-b_3)y-\epsilon+a_3\delta, b_4y+\delta, 1> = <x+(a_3+b_3)y-\epsilon+a_3\delta, b_4y+\delta, 1>$ and $f(<A\binom{x}{y}, 1>) = <-x+(a_3+b_3)y+c_1+\epsilon, b_4y+c_2+\delta, 1>$. Thus, $f(<x,y,0>) = f(<A\binom{x}{y}, 1>)$ if $\epsilon = \frac{a_3\delta+k}{2}$ where $k \in \mathbb{Z}$.

In an analogous way we obtain the following conditions for f^n, in each case of the Theorem 2.1.

$$\textit{Case I)} \quad na_1\varepsilon + na_3\delta = n\varepsilon + k, \textit{ and } na_2\varepsilon + na_4\delta = n\delta + l$$

$$\textit{Case II)} \quad \delta a_3 \sum_{i=0}^{n-1} b_4^i \in \mathbb{Z}$$

$$\textit{Case III)} \quad 2\delta \sum_{i=0}^{n-1} b_4^i \in \mathbb{Z} \textit{ and } \left(a_3 \sum_{i=0}^{n-1} b_4^i + 2b_3 \sum_{i=0}^{n-1} i b_4^{n-1-i}\right)\delta \in \mathbb{Z}$$

$$\textit{Case IV)} \quad 2\delta \sum_{i=0}^{n-1} b_4^i \in \mathbb{Z} \textit{ and } 2n\varepsilon = a_3\delta \sum_{i=0}^{n-1} b_4^i + k,$$

$$\textit{Case V)} \quad 2n\varepsilon = a_3\delta \sum_{i=0}^{n-1} b_4^i + k$$

where $k, l \in \mathbb{Z}$. Thus for each $n \in \mathbb{N}$ and ϵ, δ satisfying the conditions above the map $f^n : T \times I \to T \times I$ induces a fiber-preserving map on MA which will be represent by the same symbol. ∎

Proposition 2.5. *Let $n,\ b_3,\ b_4,\ c_1,\ c_2 \in \mathbb{Z}$, $n \geq 1$. If $c_1(b_4-1) - c_2b_3 \neq 0$ then for all $\varepsilon,\ \delta \in \mathbb{R}$ there are $k_n,\ l_n \in \mathbb{Z}$ such that $x_n = x + k_n$ and $y_n = y + l_n$ has solution $(x, y, t) \in \mathbb{R}^2 \times I$, where:*

$$x_n = x + b_3 y \sum_{i=0}^{n-1} b_4^i + (nc_1 + b_3c_2 \sum_{i=0}^{n-1} ib_4^{n-1-i})t + b_3\delta \sum_{i=0}^{n-1} ib_4^{n-1-i} + n\varepsilon;$$
$$y_n = b_4^n y + c_2 t \sum_{i=0}^{n-1} b_4^i + \delta \sum_{i=0}^{n-1} b_4^i.$$

Proof. Suppose $b_4 \neq 1$ and $b_4 \neq -1$ with n even ($b_4 = -1$ with n odd is allowed) and $c_1(b_4-1) - b_3c_2 \neq 0$ then given $\varepsilon,\ \delta \in \mathbb{R}$ we have the solutions $x \in \mathbb{R}$ and:

$$\begin{array}{rcl} t & = & \frac{nb_3\delta - n(b_4-1)\varepsilon - (b_4-1)k_n - b_3l_n}{n(c_1(b_4-1)-b_3c_2)}; \\ y & = & \frac{nc_2\varepsilon - nc_1\delta - k_nc_2}{n(c_1(b_4-1)-b_3c_2)} + l_n\left(\frac{1}{b_4^n-1} + \frac{b_3c_2}{n(b_4-1)(c_1(b_4-1)-b_3c_2)}\right) \in \mathbb{R}. \end{array}$$

