# Nielsen numbers of iterates and Nielsen type periodic numbers of periodic maps on tori and nilmanifolds

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**Abstract**

In this paper we compute the Nielsen numbers $N(f^m)$ and the Nielsen type numbers $NP_m(f)$ and $N\Phi_m(f)$ *for all* $m$, for periodic maps $f$ on tori and nilmanifolds.

For fixed $m$, there are known formulas for these numbers for arbitrary maps on tori and nilmanifolds. However when seeking to determine these numbers for all $m$ for periodic maps, fascinating patterns and shortcuts are revealed. Our method has two main thrusts. Firstly we study $N(f^m)$, $NP_m(f)$ and $N\Phi_m(f)$ on primitives (maps whose linearizations consist of primitive roots of unity), and then secondly we employ fibre techniques to give an inductive approach to the general case adding one primitive at a time. This approach is made possible by the eigen structure of the linearizations of the maps involved.

## 1 Introduction

The Nielsen numbers $N(f)$, and the Nielsen type numbers $NP_m(f)$ and $N\Phi_m(f)$ of a self map $f$ are $f$ homotopy invariant lower bounds for respectively the number of fixed points of $f$, for the number of periodic points of period exactly $m$, and for the number of periodic points of all periods dividing $m$. On tori these lower bounds are sharp ([12]) and it seems likely they are too for nilmanifolds.

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As shown in [1, 2, 7, 8, 3] (see also Theorems 2.1 and 2.2) there are simple formulas for the numbers $N(f^m)$, $NP_m(f)$ and $N\Phi_m(f)$ for fixed $m$ on tori and nilmanifolds. These formulas involve the linearization $F$ of $f$ (see [10] and section 2.1), which any self map of a torus or nilmanifold possesses. In fact this is a two way thing, since any square matrix $F$ gives rise to a map of the torus or nilmanifold, and the linearization of this map is $F$ itself. Furthermore the computations of the numbers $N(f^m)$, $NP_m(f)$ and $N\Phi_m(f)$ are independent of the choice of matrix representing the linearization (which is defined only up to conjugation). We therefore abuse notation and from now on fail to distinguish between the map $f$ and its linearization $F$. In particular, we will write $N(F^m)$, $NP_n(F)$ and $N\Phi_n(F)$ for $N(f^m)$, $NP_n(f)$ and $N\Phi_n(f)$ respectively.

Since, as we have said, for fixed $m$ there are simple formulas for all of the numbers $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ on tori and nilmanifolds, the reader may be wondering why the paper is either necessary or useful. There are, however, several points that make this study worth while.

1. We compute $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ **for all** $m$ for periodic maps.
2. There are fascinating patterns that occur among these numbers.
3. There are shortcuts for determining the $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ for all $m$ for periodic maps on tori and nilmanifolds.

To give some idea of the efficiency and simplicity of our considerations, we give an example of a periodic map of period $3,354,120$. We show that there are only 33 different possible non-zero values for the numbers $N(F^m)$, and that these values can be computed by hand without the aid of a computer. In fact the $N(F^m)$ are given as a product of easily computed powers of the primes (in this case at most 4) in the prime decomposition of the period of the map $(3,354,120)$. We show that $NP_m(F) = 0$ outside the set of $m$ for which the values of the $N(F^m)$ first occur, and use shortcuts developed in the body of the paper, to compute the $NP_m(F)$ when $m$ belongs to this same set, and to place them in a single table. Furthermore, for an arbitrary $m$, we are then able to compute $N\Phi_m(F)$ from the sum of an easily discerned subset of the set of 33 non zero values of the $NP_q(F)$. In fact when $N(F^m) \neq 0$ it is even easier, in that for this example the $N\Phi_m(F)$ can then be computed as the product of at most 4 numbers already computed in the compilation of the said table.

In terms of patterns the simplest and most fascinating, on tori and nilmanifolds, are for primitive matrices (the eigenvalues are primitive roots of unity, Definition 4.1). If $F$ is such a matrix of period $n = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$, where $p_1, \dots, p_r$ are distinct primes, and $r \geq 1$, then all three of the Numbers $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ can be easily read off from at most $1 + \sum s_i$ computations of the $N(F^m)$. With the possible exception of 1, each of the corresponding $m$ comprise of $n$ divided by the power of one of the primes $p_i$ in the prime decomposition of $n$, and the value of $N(F^m)$ for that $m$ is the same $p_i$ raised to a certain power described below. More complex periodic matrices (maps) are then computed inductively starting with a single primitive and then adding one primitive at a time. Thus the main body of the paper is divided into two parts, the first dealing with primitives, the second with the just mentioned inductive procedure.

We illustrate the part dealing with primitives in the example below, and say more about the second part afterwards. Our example is a kind of prototype for all primitives whose period $n$ is not a power of a single prime. When $n$ is a power of a single prime we need a small modification. A primitive is a matrix $F$ whose characteristic equation ($\chi_F$) is a cyclotomic polynomial $\Phi_n(x)$ for some $n$.

Rather than trying to write out the entries of very large matrices (but see Example 3.1), we point the reader to Lemma 2.6 which gives a standard method of building the "companion matrix" $\mathcal{C}(p(x))$, associated with a monic polynomial $p(x)$. As already stated this gives rise to a map whose linearization is $\mathcal{C}(p(x))$. Furthermore the characteristic equation of $\mathcal{C}(p(x))$ is $p(x)$.

**Example 1.1. Prototypical primitive example.** Let $n = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, and let $F = \mathcal{C}(\Phi_n)$ be the companion matrix (map) associated with the $n$th cyclotomic polynomial $\Phi_n$. So $F$ has as eigenvalues the 2520th primitive roots of unity, with multiplicity 1.

**Computing the** $N(F^m)$**:** Using a Maple worksheet to give approximations to the formula $N(F^m) = \chi_{F^m}(1)$ (and then rounding) we discovered, for $m$ in the range $1 \leq m \leq 1,260$, that $N(F^m) = 1$ except for $m$ in the set

$$\{280, 315, 360, 504, 560, 630, 720, 840, 945, 1008, 1080, 1120, 1260\}.$$

We organize the data in the following table which we explain below.

| $\mathcal{F}(F)$ | $m$ | 1 | 280 | 315 | 360 | 504 | 630 | 840 | 1260 |
|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{F}(F)$ | $m$ | 1 | $\widehat{3^2}$ | $\widehat{2^3}$ | $\widehat{7}$ | $\widehat{5}$ | $\widehat{2^2}$ | $\widehat{3}$ | $\widehat{2}$ |
| | $N(F^m)$ | 1 | $3^{96}$ | $2^{144}$ | $7^{96}$ | $5^{144}$ | $2^{288}$ | $3^{288}$ | $2^{576}$ |

. (1)

Using the set $\mathcal{F}(F)$ indicated in the table and defined below, we can prove (Corollary 4.8 where $(m, n)$ denotes the GCD of $m$ and $n$) that

$$N(F^m) = \begin{cases} N(F^{(m,n)}) & \text{if } (m,n) \in \mathcal{F}(F) \\ 0 & \text{if } (m,n) = n \\ 1 & \text{if } (m,n) \notin \mathcal{F}(F) \cup \{n\}. \end{cases}$$

Note we are claiming, for example, that $N(F^{280}) = N(F^{560}) = N(F^{1120})$. In fact this follows from the general principle that $N(F^m) = N(F^{(m,n)})$ for all $m$ for all periodic matrices (Proposition 3.2). Note also that of the 1,260 numbers between 1 and 1,260 that for 1,247 of them the $N(F^m)$ take the value 1. These include many numbers that divide $n$. So for example $N(F^{15}) = 1$, since $(15, n) = 15 \notin \mathcal{F}(F)$.

The set $\mathcal{F}(F) := \{1, 280, 315, 360, 504, 630, 840, 1260\}$ is the set of $m$ for which the values of the $N(F^m)$ first occur, where we interpret first with respect to division (Definition 3.3). The $\mathcal{F}$ in the notation $\mathcal{F}(F)$ is meant to remind the reader that the elements of $\mathcal{F}(F)$ are "firsts."

In order to make the second line of the table clear, we introduce a "hat" notation which we will use throughout the paper. This notation will only be used in the context that $F$ is a primitive of period often denoted $n$, and where $m|n$. In this context we use $\widehat{m}$ to denote the number

$$\widehat{m} := \tfrac{n}{m}.$$

Thus for example $504 = \widehat{5}$. In fact, as shown in the second line of the table we have, except for $m = 1$, that each $m \in \mathcal{F}(F)$ can be written as $\widehat{p^u}$ for some prime dividing $n$. Note that the "hatted" primes that appear in the first row, also appear immediately underneath in the second. Thus for $m = \widehat{2^2}$ we have that $N(F^m)$ is a power of 2, and for $m = \widehat{7}$ we have that $N(F^m)$ is a power of 7. The corresponding powers of these primes are described below.

All this, including the "hatted" composition of the sets $\mathcal{F}(F)$, generalizes to give analogous patterns for arbitrary primitives, with a small modification when $n$ is a power of a single prime. In fact it all follows from two key results for primitive matrices of period $n$. The first (see Proposition 4.5) is that

$$N(F) = \begin{cases} p & \text{if } n \text{ is a power of the single prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

The second is Theorem 4.6 which states, for $F := \mathcal{C}(\Phi_n)$, that

$$\chi_{F^q}(x) = \det(xI - F^m) = \left(\Phi_{\frac{n}{m}}(x)\right)^{\frac{\phi(n)}{\phi(\frac{n}{m})}} \text{ and } N(F^m) = \left(\Phi_{\frac{n}{m}}(1)\right)^{\frac{\phi(n)}{\phi(\frac{n}{m})}},$$

where $\phi$ is the Euler $\phi$ function. By way of illustration for $m = 360 = \widehat{7} = \frac{n}{7}$ we have that $\frac{n}{360} = 7$, so $\Phi_{\frac{n}{360}}(1) = \Phi_7(1) = 7$ from above. Moreover $\phi(n)/\phi(n/360)$ $= 96$, and so $N(F^{360}) = 7^{96}$ as shown in the table.

**Computing the** $NP_m(F)$**:** In general (not just for primitives) we will show that

$$NP_m(F) = 0 \text{ if } m \notin \mathcal{F}(F) \text{ and } NP_1(F) = N(F)$$

(Theorem 3.6, Theorem 2.2). Computation of the rest of the $NP_m(F)$ are very simple for primitives, and are given by the following formula (Corollary 4.12 rewritten). Let $m = \widehat{p^u} \neq 1$, then

$$NP_m(F) = \begin{cases} N(F^m) - N(F) & \text{if } p \nmid m \\ N(F^m) - N(F^{\frac{m}{p}}) & \text{if } p \mid m. \end{cases}$$

Thus in this same example $NP_m(F) = N(F^m) - 1$ for $m = \widehat{3^2}, \widehat{2^3}, \widehat{7}$ and $\widehat{5}$ (division of $n$ by the maximum powers in the prime decomposition) and $NP_m(F) = N(F^m) - N(F^{\frac{m}{p}})$ for $m = \widehat{2^2}, \widehat{3}$ and $\widehat{2}$. In particular, for example, $NP_{1260}(F) = 2^{576} - 2^{288}$.

