The equivalence of two methods: finding representatives of non–empty Nielsen classes

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Abstract

Let $f : X \to X$ be a self–map with X a wedge of circles or a compact surface with boundary, so that the fundamental group of X is finitely generated and free. In [3], Wagner presents an algorithm for extracting information from the homomorphism induced by f on the fundamental group. This information involves the fixed point index of f and the Nielsen classes of fixed points of f.

The step in which the representatives of Nielsen classes, Wagner tails, are calculated is equivalent to a step in the method presented by Fadell and Husseini in [1].

The Fadell–Husseini method was designed for closed two dimensional CW–complexes, but the step in which they use the Fox calculus, to determine terms in the unreduced Reidemeister trace, produces the Wagner tails and their indices.

The equivalence of these steps was stated in [2] without proof. Further developments in this area have caused continued interest in the techniques, and a clarification of the equivalence is needed. Here we provide the proof and an example.

1 Introduction

Let $f : X \to X$ be a self–map with X a wedge of circles or a compact surface with boundary, so that the fundamental group of X is finitely generated and free. Let F_r be the free group on r generators, and let $f_* : F_r \to F_r$ be the homomorphism induced by the map f.

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Consider a generator a of F_r. The word $f_*(a)$ contains information regarding algebraic representatives of Nielsen fixed point classes as well as contributions to the fixed point index of such classes. See [2] for details.

As described in Section 2, Wagner's algorithm produces initial subwords of $f_*(a)$, called initial Wagner tails $W_1, \ldots, W_n$, which are algebraic representatives of non–empty Nielsen fixed point classes. In addition, the algorithm finds contributions toward the index of each such class, with $\epsilon_j \in \{-1, 1\}$ being the contribution of W_j to the index of the class containing the word W_j.

We use here the notation that corresponds to the Reidemeister action $f_*(a) = W_j a \overline{W}_j^{-1}$. When comparing with papers that use a different notation, slight adjustments must be made.

In Section 4, we present the relevant step from the The Fadell–Husseini method from [1]. The Fox derivative applied to the word $f_*(a)$ produces an element of $\mathbb{Z}[F_r]$, and the coefficients contribute to the index of the related Nielsen fixed point classes.

Our main result is that these two calculations produce the same information.

Theorem 1.1. [Main Result] With the notation given above, we have

$$\frac{\partial f_*(a)}{\partial a} = \sum_{j=1}^{n} \epsilon_j W_j.$$

Note that, for the Fadell–Husseini method, this term contributes to the trace in dimension 1 for the alternating sum that is the Fox trace. Thus $\frac{\partial f_*(a)}{\partial a}$ appears with a coefficient of -1 in the Fox trace.

2 Determining Initial Wagner Tails

Let $\mathcal{G} = \{a_1, \ldots, a_r\}$ be a set of generators for F_r. Let $a \in \mathcal{G}$. Assume that $a^{\pm 1}$ occurs exactly n times in the word $f_*(a)$. Then there are (possibly trivial) elements $u_0, u_1, \ldots, u_n$ in the free group generated by $\mathcal{G} \backslash \{a\}$ and exponents $\epsilon_i = \pm 1$ such that $f_*(a) = u_0 a^{\epsilon_1} u_1 a^{\epsilon_2} u_2 \cdots a^{\epsilon_n} u_n$.

Let α_j be the initial segment of $f_*(a)$ that occurs before the j–th occurrence of $a^{\pm 1}$ in $f_*(a)$. That is, let $\alpha_j = u_0 a^{\epsilon_1} u_1 a^{\epsilon_2} u_2 \cdots u_{j-1}$. The initial Wagner tail W_j is given by

$$W_j = \begin{cases} \alpha_j & \text{if } \epsilon_j = 1, \\ \alpha_j a^{-1} & \text{if } \epsilon_j = -1. \end{cases}$$

Also, the corresponding fixed point has index $-\epsilon_j$.

3 Example, Part I:

Let $\mathcal{G} = \{a, b, c, d\}$. Suppose that $f_*(a) = b^5 a^3 c b^{-1} a^{-2} d$.

We have three occurrences of a and two occurrences of a^{-1} in $f_*(a)$, and thus $n = 5$. The table below provides for each j the index ϵ_j and the initial Wagner tail, W_j.

j	ϵ_j	α_j	W_j
1	1	b^5	b^5
2	1	b^5a	b^5a
3	1	b^5a^2	b^5a^2
4	-1	$b^5a^3cb^{-1}$	$b^5a^3cb^{-1}a^{-1}$
5	-1	$b^5a^3cb^{-1}a^{-1}$	$b^5a^3cb^{-1}a^{-2}$

4 Using the Fox Derivative

The Fadell–Husseini method, as described in [1] and in [2], is applied to closed surfaces with one–relator fundamental groups. By using zero as the trace in dimension two, the same method can be applied to the spaces considered by Wagner.