Thus, we need to find $k_n,\ l_n \in \mathbb{Z}$ such that $0 \leq t \leq 1$. Let $\Delta_0 = n(c_1(b_4-1) - b_3c_2) \in \mathbb{Z}$, $\Delta_0 \neq 0$, and $\Delta_1 = nb_3\delta - n(b_4-1)\varepsilon \in \mathbb{R}$, $t = \frac{\Delta_1 - (b_4-1)k_n - b_3l_n}{\Delta_0}$. If $0 \leq \Delta_1 \leq \Delta_0$ or $\Delta_0 \leq \Delta_1 \leq 0$ let $k_n = l_n = 0$, then $t = \frac{\Delta_1}{\Delta_0}$. If $0 < \Delta_0 \leq \Delta_1$ or $\Delta_1 \leq 0 < \Delta_0$ then there are $d,\ q \in \mathbb{Z}$ such that $\Delta_1 = d\Delta_0 + q$ with $0 \leq q < \Delta_0$. Let $k_n = nc_1d$ and $l_n = nc_2d$, then

$$t = \frac{d\Delta_0 + q - (b_4-1)nc_1d - b_3nc_2d}{\Delta_0} = d + \frac{q}{\Delta_0} - \frac{d\Delta_0}{\Delta_0} = \frac{q}{\Delta_0}.$$

If $\Delta_1 \leq \Delta_0 < 0$ or $\Delta_0 < 0 \leq \Delta_1$ then there are $d,\ q \in \mathbb{Z}$ such that $\Delta_1 = d\Delta_0 + q$ with $0 \leq q < |\Delta_0|$. Let $k \in \mathbb{Z}$ the least integer greater than $\frac{-q}{\Delta_0}$, $k_n = nc_1(d-k)$ and $l_n = nc_2(d-k)$, then

$$t = \frac{d\Delta_0 + q - (b_4-1)nc_1(d-k) - b_3nc_2(d-k)}{\Delta_0} = \frac{q}{\Delta_0} + k.$$

Then, $0 \leq t \leq 1$. If $b_4 = 1$ and $c_1(b_4-1) - b_3c_2 \neq 0$ then $b_3c_2 \neq 0$. Thus, given $\varepsilon,\ \delta \in \mathbb{R}$ we have the solutions $x \in \mathbb{R}$ and:

$$\begin{array}{rcl} t & = & \frac{l_n}{nc_2} - \frac{\delta}{c_2}; \\ y & = & \frac{-nc_2\varepsilon + nc_1\delta + k_nc_2}{nb_3c_2} - l_n\left(\frac{c_1}{nb_3c_2} + \frac{n-1}{2n}\right) \in \mathbb{R}. \end{array}$$

We need to find $l_n \in \mathbb{Z}$ such that $0 \leq t \leq 1$. If $c_2 > 0$ take $n\delta \leq l_n \leq n(c_2 + \delta)$ and if $c_2 < 0$ take $n\delta \geq l_n \geq n(c_2 + \delta)$.

Remark 2.2. *Note that the hypothesis $f, f^n : MA \to MA$ can be deformed to a fixed point free map over S^1, is equivalent to require that the induced homomorphisms $f_\#$ and $f_\#^n$ satisfy the conditions of the Theorem 2.3 in each case of the fiber bundle MA. But if $f_\#$ and $f_\#^n$ satisfy the conditions of the Theorem 2.3 then, by Propositions 2.2, 2.3 and 2.4, the induced homomorphism $f_\#^k$ satisfies the conditions of the Theorem 2.3 for each k positive divisor of n. Thus, the hypothesis $f, f^n : MA \to MA$ can be deformed to a fixed point free map over S^1 implies that $f^k : MA \to MA$ can be deformed to a fixed point free map over S^1, for each k positive divisor of n.*

3 Fixed points of f^n

In this section we will give the proof of the main result.

Theorem 3.1 (Main Theorem). *Let $f : MA \to MA$ be a fiber-preserving map, where MA is a T-bundle over S^1 as in the Theorem 2.1, and $n > 1$ a positive integer. Suppose $f_\#(a) = a$, $f_\#(b) = a^{b_3}b^{b_4}$ and $f_\#(c) = a^{c_1}b^{c_2}c$, and $f, f^n : MA \to MA$ can be deformed to a fixed point free map over S^1. If the following conditions are satisfied in each case bellow then f is fiberwise homotopic to a g so that g^n is fixed point free.*