**Computing the** $N\Phi_m(F)$**:** In fact for tori and nilmanifolds there are formulas for the $N\Phi_m(F)$ that involve the $N(F^q)$ and the $NP_q(F)$ for various $q|m$. For example $N\Phi_m(F) = N(F^m)$ when $N(F^m) \neq 0$ and is always given by $\sum_{q|m} NP_q(F)$ (Theorem 2.3). Thus $N\Phi_{560}(F) = N(F^{560}) = N(F^{280}) = 3^{96}$ since $3^{96} \neq 0$. But $N(F^{2520}) = 0$, so $N\Phi_{560}(F) = 7^{96} + 5^{144} + 3^{288} + 2^{576} - 3$ by the sum formula. We also have in general, for periodic matrices, that $N\Phi_m(F) = N\Phi_{(m,n)}(F)$ for all $m$ (Corollary 3.9).

We come now to indicate something of the second main thrust of the paper, where we use an inductive procedure to work out the general case. We need a definition. Let $A$ and $B$ be respectively $s \times s$ and $q \times q$ matrices respectively then we use the symbol $A \oplus B$ to denote the $(s+q) \times (s+q)$ block diagonal matrix

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

which we will call the *sum* of $A$ and $B$. In fact an arbitrary matrix of finite order is, up to conjugation, a direct sum of primitives (Theorem 2.7). Our idea is to build up the three Nielsen theories primitive by primitive in an inductive type procedure. In this scenario we are thinking of $B$ as the $k$th stage, and the addition of a primitive $A$ as the $k+1$st stage.

In order to do this, we need to regard $A \oplus B$ as a fibre preserving map of a trivial fibration. Of course we are thinking of $A$ as a self map of nilmanifold or torus. In the nilmanifold case we work with the torus model and torus model map ([6]). So without loss we can think of $A$ as a map of the $\phi(n)$ torus, and $B$ as a self map of the $\phi(q)$ torus and $A \oplus B$ as a fibre preserving map of the $\phi(n) \times \phi(q)$ torus. That is we have a commutative diagram

$$\begin{array}{ccccc} T^{\phi(n)} & \longrightarrow & T^{\phi(n)} \times T^{\phi(q)} & \longrightarrow & T^{\phi(q)} \\ A \downarrow & & \downarrow A \oplus B & & \downarrow B \\ T^{\phi(n)} & \longrightarrow & T^{\phi(n)} \times T^{\phi(q)} & \longrightarrow & T^{\phi(q)} \end{array}$$

The advantage of looking at $A \oplus B$ in this way is that we can use the philosophy and results of Nielsen fibre space theory which is to determine the various Nielsen theories of the map on the total space in terms or those on the base and fibre. Perhaps the most obvious of these is the so called naïve product Theorem which says that $N((A \oplus B)^m) = N(A^m)N(B^m)$. But of course this also follows from properties of determinants. Of more importance is the sum product formula of the author and Ed Keppelmann (see Theorem 5.1) which states in this context that

$$NP_m(A \oplus B) = \sum_{q|m} NP_q(B) NP_{\frac{m}{q}}(A^q).$$

To gain some insight into the sum product formula, consider a periodic point $x$ of $B$ of minimum period 2. Then $x \neq B(x)$ (recall we are thinking of $B$ both as a map and a matrix). In particular the fibre $T^{\phi(n)}_{B(x)}$ over $B(x)$ is not the same as the fibre $T^{\phi(n)}_x$ over $x$, but it is the same over $B^2(x) = x$. The restriction of $(A \oplus B)^2$ to $T^{\phi(n)}_x$ will then be a self map. Similarly the restriction of $(A \oplus B)^{2s}$ to $T^{\phi(n)}_x$ will then be the $s$th iterate of the same self map. Thus we need to look at $NP_{\frac{2s}{2}}(A^2)$ in the fibre, and the corresponding classes, by the so called naïve addition conditions (see for example [3]) inject into the classes of period $s$ in the total space.

The sum product formula then, allows us to perform an inductive procedure to determine the $NP_m(F)$ inductively from previous computations adding one primitive at a time. The sum product formula may, at first sight appear cumbersome, however many times many of the products $NP_q(B)NP_{\frac{m}{q}}(A^q)$ are equal to zero,. We illustrate this with an example whose details are given later (see Example 5.12 and following in section 5.2).

**Example 1.2.** Let $A := \mathcal{C}(\Phi_{20})$, $B := \mathcal{C}(\Phi_{30})$, then $\mathcal{F}(A) = \{1,4,5,10\}$, $\mathcal{F}(B = \{1,6,10,15\}$ and $\mathcal{F}(A \oplus B) = \{1,4,5,6,10,12,15\}$ (section 5.1). Also $NP_m(A \oplus B) = 0$ if $m \notin \mathcal{F}(A \oplus B)$ otherwise the sum products simplify as follows:-

| | |
|---|---|
| $NP_1(C)$ | $N(B)N(A)$ |
| $NP_4(C)$ | $N(B)NP_4(A)$ |
| $NP_5(C)$ | $N(B)NP_5(A)$ |
| $NP_6(C)$ | $NP_6(B)N(A)$ |
| $NP_{10}C)$ | $N(B)NP_{10}(A) + NP_{10}(B)N(A^{10})$ |
| $NP_{12}(C)$ | $NP_6(B)NP_2(A^2)$ |
| $NP_{15}(C)$ | $NP_{15}(B)N(A^{15})$ |

.

Though there is a corresponding formula in [4] that we could use for $N\Phi_m(A \oplus B)$ it is more convenient and more efficient to wait until the end of the induction process to calculate these numbers where we also indicate the complete identification of the $N(F^m)$ in the more complex situations.

The paper is divided as follows:- Following this introduction we give a section where we remind the reader of the linearization process and, for fixed $m$, the formulas for $N(F^m)$, $NP_m(F)$ and the $N\Phi_m(F)$ on tori and nilmanifolds. We quote a very specific form of known results on the structure of matrices of finite order that seems to be tailor made for our considerations. In section 3 we give results about the $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ that can be deduced without the main techniques of this paper. We also give the formal definition of the set $\mathcal{F}(F)$ and look at some of its properties. Section 4 brings us to one of the two main parts of the paper where we study $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ for all $m$ on primitives. We show that the patterns exhibited in Example 1.1 are easily generalizable, and that the computations of the $NP_m(F)$ for all $m$ can be read off from the values of the $N(F^m)$ for $m \in \mathcal{F}(F)$. Section 5 is devoted to the inductive step mentioned above. An outline is given at the beginning of the section where it can be more easily comprehended. In the last subsection we work the example of period $3,354,120$ mentioned earlier. This 4 stage example uses all of the techniques developed here in the paper.

I want to acknowledge the help and influence of Ed Keppelman and Chris Staecker. Ed was present at the conception of the paper which flowed out of a joint project, started way back in 1994/5, to study the Nielsen periodic point numbers on solvmanifolds. What was left over from the project was the study of these numbers on the special class of periodic maps on these spaces. In this regard, Proposition 3.2 is joint work with Ed. As can be seen from the paper, it turns out the study of these maps is already rich on tori and nilmanifolds, the building blocks for solvmanifolds. I also want to thank Chris Staecker for a lot of help during the early stages of the manuscript (see in particular Proposition 4.5 and Remark 4.7). I would also like to thank the referee for his or her comments, his careful reading of the text, his helpful suggestions and for catching a number of errors and omissions in the proofs.

## 2 Preliminaries

In this section we sketch the necessary preliminaries. Our sketch includes computational results for the numbers $N(F^m)$, $NP_m(F)$ or $N\Phi_m(F)$ rather than the definitions. The point is that for tori and nilmanifolds there is no advantage in giving the usual definitions which use orbits and sets of $m$-representatives. In other words the computational results allow us to bypass these concepts. We refer the reader to [7, 8, 3] for justification for this, and to [1] for the formula for $N(F^m)$.

This preliminary section is divided into two. In the first part we remind the reader of the concept of linearization and then use it to state the results that for fixed $m$ give formulas for the Nielsen number $N(F^m)$, and for the Nielsen type numbers $NP_m(F)$ and $N\Phi_m(F)$ for arbitrary maps on tori and nilmanifolds ([1, 7, 8]) together with a couple of results from [7, 3] (Proposition 2.4) that we will also need. In the second part of the section we look at the structure of matrices of finite order. The structure for arbitrary matrices is well known, however we quote a version for matrices of finite order presented by Koo ([11]) that particularly suits our purposes.

### 2.1 Linearization and formulas for $N(f^m)$, $NP_n(F)$ and $N\Phi_m(F)$ for fixed $m$ on arbitrary maps of tori and nilmanifolds.

In the first part of this subsection we remind the reader of the concept of linearization, of a self map of a torus or nilmanifold, details can be found in [3, 10]. If $f : T^q \to T^q$ is a map of a torus $T^q$ (a $q$ fold product of $S^1$s), then up to homotopy, $f$ can be covered by a linear map $F : \mathbb{R}^q \to \mathbb{R}^q$. By abuse we identify $F$ with the matrix with respect to the standard basis, and call $F$ the *linearization* of $f$. Another way to find $F$ of course is to look at the homomorphism $f_* : \Pi_1(T^p) \cong \mathbb{Z}^p \to \mathbb{Z}^p \cong \Pi_1(T^p)$ of of Abelian groups, and take its matrix with respect to the standard basis of $\mathbb{Z}^p$. A more usual definition of linearization does not specify the bases for $\mathbb{Z}^p$. In this case the matrix is only defined up to conjugation. However it should be clear from the formulas for the various Nielsen numbers, that they are independent of the choice among conjugate matrices.

Linearization is functorial in that the linearization of the identity map is the identity matrix, and the linearization of the composition of maps is the matrix product of the linearizations. In particular the linearization of an iterate of a map $f$ is the iterate of the linearization. In other words, if $F$ is the linearization of $f$, then $F^q$ is the linearization of $f^q$. So then the linearization of $f$ completely determines the (ordinary) Nielsen numbers $N(f^m)$ of $f$. In fact it also completely determines both $NP_m(f)$ and $N\Phi_m(f)$.

The paper [6] exhibited for each solvmanifold (and a fortiori for nilmanifolds) a model solvmanifold that had exactly the same Nielsen theory ($N(f^m)$, $NP_m(f)$ and $N\Phi_m(f)$) as any map $f$ on the original solvmanifold. Part of the process was to replace the nilmanifold in the minimal Mostow fibration by a torus. We can think of the linearization of the the map on the replaced nilmanifold as the linearization of the nilmanifold. So in fact the paper ([6]) also showed how to make a torus model for a nilmanifold (even if it was not explicitly stated). So then from now on we deal only with periodic matrices. In particular with respect to a map on a nilmanifold, we deal with the linearization on its model. Also, as mentioned in the introduction, we shall not distinguish between a map $f : X \to X$ of a torus or a nilmanifold, and its linearization $F$.

Starting with $N(F^m)$, we now give the formulas for the computation of $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ for fixed $m$. We include a couple of useful facts that will be helpful in what follows.