Fadell and Husseini use the Fox derivative of $f_*(a)$, defined below, to calculate the trace in dimension one of the Fox trace. This Fox trace is the unreduced version of the Reidemeister trace used to determine the Nielsen number of f.

Let $\mathcal{G} = \{a_1, \ldots, a_r\}$ be a set of generators for F_r. The Fox derivatives for words in F_r are defined as follows: For any $a_i, a_j \in \mathcal{G}$, $\frac{\partial 1}{\partial a_i} = 0$; $\frac{\partial a_j}{\partial a_i} = \delta_{ij}$; and, for any $u, v \in F_r$, $\frac{\partial}{\partial a_i}(uv) = \frac{\partial u}{\partial a_i} + u\frac{\partial v}{\partial a_i}$. Then $\frac{\partial}{\partial a_i}$ is a function from F_r to the free $\mathbb{Z}$ module over F_r.

Proposition 4.1. For any $w \in F_r$ and any $a \in \mathcal{G}$, we have $\frac{\partial w^{-1}}{\partial a} = -w^{-1}\frac{\partial w}{\partial a}$.

Proof. Note that $0 = \frac{\partial 1}{\partial a} = \frac{\partial}{\partial a}(w^{-1}w) = \frac{\partial w^{-1}}{\partial a} + w^{-1}\frac{\partial w}{\partial a}$. The result follows. ∎

Proposition 4.2. Let $a \in \mathcal{G}$ and $u \in F_r$ such that u is in the free group generated by $\mathcal{G}\backslash\{a\}$. Then $\frac{\partial}{\partial a}(u) = 0$, $\frac{\partial}{\partial a}(au) = 1$, and $\frac{\partial}{\partial a}(a^{-1}u) = -a^{-1}$.

Proof. If $u = 1$, the result follows from the definition and from Proposition 4.1. Otherwise, we can express u uniquely as a reduced word $x_1x_2 \ldots x_k$ so that for each i either $x_i \in \mathcal{G}\backslash\{a\}$ or $x_i^{-1} \in \mathcal{G}\backslash\{a\}$.

We induct on k. Suppose $u = x_1 \in \mathcal{G}\backslash\{a\}$. We have that $\frac{\partial u}{\partial a} = 0$ because $x_1 \neq a^{\pm 1}$. For larger k we have $u = (x_1x_2 \ldots x_{k-1})x_k$, so that by induction

$$\frac{\partial u}{\partial a} = \frac{\partial x_1x_2 \ldots x_{k-1}}{\partial a} + x_1x_2 \ldots x_{k-1}\frac{\partial x_k}{\partial a} = 0 + (x_1x_2 \ldots x_{k-1})(0) = 0.$$

Thus $\frac{\partial u}{\partial a} = 0$.

From this we have $\frac{\partial au}{\partial a} = \frac{\partial a}{\partial a} + a\frac{\partial u}{\partial a} = 1 + (a)(0) = 1$. Similarly, $\frac{\partial a^{-1}u}{\partial a} = \frac{\partial a^{-1}}{\partial a} + a^{-1}\frac{\partial u}{\partial a} = -a^{-1} + (a^{-1})(0) = -a^{-1}$. ∎

Proposition 4.3. Let $a \in \mathcal{G}$ and $u \in F_r$ such that u is in the free group generated by $\mathcal{G}\backslash\{a\}$. Then $\frac{\partial}{\partial a}(ua) = u$, and $\frac{\partial}{\partial a}(ua^{-1}) = -ua^{-1}$.

Proof. Because u is in the free group generated by $\mathcal{G}\backslash\{a\}$, the same is true for u^{-1}. By Proposition 4.1 and Proposition 4.2, we have $\frac{\partial}{\partial a}(ua) = \frac{\partial u}{\partial a} + u\frac{\partial a}{\partial a} = 0 + (u)(1) = u$.

Similarly, $\frac{\partial}{\partial a}(ua^{-1}) = \frac{\partial u}{\partial a} + u\frac{\partial a^{-1}}{\partial a} = 0 + (u)(-a^{-1}) = -ua^{-1}$. ■

Proposition 4.4. Let $a \in \mathcal{G}$ and $u_0, u_1 \in F_r$ such that u_0 and u_1 are in the free group generated by $\mathcal{G}\backslash\{a\}$. We have that $\frac{\partial}{\partial a}(u_0 a u_1) = u_0$ and $\frac{\partial}{\partial a}(u_0 a^{-1} u_1) = -u_0 a^{-1}$.

Proof.