Case I
i) $(a_1 - 1)(a_4 - 1) - a_2a_3 \neq 0$ *and* $gcd((a_4 + a_2 - 1), (a_3 + a_1 - 1)) > 1$.
ii) $(a_1 - 1)(a_4 - 1) - a_2a_3 = 0$, $c_2 \neq 0$ *and* $a_1 = 1$.
Case II
i) $c_1(b_4 - 1) - c_2b_3 = 0$, $|b_3| + |b_4 - 1| \neq 0$ *and* $b_4 \neq 1$
ii) $c_1(b_4 - 1) - c_2b_3 = 0$, $|b_3| + |b_4 - 1| \neq 0$, $b_4 = 1$ *and* a_3 *not divides* n.
iii) $c_1(b_4 - 1) - c_2b_3 = 0$, $|b_3| + |b_4 - 1| \neq 0$, $b_4 = 1$ *and* $a_3 = 0$.
Case III
$c_1(b_4 - 1) - c_2b_3 = 0$, $|b_3| + |b_4 - 1| \neq 0$.
Case IV
$c_1(b_4 - 1) - c_2b_3 = 0$, $|b_3| + |b_4 - 1| \neq 0$.
Case V
$c_1(b_4 - 1) - c_2b_3 = 0$, $|b_3| + |b_4 - 1| \neq 0$.

Remark 3.1. *Note that in the Case III, the condition $c_1(b_4 - 1) - c_2b_3 = 0$ is necessary and sufficient to deform f and f^n to a fixed point free map. Thus, if $c_1(b_4 - 1) - c_2b_3 \neq 0$ can not exist g fiberwise homotopic to f such g^n is fixed point free. The condition $|b_3| + |b_4 - 1| \neq 0$ in the cases II, III, IV and V is only to guarantee that the matrix $B = [(f_{|T})_\#]$ is not the identity matrix is these cases.*

Proof (*of the main theorem*). The technique used to proof the main theorem consists to show that for appropriated ε and δ the map $g : T \times I \to T \times I$ defined by; $g((x, y, t)) = (x + b_3y + c_1t + \varepsilon, b_4y + c_2t + \delta, t)$ induces a fiber-preserving map on MA, which we will represent by the same symbol, such that $f \sim_{S^1} g$ and g^n is a fixed point free map. Note that if $c_1(b_4 - 1) - c_2b_3 \neq 0$, then by Proposition 2.5 that map g does not works, that is, g^n will have fixed points. Thus, will use g in the situation $c_1(b_4 - 1) - c_2b_3 = 0$. From Theorem 2.4, the map g^n induces a fiber-preserving map if ε, δ satisfy the following conditions, in each case of the Theorem 2.1,

$$\begin{array}{ll}
Case\ I) & na_1\varepsilon + na_3\delta = n\varepsilon + k, \text{ and } na_2\varepsilon + na_4\delta = n\delta + l \\
Case\ II) & \delta a_3 \sum_{i=0}^{n-1} b_4^i \in \mathbb{Z} \\
Case\ III) & 2\delta \sum_{i=0}^{n-1} b_4^i \in \mathbb{Z} \text{ and } \left(a_3 \sum_{i=0}^{n-1} b_4^i + 2b_3 \sum_{i=0}^{n-1} i b_4^{n-1-i} \right) \delta \in \mathbb{Z} \\
Case\ IV) & 2\delta \sum_{i=0}^{n-1} b_4^i \in \mathbb{Z} \text{ and } 2n\varepsilon = a_3\delta \sum_{i=0}^{n-1} b_4^i + k, \\
Case\ V) & 2n\varepsilon = a_3\delta \sum_{i=0}^{n-1} b_4^i + k
\end{array}$$

where $k, l \in \mathbb{Z}$. Is important to observe that our interest is in the case $n > 1$. Let us suppose that each map $f, f^n : MA \to MA$ can be deformed to a fixed point free map over S^1.