**Theorem 2.1.** *[1, 2] Let F be the linearization of a self map of a torus or nilmanifold. Then for any positive integer m we have that*

$$N(F^m) = |det(I - F^m)| = \prod_{k=1}^{s}(1 - \lambda_k^m) = \chi_{F^m}(1),$$

*where I denotes the identity matrix, the $\lambda_k$ are the eigenvalues and $\chi_{F^m}$ is the characteristic equation of $F^m$.* ∎

**Theorem 2.2.** *([7, 8, 3]) Let F be the linearization of a self map of a torus or a nilmanifold. Then $NP_1(F) = N(F)$. If $N(F^m) = 0$, then $NP_m(F) = 0$ and if $N(F^m) \neq 0$, then*

$$NP_m(F) = \sum_{\tau \subseteq \mathbf{P}(m)} (-1)^{|\tau|} N(F^{m:\tau}),$$

*where $\mathbf{P}(m)$ is the set of prime divisors of m and $m : \tau = m \prod_{p\in\tau} p^{-1}$.* ∎

To illustrate the Theorem we note, for example, if in a particular situation we had that $N(F^{36}) \neq 0$ then $NP_{36}(F) = N(F^{36}) - N(F^{18}) - N(F^{12}) + N(F^6)$.

**Theorem 2.3.** *([3, Theorems 5.1; 5.8]) Let F be the linearization of a self map of a torus or a nilmanifold. Then*

$$N\Phi_m(F) = \begin{cases} N(F^m) & \text{if } N(F^m) \neq 0 \\ \sum_{q|m} NP_q(F) & \text{always.} \end{cases}$$

*Alternatively $N\Phi_n(F) = \sum_{\emptyset \neq \mu \subseteq M(F,n)} (-1)^{|\mu|-1} N(F^{gcd(\mu)})$ where $M(F, n)$ denotes the set of maximal divisors q of n for which $N(F^q) \neq 0$.*

The following equalities may not, perhaps, be immediately obvious from the above formulas.

$$N(F) = NP_1(F) = N\Phi_1(F).$$

We shall also need the following which, rather than following from the formulas, is used as part of their proof:-

**Proposition 2.4.** *([7, 3]) Let F be the linearization of a self map of a torus or nilmanifold, and let $q|m$, then $N(F^q) \leq N(F^m)$ and $N(F^n) \neq 0$. If in addition $q \neq m$ and $N(F^q) = N(F^m)$, then $NP_m(F) = 0$.* ∎

By way of explanation, for the reader familiar with the definitions as found in [7, 8, 3], we know that tori and nilmanifolds are $n$-toral and essentially reducible. Part of what that means is that $NP_m(F)$ is the number of irreducible essential classes (we don't need to use orbits). Since tori are Jiang spaces, and tori and nilmanifolds are $n$-toral then when $N(F^m) \neq 0$ all classes at both levels are essential, and the boosting functions are injective. This gives $N(F^q) \leq N(F^m)$. Next, when $N(F^q) = N(F^m) \neq 0$ with $q|m$ but $q \neq m$, then the boosting functions are bijective so there are no irreducible essential classes, giving that $NP_m(F) = 0$.

### 2.2 The structure of matrices of finite order

**Definition 2.5.** A map (matrix) $F$ is said to be *periodic of period* $n$ if $F^n = I$ (the identity matrix) for some positive integer $n$, and $n$ is the smallest such positive integer. A matrix is said to be *primitive* if it has eigenvalues the primitive $n$th roots of unity of multiplicity 1, for some $n$. In other words the characteristic polynomial $\chi_F$ is equal to $\Phi_n$ the $n$th cyclotomic polynomial.

The following well known lemma is useful for generating examples.

**Lemma 2.6.** *Let $p(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ be a monic polynomial, then $p(x)$ is the characteristic polynomial of the matrix $C(p(x))$ given by:*

$$C(p(x)) = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{k-1} \end{bmatrix}.$$

The matrix $C(p(x))$ in Lemma 2.6 is called the *companion matrix* of $p(x)$. This together with the characterization of periodic matrices, given below, gives a model for constructing periodic matrices.

The rational canonical form of a matrix is well known. For a periodic matrix this takes a special form, and the Theorem below represents a small tweaking of an explicit version of it that appears in [11]. The direct sum notation ($\oplus$) was introduced in the introduction.

**Theorem 2.7.** *If $A$ is an integer matrix with $A^n = I$ for some $n$, then the characteristic polynomial of $A$ is a product of cyclotomic polynomials:*

$$\chi_A = \Phi_{m_1}^{d_1} \dots \Phi_{m_r}^{d_r},$$

*where each $m_i \mid n$ and $n = \mathrm{lcm}(m_1, \dots, m_r)$.*

*Furthermore, $A$ is similar to a block diagonal matrix as follows:*

$$A \sim C(\Phi_{m_1})^{[d_1]} \oplus \cdots \oplus C(\Phi_{m_r})^{[d_r]},$$

*where each exponent $[d_i]$ indicates a $d_i$-fold direct sum of $C(\Phi_{m_i})$,*

Since the Nielsen number of $F$ is equal to the characteristic polynomial evaluated at 1, we obtain immediately:

**Corollary 2.8.** *If $F$ is an integer matrix with $A^n = I$ for some $n$, then there are integers $m_i \mid n$ and $d_i$ such that* $\mathrm{lcm}(m_1, \dots, m_r) = n$ *and*

$$N(F) = \Phi_{m_1}(1)^{d_1} \dots \Phi_{m_r}(1)^{d_r}.$$

## 3 Preliminary relationships among $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ on general periodic matrices

In this section we exhibit a number of results that can be proved directly (without the inductive procedure) for arbitrary periodic maps $F$ on tori and nilmanifolds. In addition we introduce the sets $\mathcal{F}(F)$ for arbitrary $F$, and give some simple results. We do not assume in this subsection that $F$ is a primitive matrix.

We start with an illustrative example.

**Example 3.1.** Let $F$ be the matrix below, then $F$ determines a periodic map of $\mathbb{R}^4$, and $F$ induces a self map $f$ on $T^4$ (which by abuse of notation we also call $F$) of period 10.

$$F = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

As already mentioned we identify $f$ with $F$ its linearization. Now the eigenvalues of $F$ are the primitive 10 roots of unity with multiplicity 1. In particular $F^{10} = I$ the identity matrix. Consider the following table obtained using the formula $N(F^n) = |det(F^n - I)|$.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $N(F^m)$ | 1 | 5 | 1 | 5 | 16 | 5 | 1 | 5 | 1 | 0 |

**Proposition 3.2.** *(Heath Keppelmann.) Let $F : N \to N$ be the linearization of a periodic self map of a Torus or nilmanifold of period $n > 1$, with linearization matrix $F$. If $(t, n) = 1$ then $N(F) = N(F^t)$. Moreover*

*(a) $N(F^m) = N(F^{(m,n)})$, and (b) $N(F^k) = N(F^{n-k})$ for $0 < k < n$,*

*where $m$ is arbitrary, and $(m, n)$ is the gcd of $m$ and $n$. Moreover*

*(c) $NP_m(F^{(m,n)}) = NP_m(F)$ and $N\Phi_m(F^{(m,n)}) = N\Phi_m(F)$ for all $m$.*

The equalities $N(F^2) = N(F^4) = N(F^6) = N(F^8)$ illustrate part (a), while the symmetry around $N(F^5)$ (i.e. $N(f^3) = N(f^7)$), is explained by part (b).

*Proof.* We show that $F$ and $F^t$ have the same eigenvalues for any $t$ with $(t, n) = 1$. It follows that $F^q$ and $F^{qt}$ also have the same eigenvalues for any $q$ (applying the result to $F^q$). For (a) we let $t = \frac{m}{(m,n)}$ and $q = (m, n)$. For (b) we let $t = n - 1$ and $q = r$, and then use the fact that $F^{rn-r} = F^{(r-1)n+n-r} = F^{n-r}$ (since $F^{(r-1)n} = I$). Part (c) also follows, since $F^m$ and $F^{(m,n)}$ are similar matrices for all $m$, and the result now follows from the formulas for these numbers (Theorems 2.1, 2.2 and 2.3) which are easily seen to be independent of the linearization within its congujacy class.

So let $E = \{\lambda_1, \ldots, \lambda_s\}$ be the eigenvalues of $F$ listed with multiplicities. Thus $N(F) = |\det(I - F)| = \prod_{i=1}^{s}(1 - \lambda_i)$. By Theorem 2.7, each $\lambda_i$ is a primitive $k$th root of unity for some $k \mid n$. Further, again by Theorem 2.7, all $k$th roots of unity appear as eigenvalues of $F$ with the same multiplicity. Suppose that $(t, n) = 1$, then there exist $a$ and $b$ such that $at + bn = 1$. Since $k \mid n$, we have that $t$ is relatively prime to $k$, and so $\lambda_i^t$ is also a primitive $k$-th root of unity.

This means that the function that takes $E$ to $E^t = \{\lambda_1^t, ..., \lambda_s^t\}$ takes the various primitive roots to primitive roots of the same order. In fact this function is a bijection since it has as inverse the function $E^t \to E$, that takes $\lambda^t$ to $\lambda^{at+bn}$. Thus $F^q$ and $F^{qt}$ have the same eigenvalues as claimed. ∎

As it turns out, in order to to specify $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ for all $m$ in Example 3.1, we need only the following table of values (we define $\mathcal{F}(F)$ below):-

| $\mathcal{F}(F) \cup \{10\}$ | $m$ | 1 | 2 | 5 | 10 |
|---|---|---|---|---|---|
| | $N(F^m)$ | 1 | 5 | 16 | 0 |
| | $NP_m(F)$ | 1 | 4 | 15 | 0 |
| | $N\Phi_m(F)$ | 1 | 5 | 16 | 20 |

.

The values for $NP_m(F)$ come from Theorem 2.2, while $N\Phi_{20} = 1 + 4 + 15$ from Theorem 2.3. The values for all other $m$ can be read off from the equations $N(F^m) = N(F^{(m,n)})$ and $N\Phi_m(F) = N\Phi_{(m,n)}(F)$ (Corollary 3.9).

We now make the formal definition of $\mathcal{F}(F)$.

**Definition 3.3.** *We define the set $\mathcal{F}(F)$, for an arbitrary square matrix $F$, by*

$$\mathcal{F}(F) := \{m \neq 0 \mid N(F^m) \neq 0 \text{ and } N(F^q) \neq N(F^m)\ \forall q|m,\ q \neq m\}.$$

So again the $\mathcal{F}$ in the notation $\mathcal{F}(F)$ is meant to indicate the "first" occurrence of the indicated value. Here first has to do with division, and should not be confused with the regular order of the natural numbers.

We do however have:-

**Proposition 3.4.** *Let $F$ be an arbitrary periodic matrix of period $n$, and let $N(F^n) \neq 0$. If $m$ has the property that $N(F^q) \neq N(F^m)$ for all $q < m$ then $m \in \mathcal{F}(F)$.*

*Proof.* If $n = 1$, or if $m = 1$, then there are no $q$ satisfying the condition. If $n = 1$, then $\mathcal{F}(F) = \emptyset$. If $n \neq 1$ and $m = 1$, then there is nothing to prove, since $1 \in \mathcal{F}(F)$. Let $m$ have the given property, and let $q|m$ with $q \neq m$. Then $N(F^q) \neq N(F^m)$ by hypotheses. But $N(F^q) \leq N(F^m)$ by Proposition 2.4. Thus $N(F^q) < N(F^m)$ for all $q|m$ with $q \neq m$ and so $m \in \mathcal{F}(F)$ as required. ∎

The converse of Proposition 3.4 is false as the following example shows.