Note that $\frac{\partial}{\partial a}(u_0 a u_1) = \frac{\partial}{\partial a}(u_0 a) + u_0 a \frac{\partial}{\partial a}(u_1) = u_0 + (u_0 a)(0) = u_0$. Similarly, $\frac{\partial}{\partial a}(u_0 a^{-1} u_1) = \frac{\partial}{\partial a}(u_0 a^{-1}) + u_0 a^{-1} \frac{\partial}{\partial a}(u_1) = -u_0 a^{-1} + (u_0 a^{-1})(0) = -u_0 a^{-1}$. ■

Let $a \in \mathcal{G}$ and $w \in F_r$. For some n there are possibly trivial words $u_0, u_1, \ldots, u_n$ in the free group generated by $\mathcal{G}\backslash\{a\}$ and exponents $\epsilon_i = \pm 1$ such that
$w = u_0 a^{\epsilon_1} u_1 a^{\epsilon_2} u_2 \cdots a^{\epsilon_n} u_n$.

Proposition 4.5. With w as above,

$$\frac{\partial w}{\partial a} = \sum_{j=1}^{n} \epsilon_j u_0 a^{\epsilon_1} u_1 \cdots a^{\epsilon_{j-1}} u_{j-1} a^{\gamma_j},$$

where $\gamma_j = 0$ if $\epsilon_j = 1$, and $\gamma_j = -1$ if $\epsilon_j = -1$.

Proof. We induct on n. The base case is proven in the propositions above. Suppose that $n = k+1$. Let $w = u_0 a^{\epsilon_1} u_1 a^{\epsilon_2} u_2 \cdots a^{\epsilon_{k+1}} u_{k+1}$, and let $\tilde{w} = u_0 a^{\epsilon_1} u_1 a^{\epsilon_2} u_2 \cdots a^{\epsilon_k} u_k$. Then $w = \tilde{w} a^{\epsilon_{k+1}} u_{k+1}$. We have

$$\frac{\partial w}{\partial a} = \frac{\partial \tilde{w} a^{\epsilon_{k+1}} u_{k+1}}{\partial a} = \frac{\partial \tilde{w}}{\partial a} + \tilde{w}\frac{\partial a^{\epsilon_{k+1}} u_{k+1}}{\partial a} = \frac{\partial \tilde{w}}{\partial a} + \tilde{w}(\epsilon_{k+1} a^{\gamma_{k+1}})$$

$$= \left(\sum_{j=1}^{k} \epsilon_j u_0 a^{\epsilon_1} u_1 \cdots a^{\epsilon_{j-1}} u_{j-1} a^{\gamma_j}\right)$$

$$+ (u_0 a^{\epsilon_1} u_1 \cdots a^{\epsilon_k} u_k)(\epsilon_{k+1} a^{\gamma_{k+1}})$$

$$= \sum_{j=1}^{k+1} \epsilon_j u_0 a^{\epsilon_1} u_1 \cdots a^{\epsilon_{j-1}} u_{j-1} a^{\gamma_j}. \square$$

5 Example Part 2:

Again, let $\mathcal{G} = \{a, b, c, d\}$ and $f_*(a) = b^5 a^3 c b^{-1} a^{-2} d$.

We see that $u_0 = b^5$, $u_1 = u_2 = 1$, $u_3 = cb^{-1}$, $u_4 = 1$, and $u_5 = d$. Also, $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, and $\epsilon_4 = \epsilon_5 = -1$.

By Proposition 4.5, we have

$$\frac{\partial f_*(a)}{\partial a} = b^5 + b^5 a + b^5 a^2 - b^5 a^3 c b^{-1} a^{-1} - b^5 a^3 c b^{-1} a^{-2}.$$

6 The Main Result

Theorem 1.1 With the notation given above, we have

$$\frac{\partial f_*(a)}{\partial a} = \sum_{j=1}^{n} \epsilon_j W_j.$$

Proof. First we use the earlier notation to express $f_*(a)$ as $f_*(a) = u_0 a^{\epsilon_1} u_1 a^{\epsilon_2} u_2 \cdots a^{\epsilon_n} u_n$. In Proposition 4.5, we replace w with $f_*(a)$ and note that for each j we have $\alpha_j = u_0 a^{\epsilon_1} u_1 a^{\epsilon_2} u_2 \cdots a^{\epsilon_{j-1}} u_{j-1}$ so that

$$\frac{\partial f_*(a)}{\partial a} = \sum_{j=1}^{n} \epsilon_j \alpha_j a^{\gamma_j},$$

where $\gamma_j = 0$ if $\epsilon_j = 1$, and $\gamma_j = -1$ if $\epsilon_j = -1$. For each j, by definition, the initial Wagner tail is $W_j = \alpha_j a^{\gamma_j}$. ∎

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