(Case I) For each map f such that $(f_{|T})_\# = Id$ consider the map g' fiberwise homotopic to f given by: $g'(< x, y, t >) =< x + c_1t + \epsilon, y + c_2t + \delta, t >$, with ϵ, δ satisfying the conditions;

$$(I) \quad \begin{cases} na_1\epsilon + na_3\delta & = & n\epsilon + k_n \\ na_2\epsilon + na_4\delta & = & n\delta + l_n \end{cases}$$

for some $k_n, l_n \in \mathbb{Z}$. If $det = (a_1 - 1)(a_4 - 1) - a_2a_3 \neq 0$, we obtain

$$(II) \quad \epsilon = \frac{k_n(a_4 - 1) - a_3l_n}{ndet} \quad and \quad \delta = \frac{l_n(a_1 - 1) - a_2k_n}{ndet}$$

Note that g' is fiberwise homotopic to the map g defined by:

$$g(< x, y, t >) = \begin{cases} < x + 2c_1t + \epsilon, y + \delta, t > & if \quad 0 \le t \le \frac{1}{2} \\ < x + c_1 + \epsilon, y + c_2(2t - 1) + \delta, t > & if \quad \frac{1}{2} \le t \le 1 \end{cases}$$

In fact, $H : MA \times I \to MA$ defined by:

$$H(< x, y, t >, s) = \begin{cases} < x + c_1t + \epsilon, y + c_2t + \delta, t > & if \quad 0 \le t \le s \\ < x + c_1(2t - s) + \epsilon, y + c_2s + \delta, t > & if \quad s \le t \le \frac{s+1}{2} \\ < x + c_1 + \epsilon, y + c_2(2t - 1) + \delta, t > & if \quad \frac{s+1}{2} \le t \le 1 \end{cases}$$

is a homotopy between g' and g. Note that,

$$g^n(< x, y, t >) = \begin{cases} < x + n2c_1t + n\epsilon, y + n\delta, t > & if \quad 0 \le t \le \frac{1}{2} \\ < x + nc_1 + n\epsilon, y + nc_2(2t - 1) + n\delta, t > & if \quad \frac{1}{2} \le t \le 1 \end{cases}$$

i) Suppose $det \neq 0$ and $d = gcd((a_4 + a_2 - 1), (a_3 + a_1 - 1)) > 1$. Choose $\epsilon = \delta = \dfrac{1}{nd}$. This values satisfy the system (I) and $n\epsilon = n\delta = \dfrac{1}{d} \in \mathbb{Q} - \mathbb{Z}$. If g^n has a fixed point for $0 \le t \le \frac{1}{2}$ then we must have $n\delta \in \mathbb{Z}$. Also, if g^n has a fixed point for $\frac{1}{2} \le t \le 1$ then we must have $n\epsilon \in \mathbb{Z}$, which is a contradiction, that is, $Fix(g^n) = \emptyset$.

ii) Suppose $0 = det = (a_1 - 1)(a_4 - 1) - a_2a_3$, $c_2 \neq 0$ and $a_1 = 1$. Thus, we must have $a_2 = 0$. From system (I) we obtain the equations; $na_3\delta = k_n$ and $n(a_4 - 1)\delta = l_n$, for some $k_n, l_n \in \mathbb{Z}$. This equations do not depend of ϵ, therefore we can choose ϵ an irrational number. Thus, we choose ϵ an irrational number and $\delta = \frac{1}{n}$.

We observe that both g and g' are fiberwise homotopic to the given map f, and $(g')^n(< x, y, t >) =< x + nc_1t + n\epsilon, y + nc_2t + n\delta, t >$, with ϵ, δ satisfying the conditions of the system (I). If $(g')^n$ has a fixed point then we must have $nc_1t + n\epsilon = p_n$ and $nc_2t + n\delta = q_n$ for some $p_n, q_n \in \mathbb{Z}$. If $c_1 = 0$ we have a contradiction because ϵ is an irrational number. If $c_1 \neq 0$ and $c_2 \neq 0$ then we have $nc_2\epsilon - nc_1\delta = c_2p_n - c_1q_n$, which is a contradiction because ϵ is an irrational number and $\delta = \frac{1}{n}$. Therefore, $(g')^n$ can not have a fixed point.