**Example 3.5.** Let $F = \mathcal{C}(\Phi_{20}\Phi_{30})$ then the table for $\mathcal{F}(F)$ is given below.

| $\mathcal{F}(F)$ | $m$ | 1 | 4 | 5 | 6 | 10 | 12 | 15 |
|---|---|---|---|---|---|---|---|---|
| | $N(F^m)$ | 1 | $5^2$ | $2^4$ | $5^2$ | $3^4 2^8$ | $5^4$ | $2^7$ |

Note that both 4 and 6 satisfy Definition 3.3 and so belong to $\mathcal{F}(F)$. This makes sense geometrically, since $N(F^4)$ is detecting 24 (=25 -1) periodic points of least period 4, and $N(F^6)$ is detecting 24 periodic points of least period 6. Clearly these cannot be the same points. What we are exhibiting here is that

$$\mathcal{F}(F) \neq \{m|n \mid N(F^m) \neq 0 \text{ and } N(F^q) \neq N(F^m)\ \forall q < m\}$$

So again we are stressing that $\mathcal{F}$ refers to "first" with respect to division, not with respect to the usual order on the natural numbers.

**Theorem 3.6.** *Let F be an arbitrary periodic matrix of order n. Then $\mathcal{F}(F)$ is finite, and*

$$NP_m(F) = 0 \textit{ if } m \notin \mathcal{F}(F).$$

*In particular $NP_m(F) = 0$ if $m \nmid n$, or if $mp_i > n$ where $p_i$ is the smallest prime dividing n.*

In all our examples we have that $NP_m(F) \neq 0$ if $m \in \mathcal{F}(F)$, but a proof of this for general periodic maps on tori and nilmanifolds is elusive. It is not true in general as the example $f = -1 \vee 1 : S^1 \vee S^1 \to S^1 \vee S^1$ indicates. Here $\mathcal{F}(f) = \{1, 2\}$ but $NP_2(f) = 0$, since the unique class at level 2, though essential, is in fact reducible. As an illustration of the last part of the Theorem, if $n = 3^2 \cdot 5$ then $p_i = 3$, and $NP_m(F) = 0$ for $m > 15$.

*Proof.* To see that $\mathcal{F}(F)$ is finite, recall that $N(F^m) = N(F^{(m,n)})$ for all $m$, so that there cannot be more firsts than the number of divisors of $n$. To see that $NP_m(F) = 0$ if $m \notin \mathcal{F}(F)$, let $m$ be such that $m \notin \mathcal{F}(F)$. If $N(F^m) = 0$, then $NP_m(F) = 0$ by definition. If $N(F^m) \neq 0$ then $m$ is not a first. In particular there is a $q \mid m$ with $q \neq m$ and $N(F^m) = N(F^q)$. But then $NP_m(F) = 0$ from Proposition 2.4.

For the "In particular" part, in either case $m \nmid n$. But then $N(F^m) = N(F^{(m,n)})$ and clearly $(m, n) < m$, so $m$ cannot be a first and so cannot belong to $\mathcal{F}(F)$. ∎

The next result is a small refinement of Theorem 2.3 and a small tweaking of Theorem 2.2. We need some notation. Let $m$ be a positive integer, we define

$$m!\mathcal{F}(F) := \{q|m \mid q \in \mathcal{F}(F)\}.$$

**Theorem 3.7.** *Let F be an arbitrary periodic matrix. Then*

$$N\Phi_m(F) = \begin{cases} N(F^m) & \textit{if } N(F^m) \neq 0 \\ \sum_{q \in m!\mathcal{F}(F)} NP_q(F) & \textit{always.} \end{cases}$$

*Moreover if $N(F^m) \neq 0$, then $N(F^m) = \sum_{q \in m!\mathcal{F}(F)} NP_q(F)$, and*

$$NP_m(F) = N(F^m) - \sum_{q \in m!\mathcal{F}(F) - \{m\}} NP_q(F).$$

*Proof.* That $N\Phi_m(F) = N(F^m)$ if $N(F^m) \neq 0$ is not new, but is included for completeness. The sum formula in the first part is a restatement of the sum formula in Theorem 2.3 which simply omits those $q$ for which $NP_q(F) = 0$. This of course occurs if $q \notin \mathcal{F}(F)$. For the second formula, if $N(F^m) \neq 0$ then $N(F^m) = N\Phi_m(F) = \sum_{q \in m!\mathcal{F}(F)} NP_q(F)$, and the given formula is just a rearrangement of the sum formula in the first part. ∎

**Example 3.8.** Let $F = \mathcal{C}(\Phi_{20}\Phi_{30})$ be as in Example 3.5. We compute $N\Phi_{40}(F)$. Now $N(F^{40}) = 0$ since any multiple of either 20 or 30 has Nielsen number 0. So we must use the addition formula. Now $40!\mathcal{F}(F) = \{1, 4, 5, 10\}$ and $NP_1(F) = 1$, $NP_4(F) = 5^2 - 1$, $NP_5(F) = 2^4 - 1$ while $NP_{10}(F) = 3^4 2^8 - 2^4 + 1$ by Theorem 2.2. So $N\Phi_{40}(F) = 3^4 2^8 - 2^4 + 1 + 2^4 - 1 + 5^2 - 1 + 1 = 3^4 2^8 + 5^2$.

We compute $NP_{12}(F)$ by the second formula in Theorem 3.7. Note that $12!\mathcal{F}(F) - \{12\} = \{1, 4, 6\}$. So $NP_{12}(F) - \sum_{q \in \{1,4,6\}} NP_q(F)$ $=5^4 - (1 + 5^2 - 1 + 5^2 - 1)$ $=5^4 - 2 \cdot 5^2 + 1$.

**Corollary 3.9.** *Let $F$ be an arbitrary periodic matrix of order $n$. Then for any positive integer $m$ we have that*

$$N\Phi_m(F) = N\Phi_{(m,n)}(F).$$

*Proof.* By Theorem 3.7 we need only show that $(m, n)!\mathcal{F}(F) = m!\mathcal{F}(F)$. So let $q \in (m, n)!\mathcal{F}(F)$ then $q|(m, n)$ and $q$ is a first. That is $N(F^q) \neq 0$ and there is no $t|q$ with $t \neq q$ and with $N(F^t) = N(F^q)$. But $q|(m, n)$ implies that $q|m$ and as already seen $q$ is a first and so belongs to $m!\mathcal{F}(F)$. That is $(m, n)!\mathcal{F}(F) \subseteq m!\mathcal{F}(F)$. On the other hand if $q|m$ but $q \nmid (m, n)$, then $q \nmid n$, and $NP_q(F) = 0$. So $q$ does not belong to $\mathcal{F}(F)$ and hence cannot belong to either $m!\mathcal{F}(F)$ or $(m, n)!\mathcal{F}(F)$. So any $q \in m!\mathcal{F}(F)$ must divide $(m, n)$ and is, of course a first, so $q \in (m, n)!\mathcal{F}(F)$ by definition. ∎

**Remark 3.10.** So Theorem 3.7 gives us a way of computing $N\Phi_m(F)$ at every stage of what will be our inductive procedure. It is however more efficient to wait until the end of the process to perform the computations, since it is possible that the size of $\mathcal{F}(F)$ can actually be smaller than those of the various stages along the way (see step 4 of the final Example 5.28).

## 4 Patterns among $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ for primitives

In this section we study the numbers $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ for all $m$ on primitives. It is convenient to give the formal definitions here.

**Definition 4.1.** We call a square matrix $F$ a *primitive matrix or map of order $n$*, if it has only primitive $n$th roots of unity as eigenvalues with multiplicity 1.

**Proposition 4.2.** *A primitive matrix $F$, of period $n$, is a $\phi(n) \times \phi(n)$ square matrix, where $\phi$ is the Euler $\phi$ function. Any other primitive matrix of period $n$, is similar over the complex numbers to $F$.* ∎

**Proposition 4.3.** *If $F$ is a primitive matrix of order $n$, then the characteristic equation* $\det(xI - F)$ *of $F$ is equal to $\Phi_n(x)$ the nth cyclotomic polynomial. Conversely if the characteristic equation of $F$ is equal to $\Phi_n(x)$ then $F$ is a primitive $n$ matrix. In particular*

$$N(F) = \Phi_n(1).$$

In all the examples in the introduction we had $N(F) = \Phi_n(1) = 1$, but this is not always the case as the next example shows.

**Example 4.4.** Let $n = 2^4 = 16$, and let $F$ be a primitive period 16 matrix. We deviate slightly from pattern of the example in the introduction. The values of $N(F^m)$ were originally found using a Maple worksheet.

| $\mathcal{F}(F) \cup \{16\}$ | $m$ | $1\ (=\widehat{2^4})$ | $\widehat{2^3}$ | $\widehat{2^2}$ | $\widehat{2}$ | $\widehat{2^0}\ (= 16)$ |
|---|---|---|---|---|---|---|
| | $N(F^m)$ | $2$ | $2^2$ | $2^4$ | $2^8$ | $0$ |
| | $NP_m(F)$ | $2$ | $2^2 - 2$ | $2^4 - 2^2$ | $2^8 - 2^4$ | $0$ |
| | $N\Phi_m(F)$ | $2$ | $2^2$ | $2^4$ | $2^8$ | $2^8$ |

Of course $N(F^m) = N(F^{(m,16)})$ for all $m$, $NP_m(F) = 0$ if $m \notin \mathcal{F}(F)$ and $N\Phi_m(F) = N\Phi_{(m,n)}(F)$ for all $m$.

So here is our first example where $N(F) \neq 1$. The following Proposition gives the general situation for $N(F)$ for primitives.

**Proposition 4.5.** *(Heath Staecker) Let $F = \mathcal{C}(\Phi_n)$ where $n = p_1^{s_1} \cdots p_r^{s_r}$, with $r \geq 1$, then*

$$N(F) = \begin{cases} p_1 & \textit{if } r = 1 \\ 1 & \textit{if } r > 1. \end{cases}$$

*Proof.* We use the following two well known fundamental relations

$$\Phi_{p^k}(x) = \sum_{i=0}^{p-1} x^{ip^{k-1}} \text{ and } x^n - 1 = \prod_{d|n} \Phi_d(x).$$

The result that $N(F) = p_1$ when $r = 1$ comes from Proposition 4.3 by simply putting $x = 1$ in the first equation.

It is convenient to abbreviate $\Phi_n(1)$ by $\mathcal{C}_n$ for our proof that $N(F) = 1$ when $r > 1$. By dividing both sides of the second equation above by $x - 1$ we see that $1 + x + x^2 + \cdots + x^{n-1} = \prod_{d|n:d\neq 1} \Phi_d(x)$, so by putting $x = 1$, we have that $n = \prod_{d|n:d\neq 1} \mathcal{C}_d$. Using this and strong induction on $n$ we prove that for $n$ a product of prime powers of more than one prime, we have that $\mathcal{C}_n = 1$.

Since $\Phi_6(x) = x^2 - x + 1$ then $\mathcal{C}_6 = 1$ and we can start the induction here. Let $n = p_1^{s_1} \cdots p_r^{s_r}$ with $r > 1$ and $p_1, \ldots, p_r$ distinct primes, and suppose we have proved the result for all cases less than $n$. Now

$$n = \prod_{d|n;d\neq 1} \mathcal{C}_d = \mathcal{C}_n \prod_{d|n;1<d<n} \mathcal{C}_d.$$

In the right hand product, if $d$ is not a power of a prime then we have $\mathcal{C}_d = 1$ by the induction hypotheses. Thus we may remove all factors in the product which are not prime powers, and the above reads

$$n = \mathcal{C}_n \cdot (\mathcal{C}_{p_1}\mathcal{C}_{p_1^2} \dots \mathcal{C}_{p_1^{s_1}}) \dots (\mathcal{C}_{p_r}\mathcal{C}_{p_r^2} \dots \mathcal{C}_{p_r^{s_r}}).$$

By the first part $\mathcal{C}_{p_i^k} = p_i$ for each prime $p_i$, so that the last equation gives

$$n = \mathcal{C}_n \cdot (p_1^{s_1} \dots p_r^{s_r}) = \mathcal{C}_n \cdot n,$$

and so $\mathcal{C}_n = \Phi_n(1) = N(F) = 1$ as desired. ∎

Proposition 4.5 together with the results of section 3.1 and the Theorem below will give us all we need to compute the Nielsen numbers of iterates of primitive periodic matrices.

**Theorem 4.6.** *Let $F$ be primitive of period $n$. If $q|n$ then*

$$\chi_{F^q}(x) = \det(xI - F^q) = \left(\Phi_{\frac{n}{q}}(x)\right)^{\frac{\phi(n)}{\phi(\frac{n}{q})}}, \textit{ and so } N(F^q) = \left(\Phi_{\frac{n}{q}}(1)\right)^{\frac{\phi(n)}{\phi(\frac{n}{q})}}.$$

*Proof.* Let $E_1 := \{\lambda_1, \cdots \lambda_{\phi(n)}\}$ denote eigenvalues of $F$, which are in fact the primitive $n$th roots of unity. Then the eigenvalues of $F^q$ are $E_q := \{\lambda_1^q, \cdots \lambda_{\phi(n)}^q\}$. Clearly, if $(k, n) = 1$ then $(k, n/q) = 1$ for any $q \mid n$. So if $\lambda = \exp(2k\pi i/n)$ is a primitive $n$th root of unity and $q \mid n$, then $\lambda^q = \exp(2k\pi i/(n/q))$ is a primitive $(n/q)$th root of unity. Thus the eigenvalues of $F^q$ are all primitive $(n/q)$th roots of unity.

Since $F$ is a matrix of order $n$ and $q \mid n$, then $F^q$ is a matrix of order $n/q$ and so Theorem 2.7 applies to $F^q$. In particular the characteristic polynomial of $F^q$ is a product of cyclotomic polynomials. Since all eigenvalues of $F^q$ are primitive $n/q$ roots of unity, it must be the case that the characteristic polynomial is $(\Phi_{n/q})^t$ for some positive integer $t$. It remains only to show that $t = \phi(n)/\phi(n/q)$.

Because $F$ is a square matrix of size $\phi(n)$, so too is $F^q$, and thus $\phi(n)$ is the degree of its characteristic polynomial $(\Phi_{n/q})^t$. Since $\deg(\Phi_{n/q}) = \phi(n/q)$, we have $\phi(n) = t\phi(n/q)$, and so $t = \phi(n)/\phi(n/q)$ as desired. ∎

**Remark 4.7.** Theorem 4.6 was conjectured after running a number of Maple worksheets looking for patterns. For the longest time a proof was elusive. Of course it was obvious that the $\lambda^q$ were primitive $(n/q)$th roots of unity, but it was how to get the distribution right that was elusive. In fact, as the proof shows, it follows from a small tweaking of a special case (for matrices of finite order) of the rational canonical form. Chris Staecker found this by typing "matrices of finite order" into Google, something that continues to amaze me. The second entry in Chris' search ([11]) puts the rational canonical form in the precise form that we need, and it also turns out to be a convenient reference for section 2.2. I would like to thank Chris for his help here, and in many other places. In addition I would like to thank Tom Baird, Mikhail Kotchetov and Mike Parmenter for interesting prior conversations aimed at proving the Theorem directly, including a proposed Galois theory proof. In fact the correct distribution also follows from the fact that $q_* : \mathbb{Z}_n^* \to \mathbb{Z}_{\frac{n}{q}}^*$ is a homomorphism, but not with respect to multiplication of eigenvalues.

When dealing with primitives of order $n = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$, it is convenient to separate the cases $r > 1$ (Corollary 4.8) and $r = 1$ (Corollary 4.11). Part of this is that we can be more specific when $n = p^s$ is a power of a single prime.

**Corollary 4.8.** *Let $F = \mathcal{C}(\Phi_n)$ be a primitive matrix with $n = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$, where $p_1, \ldots, p_r$ are distinct primes, and $r > 1$. Then*

$$\mathcal{F}(F) = \{1\} \cup \{\widehat{p_i^u} \mid \text{ where } i = 1, \cdots, r \text{ and } 1 \le u \le s_i\}.$$

*If $(m, n) = n$, then $N(F^m) = 0$. If $(m, n) \notin \mathcal{F}(F) \cup \{n\}$, then $N(F^m) = 1$.*

$$\text{Finally if } (m, n) = \widehat{p_i^u} \text{ for } 1 \le u \le s_i, \text{ then } N(F^m) = p_i^{\left(\frac{\phi(n)}{p_i^u - p_i^{u-1}}\right)}.$$

Note for $r > 1$ we have that $\#(\mathcal{F}(F)) = 1 + \sum_{i=1}^r s_i$.

*Proof.* Since $N(F^m) = N(F^{(m,n)})$ we can, without loss, replace $(m, n)$ by $m \mid n$ in the Corollary. Also since $N(F^n) = 0$ from Theorem 2.1 we need consider only $m|n$ and $m < n$.

We prove first that if $m = \widehat{p_i^u}$ for $1 \le u \le s_i$, then $N(F^m)$ is as shown. So let $m = \widehat{p_i^u}$, then $\frac{n}{m} = p_i^u$ and $\Phi_{\frac{n}{m}}(1) = p_i$ by Proposition 4.5. So $N(F^m)$ is equal to $p_i$ to the power $\frac{\phi(n)}{\phi(\frac{n}{m})}$ by Theorem 4.6. But $\phi(\frac{n}{m}) = \phi(p_i^u) = p_i^u - p_i^{u-1}$ and the description of $N(F^m)$ as given is proved. In particular $N(F^m) \neq 1$ for such $m$.

We next show that $\{1\} \cup \{\widehat{p_i^u} \mid \text{ where } i = 1, \cdots r \text{ and } 1 \le u \le s_i\} \subseteq \mathcal{F}(F)$. Now clearly 1 is a first and so belongs to $\mathcal{F}(F)$. To see that $1 \neq m = \widehat{p_i^u}$ is a first, we must show that $N(F^q) \neq N(F^m)$ for any $q|m$ with $q \neq m$. If $q = 1$ then, as we have already seen $N(F^m) \neq N(F) = 1$. Next, if $q = \widehat{p_i^v}$ with $u < v \le s_i$ then $\phi(\frac{n}{v}) = p^v - p^{v-1} > p^u - p^{u-1}$ so $N(F^q) < N(F^m)$, by the description of $N(F^q)$ and $N(F^m)$ already seen. If $q|m$ and $q \neq \widehat{p_i^v}$ for any $v$ with $u < v \le s_i$ then $q\ell = m$ for some $\ell$ with $p_j|\ell$ for some $p_j \neq p_i$ (this can only happen if $r > 1$). In this case $\frac{n}{q} = \ell p^u$ is a composite so $N(F^q) = 1 \neq N(F^m)$. So in each case if $q|m$ and $q \neq m$, then $N(F^q) \neq N(F^m)$, and $m$ is indeed a first.

Next if $m \neq n$ and $m|n$ and $1 \neq m \neq \widehat{p_i^u}$ for $1 \le u \le s_i$ for any $i$, then $m$ must divide at least one of the $\widehat{p_i^u}$ but not be equal any of them. But we have already proved for such an integer that $N(F^m) = 1$ and so is not a first. We have actually shown, if $m \neq n$ and $m|n$ and $m \notin \{1\} \cup \{\widehat{p_i^u} \mid \text{ where } i = 1, \cdots r \text{ and } 1 \le u \le s_i\}$, then both $N(F^m) = 1$ and $m \notin \mathcal{F}(F)$, and we are done. ■

Before giving the Corollary for the case $r = 1$, we give a couple of examples that pertain to Corollary 4.8. Recall first, that if $q = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$ then the Euler $\phi$ function is given by $\phi(q) = \phi(p_1^{k_1})\phi(p_2^{k_2}) \cdots \phi(p_s^{k_s})$, where for $\phi(p_i^{k_i}) = p_i^{k_i} - p_i^{k_i-1}$.

**Example 4.9.** Let $\mathcal{C}(n)$, where $n = 2520 = 3^2 2^3(5)(7)$ be as in Example 1.1. Then $\mathcal{F}(F)$= $\{1, 280, 840, 315, 630, 1260, 504, 360\}$. Now $\phi(2520) = \phi(2^3 3^2(5)(7))$ $= (2^3 - 2^2)(3^2 - 3)(5 - 1)(7 - 1) = 576$. So for example to compute $N(F^{840})$, we recall that $840 = \widehat{3}$, so $p = 3$ and $u = 1$ so $p^u - p^{u-1} = 2$. From Corollary 4.8, $N(F^{840}) = 3^{\frac{576}{2}} = 3^{288}$ as shown in that example. Similarly $N(F^{280}) = 3^{\frac{576}{3^2-3^1}} = 3^{\frac{576}{6}} = 3^{96}$.

The next example shows that we cannot remove the condition that $F$ is primitive from the conclusion that $N(F^m) = N(F)$ if $(m,n) \notin \mathcal{F}(F) \cup \{n\}$.

**Example 4.10.** Let $F = \mathcal{C}(\Phi_{10}\Phi_{35})$, then $\mathcal{F}(F) = \{1,2,3,5,6,7,14,15\}$, and $F$ has period 210. Now $(30,210) = 30 \notin \mathcal{F}(F) \cup \{n\}$, but $N(F^{30}) = 0 \neq N(F)$.

The difference in cardinality of $\mathcal{F}(F)$ (from Corollary 4.8) in the Corollary below comes because 1 is already present as $\widehat{p^s}$. Note also that $N(F) = p$ rather than 1 in Corollary 4.8. These points will also make a difference when it comes to discussing the $NP_m(F)$.