(Case II) Let $g : MA \to MA$ be the map fiberwise homotopy to f given by $g(< x,y,t >) =< x + b_3y + c_1t + \varepsilon, b_4y + c_2t + \delta, t >$, where $a_3\delta \in \mathbb{Z}$. If $b_4 = 1$ then $c_2b_3 = 0$, but if $b_3 = 0$ then the matrix B of $(f_{|T})_\#$ is the identity, contradicting a hypothesis. Suppose $b_4 = 1$, $b_3 \neq 0$ and $c_2 = 0$, by Theorem 2.4, $g^n(< x,y,t >) =< x_n, y_n, t >$ for each $n \in \mathbb{N}$ where,

$$\begin{cases} x_n &= x + nb_3y + nc_1t + \left(\frac{n(n-1)}{2}\right) b_3\delta + n\varepsilon, \\ y_n &= y + n\delta. \end{cases}$$

If $g^n : MA \to MA$ has a fixed point $< x,y,t >$ then $x_n = x + k_n$ and $y_n = y + l_n$ for some $k_n, l_n \in \mathbb{Z}$. By the second equation of the system above we must have $n\delta = l_n$ for some $l_n \in \mathbb{Z}$. Therefore, $g^n : MA \to MA$ is fixed point free if $a_3 = 0$ and $\delta \in \mathbb{R} - \mathbb{Q}$ or if a_3 not divides n and $\delta = \frac{1}{a_3}$.

Now we suppose $b_4 \neq 1$ and we choose $\delta = 0$ then $g^n(< x,y,t >) = < x_n, y_n, t >$, where

$$\begin{aligned} x_n &= x + b_3y\sum_{i=0}^{n-1} b_4^i + \left(nc_1 + b_3c_2\sum_{i=0}^{n-1} ib_4^{n-1-i}\right) t + n\varepsilon \\ &= x + \left(\frac{b_4^n-1}{b_4-1}\right) b_3y + \left(nc_1 + \frac{b_3c_2(b_4^n-1+n(1-b_4))}{(b_4-1)^2}\right) t + n\varepsilon \\ &= x + \left(\frac{b_4^n-1}{b_4-1}\right) b_3y + \left(\frac{b_4^n-1}{b_4-1}\right) c_1t + n\varepsilon; \\ y_n &= b_4^ny + c_2t\sum_{i=0}^{n-1} b_4^i \;=\; b_4^ny + \left(\frac{b_4^n-1}{b_4-1}\right) c_2t. \end{aligned}$$

If $b_4 = -1$ and n is even then $g^n : MA \to MA$ is fixed point free for $\varepsilon \in \mathbb{R} - \mathbb{Q}$, otherwise we had $n\varepsilon = k_n \in \mathbb{Z}$. Suppose $b_4 \neq 1$ or $b_4 = -1$ with n odd. If $c_2 \neq 0$ we have:

$$t = \frac{l_n(b_4-1)}{c_2(b_4^n-1)} + \frac{1-b_4}{c_2}y.$$

$$\begin{aligned} \Rightarrow x_n &= x + \left(\frac{b_4^n-1}{b_4-1}\right) b_3y + \left(\frac{b_4^n-1}{b_4-1}\right) c_1t + n\varepsilon \\ &= x - \left(\frac{(b_4^n-1)((b_4-1)c_1-c_2b_3)}{(b_4-1)c_2}\right) y + n\varepsilon + \frac{c_1l_n}{c_2} \\ &= x + n\varepsilon + \frac{c_1l_n}{c_2} \\ \Rightarrow x + k_n &= x + n\varepsilon + \frac{c_1l_n}{c_2} \Rightarrow k_n = n\varepsilon + \frac{c_1l_n}{c_2}. \end{aligned}$$

Hence, then $g^n : MA \to MA$ is fixed point free for $\varepsilon \in \mathbb{R} - \mathbb{Q}$. On the other hand, if $c_2 = 0$ then $c_1 = 0$ because $b_4 \neq 1$. Therefore,

$$\begin{aligned} x_n &= x + \left(\frac{b_4^n-1}{b_4-1}\right) b_3y + \left(\frac{b_4^n-1}{b_4-1}\right) c_1t + n\varepsilon \\ &= x + \left(\frac{b_4^n-1}{b_4-1}\right) b_3y + n\varepsilon; \\ y_n &= b_4^ny + \left(\frac{c_2(b_4^n-1)}{b_4-1}\right) t \;=\; b_4^ny. \end{aligned}$$

So, $g^n : MA \to MA$ is fixed point free for $\varepsilon \in \mathbb{R} - \mathbb{Q}$, otherwise $y = \frac{l_n}{b_4^n-1}$,