**Corollary 4.11.** *Let $F = \mathcal{C}(\Phi_n)$ with $n = p^s$, then*

$$\mathcal{F}(F) = \{1, p, p^2, \cdots, p^{s-1}\}, \textit{ and } \#(\mathcal{F}(F)) = s,$$

$$\textit{and if } (m,n) \in \mathcal{F}(F), \textit{ then } N(F^m) = p^{(m,n)}.$$

*If $(m,n) \notin \mathcal{F}(F) \cup \{n\}$, then $N(F^m) = p$, and $N(F^m) = 0$ if $(m,n) = n$.*

*Proof.* Note first that we can write $\{1, p, p^2, \cdots, p^{s-1}\}$ as $\{\widehat{p^s}, \widehat{p^{s-1}}, \cdots, \widehat{p}\}$, so the proof that $\mathcal{F}(F)$ is as shown, is similar to the proof in Corollary 4.8. Recall that $N(F^m) = N(F^{(m,n)})$, so (putting $(m,n) = q$), we show that $N(F^q) = p^q$ for $q = \widehat{p^u} \in \mathcal{F}(F)$. So as in Corollary 4.8 $N(F^q)$ is $p$ to the power $\frac{\phi(n)}{p^u - p^{u-1}}$. But $\frac{\phi(n)}{p^u - p^{u-1}} = \frac{p^s - p^{s-1}}{p^u - p^{u-1}} = \frac{p^s}{p^u} = \widehat{p^u} = q$ as required.

The rest is trivial. ∎

Having dealt with the $\mathcal{F}(F)$ and the $N(F^m)$, we now come to the $NP_m(F)$.

**Corollary 4.12.** *Let $F = \mathcal{C}(\Phi_n)$ where $n = p_1^{s_1} \cdots p_r^{s_r}$ with $r \geq 1$, then $NP_m(F) = 0$ if and only if $m \notin \mathcal{F}(F)$.*

*Let $r > 1$. If $m = 1$, then $NP_m(F) = 1$. Let $m = \widehat{p_i^u} \in \mathcal{F}(F)$ where $1 \leq u \leq s_i$. Then*

$$NP_m(F) = \begin{cases} N(F^m) - 1 & \textit{if } u = s_i \\ N(F^m) - N(F^{\frac{m}{p_i}}) & \textit{if } u < s_i, \end{cases}$$

*where the $N(F^q)$ are given in Corollary 4.8.*

*Let $r = 1$ and $m = \widehat{p_i^u} \in \mathcal{F}(F)$, where $1 \leq u \leq s_i$. Then*

$$NP_m(F) = \begin{cases} p_1 & \textit{if } u = s_i, \\ p_1^m - p_1^{\frac{m}{p}} & \textit{if } u < s_i. \end{cases}$$

*In particular if $n = p^s$, then $NP_p(F) = p^p - p$ independent of $s$.*

*Proof.* We deal first with the case that $r > 1$. Now for $m = 1$ we have that $NP_1(F) = N(F) = 1$ by Theorem 2.2 and Proposition 4.5. For $m = \widehat{p_i^u} \in \mathcal{F}(F)$ with $1 \leq u \leq s_i$ we use Möbius inversion (Theorem 2.2). Suppose first that $m = \widehat{p_i^{s_i}}$. Now if $q$ is a proper divisor of $m$, then $q \notin \mathcal{F}(F)$ and so $N(F^q) = 1$ (Corollary 4.8). In particular for all subsets $\tau$ of $\mathbf{P}(m)$ except for the empty set we have that $N(F^{m:\tau}) = 1$, and of course the empty set gives us $N(F^m)$. Now $\#(\mathbf{P}(m)) = r$, so there are $\binom{r}{1}$ subsets of $\mathbf{P}(m)$ with just one element, and there are $\binom{r}{2}$ two element subsets etc. So by Möbius inversion

$$NP_m(F) = \binom{r}{0}N(F^m) + \left(-\binom{r}{1} + \binom{r}{2} - \binom{r}{3} \cdots + (-1)^r\binom{r}{r}\right),$$

which is equal to $N(F^m) - 1 + ((+1) + (-1))^r = N(F^m) - 1$ as required.

For $u < s_i$ we have that $N(F^{m:\tau}) = 1$ except for $\tau = \emptyset$ and $\tau = \{p_i\}$, and the above argument is easily modified[1] to give $NP_m(F) = N(F^m) - N(F^{m/p_i})$.

For the case $r = 1$ we note again that $\widehat{p_1^{s_1}} = 1$, and $NP_1(F) = N(F) = p_1$ from Theorem 2.2 and Proposition 4.5. Also $\mathbf{P}(m) = \{p_1\}$ and $p_1 \mid m$ for all other $m \in \mathcal{F}(F) = \{\widehat{p_1^{s_1}}, \widehat{p^{s_1-1}}, \cdots, p\}$. Thus $NP_m(F) = N(F^m) - N(F^{m/p_1})$ as required, using Theorem 2.2 again. ∎

**Corollary 4.13.** *Let $F$ be a primitive matrix of period $n$ where $n = p_1^{s_1} \cdots p_r^{s_r}$ with $r \geq 1$, then*

$$N\Phi_m(F) = \begin{cases} N(F) & \text{if } (m,n) \notin \mathcal{F}(F) \cup \{n\} \\ N(F^{(m,n)}) = p^{\frac{\phi(n)}{\phi(n/(m,n))}} & \text{if } (m,n) = \widehat{p^u} \in \mathcal{F}(F) \\ \sum_{p\in\mathbf{P}(m)} N(F^{\frac{n}{p}}) - r + 1 & \text{if } (m,n) = n. \end{cases}$$

*Proof.* That $N\Phi_m(F) = N(F)$ for the indicated $m$ follows from Theorem 2.3 and Corollaries 4.8 and 4.11, as does the equation $N(F^{(m,n)}) = p^{\frac{\phi(n)}{\phi(n/(m,n))}}$ for $(m,n) = \widehat{p^u} \in \mathcal{F}(F)$.

When $(m,n) = n$, then $N(F^{(m,n)}) = 0$, and we need to use the sum formula in Theorem 2.3. When $r = 1$ we have that $M(F,n) = \{p^{s-1}\}$, and the sum formula gives $N\Phi_n(F) = N(F^{\frac{n}{p}})$ directly. Next when $r > 1$ the maximal divisors of $n$ are simply the $\widehat{p_i}$, and there are $r$ of them. Now $\mathrm{GCD}(\mu_i) = p_i$ for the $r$ one element subsets $\mu_i := \{p_i\}$. When $\#(\mu) \geq 2$, on the other hand, we have that $\mathrm{GCD}(\mu_i) = 1$ and of course for such $\mu$ we have that $N(F^{\mathrm{GCD}(\mu)}) = 1$. We are now in a very similar situation to Corollary 4.12, but with one less term and the signs reversed. Thus $N\Phi_n(F) = \sum_{p\in\mathbf{P}(m)} N(F^{\frac{n}{p}}) - \binom{r}{2} + \binom{r}{3} \cdots + (-1)^{r-1}\binom{r}{r}$, which is equal to $\sum_{p\in\mathbf{P}(m)} N(F^{\frac{n}{p}}) - r + 1 - \binom{r}{0} + \binom{r}{1} - \binom{r}{2} + \binom{r}{3} \cdots + (-1)^{r+1}\binom{r}{r} = \sum_{p\in\mathbf{P}(m)} N(F^{\frac{n}{p}}) - r + 1 - ((+1) + (-1))^r = \sum_{p\in\mathbf{P}(m)} N(F^{\frac{n}{p}}) - r + 1$ as required. ∎

[1] $NP_m(F) = \binom{\ell}{0}N(F^m) - N(F^{m/p_i}) - (\binom{\ell}{1} - 1) + \binom{\ell}{2} - \binom{\ell}{3} \cdots + (-1)^\ell\binom{\ell}{\ell}$.

## 5 Computing $N(F^m)$, $NP_m(F)$ and $N\Phi_m(F)$ inductively

In this section we use the natural eigenspace structure of the maps on the universal covering space to split our map on the torus (the torus model in the nilmanifold case) into a series of product maps on trivial fibrations. The aim is to build up the numbers $N(F^m)$ and $NP_m(F)$ inductively adding (direct sum) one primitive at a time. As mentioned earlier we reserve the computation of the $N\Phi_m(F)$ until the end of the finite induction process.

More precisely let $A$ be a primitive of period $n_A$, and $B$ an arbitrary periodic matrix of period $n_B$. Then as seen in the introduction, we can think of $A \oplus B$ as a product map $A \oplus B : T^{\phi(n_A)} \times T^{\phi(n_B)} \to T^{\phi(n_A)} \times T^{\phi(n_B)}$. This map is moreover fibre preserving with respect to the trivial fibration $T^{\phi(n_A)} \times T^{\phi(n_B)} \to T^{\phi(n_B)}$ with $A$ being the restriction of $A \oplus B$ to the fibre $T^{\phi(n_A)}$. In the inductive step we are assuming we already know the $N(B^q)$, $NP_q(B)$ for all $q$ and we wish to use this, the information from the primitive $A$ and the sum product (Theorem 5.1 below) to compute $N((A \oplus B)^q)$, $NP_q(A \oplus B)$ for all $q$. So the inductive step is to add (direct sum) a primitive $A$ using the results of section 4 to compute $N(A^q)$ and $NP_q(A)$ for all $q$. In order to use the sum product, however we will also need to study the numbers $NP_{\frac{m}{q}}(F^q)$ for various $m$ and $q|m$.

The first formula below follows from properties of determinants (or the naïve product Theorem). The second formula is an easy adaption of [4, Corollary 4.6] using the trivial fibration described in the introduction.

**Theorem 5.1.** *Let $A$ and $B$ be periodic matrices with $N(B^m) \neq 0$. Then*

$$N((A \oplus B)^m) = N(A^m)N(B^m) \text{ and}$$

$$NP_m(A \oplus B) = \sum_{q|m} NP_q(B) NP_{\frac{m}{q}}(A^q).$$

We call the second formula in Theorem 5.1 the sum product formula, or simply the sum product.

The section is divided into four subsections. In section 5.1 we seek to determine for which $m$ we automatically have that $NP_m(A \oplus B) = 0$ without having to compute the sum product. In fact we make an approximation $\mathcal{F}(A) \otimes \mathcal{F}(B)$ to $\mathcal{F}(A \oplus B)$, and show $NP_m(A \oplus B) = 0$ if $m \notin \mathcal{F}(A) \otimes \mathcal{F}(B)$. Some clarifying results about $\mathcal{F}(A) \otimes \mathcal{F}(B)$ are also given here. In section 5.2, having determined for which $m$ we have $NP_m(A \oplus B) = 0$, we turn our attention to determining which of the terms in the sum product are zero. As part of this we study the sets $\mathcal{F}(A^q)$ and compare them with $\mathcal{F}(A)$. This enables us, for each $m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$, to define a set $m^{\mathcal{F}}$ which has the property that the product $NP_q(B)NP_{\frac{m}{q}}(A^q) = 0$ if $q \notin m^{\mathcal{F}}$. In section 5.3 we study the numbers $NP_{\frac{m}{q}}(A^q)$, determine when they are equal to $NP_m(A)$, and what they are when they are not. Finally in section 5.4 we work a specific example of the inductive procedure with four stages.

### 5.1 Inductive approximations to the $\mathcal{F}(A \oplus B)$

We use an example to help motivate our definitions. The example was initially worked using a maple worksheet.

**Example 5.2.** Let $A := \mathcal{C}(\Phi_{n_A})$ where $n_A = 2^2 \times 3 \times 5 = 60$, and $B = \mathcal{C}(\Phi_{n_B})$ where $n_B = 2 \times 3^2 \times 5 = 90$. Then $A \oplus B$ has order 180, and the tables for $\mathcal{F}(A)$, $\mathcal{F}(B)$ and $\mathcal{F}(A \oplus B)$ are given as follows:-

| $\mathcal{F}(A)$ | $q$ | 1 | 12 | 15 | 20 | 30 |
|---|---|---|---|---|---|---|
| | $N(A^q)$ | 1 | $5^4$ | $2^8$ | $3^8$ | $2^{16}$ |

| $\mathcal{F}(B)$ | $q$ | 1 | 10 | 18 | 30 | 45 |
|---|---|---|---|---|---|---|
| | $N(B^q)$ | 1 | $3^4$ | $5^6$ | $3^{12}$ | $2^{24}$ |

.

| $\mathcal{F}(A \oplus B)$ | $q$ | 1 | 10 | 12 | 15 | 18 | 20 | 30 | 36 | 45 |
|---|---|---|---|---|---|---|---|---|---|---|
| | $N((A \oplus B)^q)$ | 1 | $3^4$ | $5^4$ | $2^8$ | $5^6$ | $3^4 3^8$ | $2^{16} 3^{12}$ | $5^4 5^6$ | $2^8 2^{24}$ |

.

Note that $36 \in \mathcal{F}(A \oplus B)$ because $N((A \oplus B)^{36}) = N(A^{36})N(B^{36}) = N(A^{12})N(B^{18}) = 5^4 \cdot 5^6 = 5^{10}$. So 36 is a first that belongs to neither $\mathcal{F}(A)$ nor $\mathcal{F}(B)$. Note however that new values occur not only at $m = 36 = lcm(12, 18)$, but also at $m = 20 = lcm(10, 20)$.

It should therefore be obvious that the lcm of pairs taken from $\mathcal{F}(A)$ and $\mathcal{F}(B)$ have the potential to give new firsts. In fact we define the set

$$\mathcal{LCM}(A, B) := \{lcm(a, b) \mid a \in \mathcal{F}(A), b \in \mathcal{F}(B)\}.$$

Note however, that not every lcm of pairs of elements of $\mathcal{F}(A)$ and $\mathcal{F}(B)$ appear in $\mathcal{F}(A \oplus B)$. In particular $60 = lcm(15, 12)$, but $N((A \oplus B)^{60}) = 0$ and of course we exclude zeros from $\mathcal{F}(F)$. Accordingly we set

$$\mathcal{Z}(A, B) := \{q \in \mathcal{LCM}(A, B) \mid N(A^q) \cdot N(B^q) = 0\},$$

and make the following formal definition.

**Definition 5.3.** Suppose that $A$ and $B$ are periodic matrices with $\mathcal{F}(A)$ and $\mathcal{F}(B)$ already defined, we define $\mathcal{F}(A) \otimes \mathcal{F}(B)$ to be the set

$$\mathcal{F}(A) \otimes \mathcal{F}(B) := \mathcal{LCM}(A, B)) - \mathcal{Z}(A, B).$$

In the example we have that $\mathcal{LCM}(A, B) = \mathcal{F}(A) \cup \mathcal{F}(B) \cup \{36, 60, 90\}$, and $\mathcal{Z}(A, B) = \{60, 90\}$. Note that we are not considering all zeros here, only those that occur in $\mathcal{LCM}(A, B)$. So for this example we have $\mathcal{F}(A) \otimes \mathcal{F}(B) = \mathcal{F}(A \oplus B) = \mathcal{F}(A) \cup \mathcal{F}(B) \cup \{36\}$.

**Proposition 5.4.** *Let $A$ and $B$ be periodic matrices, then*

$$\mathcal{F}(A \oplus B) \subseteq \mathcal{F}(A) \otimes \mathcal{F}(B), \text{ and } \mathcal{F}(A) \otimes \mathcal{F}(B) = \mathcal{F}(B) \otimes \mathcal{F}(A).$$

*Moreover if $m \notin \mathcal{F}(A) \otimes \mathcal{F}(B)$, then $NP_m(A \oplus B) = 0$.*

*Proof.* Commutativity is trivial. To prove the inclusion, let $m \in \mathcal{F}(A \oplus B)$, then $N((A \oplus B)^m) = N(A^m)N(B^m) \neq 0$ by definition. Thus we need only show that $m \in \mathcal{LCM}(A, B)$. If $m \in \mathcal{F}(A)$ or $m \in \mathcal{F}(B)$, then $m = lcm(m, 1) \in \mathcal{LCM}(A, B)$. In any case, there are integers $a \in \mathcal{F}(A)$ and $b \in \mathcal{F}(B)$ with $a|m$ and $b|m$ and such that $N(A^a) = N(A^m)$ and $N(B^b) = N(B^m)$. Let $c = lcm(a, b)$ then using Proposition 2.4 we have that $N(A^a) \leq N(A^c)$ and $N(B^b) \leq N(B^c)$. Moreover $a|m$ and $b|m$, so $c|m$. Thus $N(A^a)N(B^b) \leq N(A^c)N(B^c) \leq N(A^m)N(B^m)$ and therefore $N(A^c)N(B^c) = N(A^m)N(B^m)$. So $m = c$ since if not, $m$ would not be a first. Thus $m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$ as required.

For the last part if $m \notin \mathcal{F}(A) \otimes \mathcal{F}(B)$, then $m \notin \mathcal{F}(A \oplus B)$ by the first part. So $m$ is not a first. Thus there is a $q|m$ with $q < m$ and $N((A \oplus B)^q) = N((A \oplus B)^m)$. By Proposition 2.4 again we have that $NP_m(A \oplus B) = 0$. ∎

**Remark 5.5.** Though we have no counter example for the reverse of the inclusion in Proposition 5.4 a proof that it holds is elusive.

In spite of the remark we do have

**Proposition 5.6.** *If for all $m_1, m_2 \in \mathcal{F}(A) \otimes \mathcal{F}(B)$ with $m_1 \neq m_2$ we have that $N((A \oplus b)^{m_1}) \neq N((A \oplus B)^{m_2})$ then*

$$\mathcal{F}(A) \otimes \mathcal{F}(B) = \mathcal{F}(A \oplus B).$$

*Proof.* Let $m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$, then $N((A \oplus b)^q) \neq N((A \oplus B)^m)$for all $q < m$ by the hypothesis. So $m \in \mathcal{F}(A \oplus B)$ by Proposition 3.4 . ∎

The following example shows it is not always true that $\mathcal{F}(A) \cup \mathcal{F}(B) \subseteq \mathcal{F}(A \oplus B)$.

**Example 5.7.** Example where $\mathcal{F}(A) \cup \mathcal{F}(B) \not\subseteq \mathcal{F}(A \oplus B)$. Let $A := \mathcal{C}(\Phi_{30})$ and $B = \mathcal{C}(\Phi_6)$, then $\mathcal{F}(A) = \{1, 6, 10, 15\}$ and $\mathcal{F}(B) = \{1, 2, 3\}$ while $\mathcal{F}(A \oplus B) = \{1, 2, 3, 10, 12, 15\}$. The number $6 \in \mathcal{F}(A)$ is not in $\mathcal{F}(A \oplus B)$, because $N(B^6) = 0$ so $N((A \oplus B)^6) = N(A^6)N(B^6) = 0$ too.

We saw in Example 5.7 that $\mathcal{F}(A)$ may not be contained in $\mathcal{F}(A) \otimes \mathcal{F}(B)$. On the other hand if $A = B$ we have no problem determining $\mathcal{F}(A \oplus B)$

**Proposition 5.8.** *Let $A$ be an arbitrary periodic matrix, then*

$$\mathcal{F}(A \oplus A) = \mathcal{F}(A).$$

*Proof.* The result essentially follows from the fact that $N((A \oplus A)^m) = (N(A^m))^2$. So for example if $m \in \mathcal{F}(A \oplus A)$ then $(N(A^m))^2 \neq 0$, and for all proper divisors $q$ of $m$ we have that $(N(A^q))^2 < (N(A^m))^2$. Clearly by taking square roots we have that $N(A^m) \neq 0$ and $N(A^q) < N(A^m)$ for all proper divisors of $m$ so $m$ is a first. The converse is similar. ∎

The following Corollary will allow us, in our inductive procedure, to add a $k$th power of a primitive in one fell swoop, rather than having to do it $k$ times. Its proof essentially boils down to the fact that for each $q$ we have that $N(\oplus_{i=1}^k A)^q) = (N(A^q))^k$, and is omitted.

**Corollary 5.9. Powers of a primitive** *Let $F = \oplus_{i=1}^{k} A$, where $A = \mathcal{C}(\Phi_n))$ with $n = p_1^{s_1} \cdots p_r^{s_r}$ and $r \geq 1$. Then For all $m$ we have that $N(F^m) = (N(A^m))^k$, where the $N(A^m)$ are given in Corollary 4.8. Furthermore $NP_m(F) = 0$ if and only if $m \notin \mathcal{F}(A)$.*
*Let $r > 1$. If $m = 1$, then $NP_m(F) = 1$ otherwise let $m = \widehat{p_i^u} \in \mathcal{F}(A)$. Then*

$$NP_m(F) = \begin{cases} (N(A^m))^k - 1 & \text{if } u = s_i \\ (N(A^m))^k - N((A^{\frac{m}{p_i}})^k & \text{if } u < s_1. \end{cases}$$

*Let $r = 1$ and $m = \widehat{p_1^u} \in \mathcal{F}(A)$. Then*

$$NP_m(F) = \begin{cases} p_1^k & \text{if } u = s_i, \\ p_1^{km} - p_1^{k\frac{m}{p}} & \text{if } u < s_1. \end{cases}$$

*In particular if $n = p^s$, then $NP_p(F) = p^{kp} - p^k$ independent of $s$.* ∎

There is of course a similar result for $N\Phi_m(\mathcal{C}(\Phi_n)^{[k]})$ which we omit.
The proof of the following Proposition is left to the reader.

**Proposition 5.10.** $(\mathcal{F}(A) \otimes \mathcal{F}(B)) \otimes \mathcal{F}(C) = \mathcal{F}(A) \otimes (\mathcal{F}(B) \otimes \mathcal{F}(C))$.

The final result of this subsection, whose proof combines the first parts of Propositions 3.2 and 5.1, and can obviously be extended to more than two factors.

**Proposition 5.11.** *Let $A$ and $B$ be arbitrary periodic matrices of periods $n_A$ and $n_B$ respectively, then*

$$N((A \oplus B)^m) = N(A^{(m,n_A)})N(B^{(m,n_B)}).$$ ∎

In Corollaries 4.8 and 4.12 we saw for primitives that if $(m, n) \notin \mathcal{F}(F) \cup \{n\}$, then $N(F^m) = N(F)$. Proposition 5.11 helps us to see why this is not true for arbitrary periodic matrices. To give a concrete example consider $m = 40$ in Example 3.5 where $F = \mathcal{C}(\Phi_{20}\Phi_{30})$ with period 60. Now $(40, 60) = 20$, $(40, 20) = 20$, $(40, 30) = 10$, and $20 \notin \mathcal{F}(F) \cup \{60\} = \{1, 4, 5, 6, 10, 12, 15, 60\}$ but $N(F^{20}) = N((\mathcal{C}(\Phi_{20})^{20})N((\mathcal{C}(\Phi_{30})^{10}) = 0 \neq 1 = N(\mathcal{C}(\Phi_{20}))N(\mathcal{C}(\Phi_{30}))$.