$x_n = x + n\varepsilon + \frac{b_3 l_n}{b_4 - 1}$ and

$$x + n\varepsilon + \frac{b_3 l_n}{b_4 - 1} = x + k_n \Rightarrow \underbrace{\varepsilon}_{\in \mathbb{R}-\mathbb{Q}} = \underbrace{\frac{k_n}{n} - \frac{b_3 l_n}{n(b_4 - 1)}}_{\in \mathbb{Q}}.$$

(Case III) The proof in this case is similar to the case (2), but here we consider $a_3(b_4 - 1) = -2b_3$, $\delta = \frac{k}{2}$ with $a_3\delta \in \mathbb{Z}$ and $k \in \mathbb{Z}$. If $b_4 = 1$ then $b_3 = 0$, and this situation we will have $|b_3| + |b_4 - 1| = 0$ contradicting a hypothesis. If $b_4 \neq 1$ then $g^n : MA \to MA$ is fixed point free for $\varepsilon \in \mathbb{R} - \mathbb{Q}$ and the proof is the same of the case II.

(Case IV) Suppose $g(< x, y, t >) =< x + b_3 y + c_1 t + \varepsilon, b_4 y + c_2 t + \delta, t >$ such that $b_4(nb_3 + 1) \equiv 1 \; mod 2$, $a_3(b_4 - 1) = 0$, $\delta = \frac{m}{2}$ and $\varepsilon = \frac{a_3 m + 2r}{4}$, $m, r \in \mathbb{Z}$. Thus, given $n \geq 1$ and $g^n(< x, y, t >) =< x_n, y_n, t >$ we want to know when g^n has a fixed point, i.e., there are $k_n, l_n \in \mathbb{Z}$ such that $x_n = x + k_n$ and $y_n = y + l_n$.

Note that the expression $b_4(nb_3 + 1) \equiv 1 \; mod \; 2$ follows from item 3 of Theorem 2.3 as below

$$\begin{aligned} & n(b_4(b_3 + 1) - 1 - \underbrace{c_1(b_4 - 1) + b_3 c_2}_{=0}) - (n - 1)(b_4 - 1) \equiv 0 \; mod 2 \\ \Rightarrow \quad & nb_4 b_3 + nb_4 - n - (n - 1)b_4 + (n - 1) \equiv 0 \; mod 2 \\ \Rightarrow \quad & nb_4 b_3 + b_4 - 1 \equiv 0 \; mod 2. \end{aligned}$$

If $b_4 = 1$ and n is odd then we must have $c_2 = 0$ because if $b_3 = 0$ then we would have $|b_3| + |b_4 - 1| = 0$. So, $g^n : MA \to MA$ has not a fixed point $< x, y, t >$ for $\delta = \frac{1}{2}$, otherwise we had $y + l_n = y + \frac{n}{2}$ and $l_n = \frac{n}{2} \in \mathbb{Z}$. Note that we have a exception if $b_4 = 1$ and n even, because $c_2 = 0$. Hence, g^n is fixed point free if $b_4 = 1$ and $\delta = \frac{1}{2}$.

Suppose $b_4 \neq 1$. From expression $b_4(nb_3 + 1) \equiv 1 \; mod \; 2$, proved above, we must have b_4 odd. Thus, we have $a_3 = 0$ and $[(nc_1 + \frac{n(n-1)}{2} b_3 b_4 c_2, c_2 + (n - 1)b_4 c_2)] = [(nc_1, nc_2)] \neq [(0, 0)] \in \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (2,0),(0,2) \rangle}$. If $g^n : MA \to MA$ has a fixed point $< x, y, t >$ then

$$\begin{aligned} y + l_n &= b_4^n y + c_2 \left(\frac{b_4^n - 1}{b_4 - 1} \right) t + \left(\frac{b_4^n - 1}{b_4 - 1} \right) \delta \\ \Rightarrow y &= \frac{l_n(b_4 - 1) - (\delta + c_2 t)(b_4^n - 1)}{(b_4 - 1)(b_4^n - 1)} \end{aligned}$$

$$\Rightarrow x_n = x + \frac{b_3(l_n - n\delta)}{(b_4 - 1)} + n\varepsilon \quad \Rightarrow k_n = \frac{b_3(l_n - n\delta)}{(b_4 - 1)} + n\varepsilon.$$

So, $k_n \notin \mathbb{Z}$ for appropriates δ and ε, $n \in \mathbb{N}$. Therefore, $g^n : MA \to MA$ is fixed point free.

(Case V) Suppose $g(< x, y, t >) =< x + b_3 y + c_1 t + \varepsilon, b_4 y + c_2 t + \delta, t >$ such that $a_3(b_4 - 1) = 2b_3$, $\varepsilon = \frac{a_3\delta + 1}{2}$, $m \in \mathbb{Z}$. We must consider $b_4 \neq 1$, otherwise we will obtain $b_3 = 0$ since $a_3(b_4 - 1) = 2b_3$, therefore $|b_3| + |b_4 - 1| = 0$ contradicting our hypothesis. Suppose $b_4 \neq 1$. From Theorem 2.4 we have two equations;

$$\begin{aligned}
(I)\quad x_n &= x + b_3 y \sum_{i=0}^{n-1} b_4^i + (nc_1 + b_3 c_2 \sum_{i=0}^{n-1} i b_4^{n-1-i})t + b_3\delta \sum_{i=0}^{n-1} i b_4^{n-1-i} \\
&\quad + n\varepsilon \\
&= x + b_3 y \sum_{i=0}^{n-1} b_4^i + c_1 t \sum_{i=0}^{n-1} b_4^i + b_3\delta \sum_{i=0}^{n-1} i b_4^{n-1-i} + n\varepsilon \\
(II)\quad y_n &= b_4^n y + c_2 t \sum_{i=0}^{n-1} b_4^i + \delta \sum_{i=0}^{n-1} b_4^i.
\end{aligned}$$

If $b_4 = -1$ and n is even then $g^n : MA \to MA$ has not a fixed point $< x, y, t >$ for $\delta \in \mathbb{R} - \mathbb{Q}$ and $\varepsilon = \frac{a_3\delta+1}{2}$, otherwise

$$\begin{aligned}
x + k_n &= x - \tfrac{nb_3\delta}{2} + n\varepsilon,\ k_n \in \mathbb{Z} \\
\Rightarrow k_n &= \tfrac{n\delta(a_3-b_3)+1}{2} \notin \mathbb{Z}.
\end{aligned}$$

Now suppose $n > 1$ any natural number with $b_4 \neq 1$, (except $b_4 = -1$ and n even, which was already made). In this situation $g^n : MA \to MA$ has not a fixed point $< x, y, t >$ for $\delta \in \mathbb{R} - \mathbb{Q}$ and $\varepsilon = \frac{a_3\delta+1}{2n}$, otherwise we will obtain $x_n = x + k_n$ and $y_n = y + l_n$ with $k_n, l_n \in \mathbb{Z}$. From equation (II) we obtain

$$\begin{aligned}
y + l_n &= b_4^n y + c_2 \left(\tfrac{b_4^n - 1}{b_4 - 1}\right) t + \left(\tfrac{b_4^n - 1}{b_4 - 1}\right) \delta \\
\Rightarrow y &= \frac{l_n(b_4 - 1) - (\delta + c_2 t)(b_4^n - 1)}{(b_4 - 1)(b_4^n - 1)}
\end{aligned}$$

Replacing the value of y of the last equation into equation (I), and using $\varepsilon = \frac{a_3\delta+1}{2n}$, we will obtain;

$$x_n = x + \frac{b_3 l_n (b_4^n - 1)}{(b_4 - 1)} - \delta \frac{b_3(n-1)}{b_4 - 1} + \frac{1}{2}$$

Replacing this value into the equation $x_n = x + k_n$ we obtain;

$$k_n = \frac{b_3 l_n (b_4^n - 1)}{(b_4 - 1)^2} - \delta \frac{b_3(n-1)}{b_4 - 1} + \frac{1}{2}$$

When $b_3 \neq 0$ we have a contradiction because $\delta \in \mathbb{R} - \mathbb{Q}$. When $b_3 = 0$ we have a contradiction because $k_n \in \mathbb{Z}$. Therefore, $g^n : MA \to MA$ is a fixed point free map. ■

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