## 5.2 The sets $\mathcal{F}(A^q)$ and $m^{\mathcal{F}}$ for $q \mid m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$

In this subsection $B$ will be an arbitrary periodic matrix, and $A$ a primitive. We are thinking of $B$ as the $k$th stage of our induction procedure and $A \oplus B$ as the $k+1$st stage. In this subsection we are looking to simplify the sum product formula (the second part of Theorem 5.1). In particular we want to know when the various $NP_q(B)NP_{\frac{m}{q}}(A^q)$ are zero. We show for a fixed $m$, rather than having to take the sum of the products over all $q \mid m$, that we need only take the sum over a subset of such $q$ which we denote by $m^{\mathcal{F}}$ (Notation 5.17). The point is if $q \notin m^{\mathcal{F}}$ then the product $NP_q(B)NP_{\frac{m}{q}}(A^q) = 0$. We verify the simplification advertised in Example 1.2 given in the introduction, which we restate below. In the next subsection we will need to figure out the numbers $NP_{\frac{m}{q}}(A^q)$. After reminding the reader of Example 1.2 we start by looking at $\mathcal{F}(A^q)$ and its comparison with $\mathcal{F}(A)$ for various $q$. We continue with the definition and the study of the sets $m^{\mathcal{F}}$.

**Example 5.12. Example 1.2 revisited**. In the introduction we gave (without proof) the table below of values of $NP_m(A \oplus B)$ for $A := \mathcal{C}(\Phi_{20})$ and $B := \mathcal{C}(\Phi_{30})$. From Example 3.5 we have that $\mathcal{F}(A) = \{1,4,5,10\}$, $\mathcal{F}(B) = \{1,6,10,15\}$ and $\mathcal{F}(A \oplus B) = \{1,4,5,6,10,12,15\}$. We verify the table below in this subsection. In particular we will see, for $m = 1$, $m = 4$ and $m = 5$ that $m^{\mathcal{F}} = \{1\}$ (Notation 5.17, Example 5.22).

| | |
|---|---|
| $NP_1(A \oplus B)$ | $N(B)N(A)$ |
| $NP_4(A \oplus B)$ | $N(B)NP_4(A)$ |
| $NP_5(A \oplus B)$ | $N(B)NP_5(A)$ |
| $NP_6(A \oplus B)$ | $NP_6(B)N(A)$ |
| $NP_{10}(A \oplus B)$ | $N(B)NP_{10}(A) + NP_{10}(B)N(A^{10})$ |
| $NP_{12}(A \oplus B)$ | $NP_6(B)NP_2(A^4)$ |
| $NP_{15}(A \oplus B)$ | $NP_{15}(B)N(A^{15})$ |

**Proposition 5.13.** *Let $F = (\mathcal{C}(\Phi_n))$, then $\mathcal{F}(F^q) = \mathcal{F}(\mathcal{C}(\Phi_{\frac{n}{(n,q)}}))$.*

*Proof.* By Proposition 3.2 we can work with $F^{(n,q)}$ rather than $F^q$. By Theorem 2.7 we have that $F^{(n,q)} = \mathcal{C}(\Phi_{\frac{n}{(n,q)}})^{[\frac{\phi(n)}{\phi(\frac{n}{(n,q)})}]}$. Thus $F^{(n,q)}$ is a sum of $\frac{\phi(n)}{\phi(\frac{n}{(n,q)})}$ copies of the matrix $\mathcal{C}(\Phi_{\frac{n}{(n,q)}})$. So by Proposition 5.8 we have that $\mathcal{F}(F^{(n,q)}) = \mathcal{F}(\mathcal{C}(\Phi_{\frac{n}{(n,q)}}))$ as stated. ∎

The next example not only illustrates Proposition 5.13, but compares various $\mathcal{F}(F^q)$ with $\mathcal{F}(F)$.

**Example 5.14.** Let $F = \mathcal{C}(\Phi_{2520})$ be as in Example 1.1. The table below compares $\mathcal{F}(F)$ with $\mathcal{F}(F^{42}) = \mathcal{F}(\mathcal{C}(\Phi_{60}))$, $\mathcal{F}(F^{105}) = \mathcal{F}(\mathcal{C}(\Phi_{12})))$, $\mathcal{F}(F^{955}) = \mathcal{F}\mathcal{C}(\Phi_8))$ and $\mathcal{F}(F^{630}) = \mathcal{C}(\Phi_4))$.

From Proposition 5.13 we have that $\mathcal{F}(F^{42}) = \mathcal{F}(\mathcal{C}(\Phi_{60})) = \{1,12,15,20,30\}$, and from Proposition 3.2 that $N(F^{945}) = N(F^{(945,n)}) = N(F^{315})$ so as above $\mathcal{F}(F^{945})) = \{1,2,4\}$. From the table we can see also see that this last set is equal to

$$\{1\} \cup \{v \in \mathbb{N} \mid v(945,n) \in \mathcal{F}(F)\}.$$

This illustrates the following alternative way of describing the $\mathcal{F}(F^q)$.

**Proposition 5.15.** *Let $F = \mathcal{C}(\Phi_n)$, and let $q$ be arbitrary. If $(q,n) = n$ then $\mathcal{F}(F^q) = \emptyset$, otherwise*

$$\mathcal{F}(F^q) = \{1\} \cup \{v \in \mathbb{N} \mid v(q,n) \in \mathcal{F}(F)\},$$

*and of course if $v \in \mathcal{F}(F^q)$, then $N((F^q)^v) = N(F^{qv}) = N(F^{v(q,n)})$.*

*Moreover if $\frac{m}{q} \in \mathbb{N}$, then $\frac{m}{q} \in \mathcal{F}(F^q) \Leftrightarrow q = m$ or $\frac{m}{q}(n,q) \in \mathcal{F}(F)$. If $q|n$ with $q \neq n$ then*

$$\mathcal{F}(F^q) = \{1\} \cup \{v \in \mathbb{N} \mid qv \in \mathcal{F}(F)\},$$

*and if $m|n$ and $\frac{m}{q} \in \mathbb{N}$, then $\frac{m}{q} \in \mathcal{F}(F^q) \Leftrightarrow q = m$ or $m \in \mathcal{F}(F)$.*

| $\mathcal{F}(F)$ | $m$ | 1 | 280 | 315 | 360 | 504 | 630 | 840 | 1260 |
|---|---|---|---|---|---|---|---|---|---|
| | | 1 | $\widehat{3^2}$ | $\widehat{2^3}$ | $\widehat{7}$ | $\widehat{5}$ | $\widehat{2^2}$ | $\widehat{3}$ | $\widehat{2}$ |
| | $N(F^m)$ | 1 | $3^{96}$ | $2^{144}$ | $7^{96}$ | $5^{144}$ | $2^{288}$ | $3^{288}$ | $2^{576}$ |
| $\mathcal{F}(F^{42})$ | $t$ | 1 | | | | 12 | 15 | 20 | 30 |
| | $N((F^{42})^t)$ | 1 | | | | $5^{144}$ | $2^{288}$ | $3^{288}$ | $2^{576}$ |
| $\mathcal{F}(F^{105})$ | $t$ | 1 | | 3 | | | 6 | 8 | 12 |
| | $N((F^{105})^t)$ | 1 | | $2^{144}$ | | | $2^{288}$ | $3^{288}$ | $2^{576}$ |
| $\mathcal{F}(F^{945}) = \mathcal{F}(F^{315})$ | $t$ | | | 1 | | | 2 | | 4 |
| $(945, n) = 315$ | $N((F^{315})^t)$ | | | $2^{144}$ | | | $2^{288}$ | | $2^{576}$ |
| $\mathcal{F}(F^{630})$ | $t$ | | | | | | 1 | | 2 |
| | $N((F^{630})^t)$ | | | | | | $2^{288}$ | | $2^{576}$ |

Note that if $(q, n) \neq 1$, then $\#(\mathcal{F}(F^q) < \#(\mathcal{F}(F))$

*Proof of 5.15.* If $(q, n) = n$ then $N(F^{qm}) = 0$ for all $m$ and so $\mathcal{F}(F^q) = \emptyset$ as stated. Otherwise let $h = (q, n)$, then $h|n$, and by the proof of Proposition 3.2, $F^q$ and $F^h$ have the same eigenvalues, so $\mathcal{F}(F^q) = \mathcal{F}(F^h)$. Next, let $B = \mathcal{C}(\Phi_{\frac{n}{h}})$, then $B$ is primitive, and by Theorem 4.6 we can think of $F^h$ as a finite direct sum of copies of $B$. By Proposition 5.8 we have that $\mathcal{F}(F^h) = \mathcal{F}(B)$. We show $\mathcal{F}(B) = \{1\} \cup \{v \in \mathbb{N} \mid hv \in \mathcal{F}(F)\}$.

Let $n = p_1^{s_1} \cdots p_r^{s_r}$ with $r \geq 1$. Then $\frac{n}{h} = p_1^{t_1} \cdots p_r^{t_r}$ for some $t_i$, with $0 \leq t_i \leq s_i$ for all $i$. Now $\mathcal{F}(B) = \{1\} \cup \{(n/h)/p_i^\ell \mid 1 \leq \ell \leq t_i\}$ by Corollary 4.8. We show that $\mathcal{F}(B) \subseteq \{1\} \cup \{v \mid hv \in \mathcal{F}(F)\}$ and vice versa. Clearly $1 \in \{1\} \cup \{v \in \mathbb{N} \mid hv \in \mathcal{F}(F)\}$, so let $w = (n/h)/p_i^\ell \in \mathcal{F}(B)$, then $hw = h(n/h)/p_i^\ell = n/p_i^\ell = \widehat{p_i^\ell} \in \mathcal{F}(F)$, so $w \in \{1\} \cup \{v \in \mathbb{N} \mid hv \in \mathcal{F}(F)\}$, and $\mathcal{F}(B) \subseteq \{1\} \cup \{v \in \mathbb{N} \mid hv \in \mathcal{F}(F)\}$.

Conversely let $w \in \{1\} \cup \{v \in \mathbb{N} \mid hv \in \mathcal{F}(F)\}$. If $w = 1$ then $w \in \{1\} \cup \{v \in \mathbb{N} \mid hv \in \mathcal{F}(F)\}$, otherwise $hw = \widehat{p_i^\ell}$ for some $i$ with $1 \leq \ell \leq s_i$. Clearly $w = (n/h)/p_i^\ell$. We must show that $1 \leq \ell \leq t_i$. Now $w \in \mathbb{N}$ and this would not be the case if $\ell > t_i$, so $1 \leq \ell \leq t_i$, and $w = (n/h)/p_i^\ell \in \mathcal{F}(B)$ as required.

The rest is trivial. ∎

**Corollary 5.16.** *Let $B$ be periodic, $A$ primitive of period $n$, and $m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$. Then*

$$NP_q(B)NP_{\frac{m}{q}}(A^q) = 0$$

*if either $q \notin \mathcal{F}(B)$, or $\frac{m}{q} \notin \mathcal{F}(A^q)$. In particular if $m|n$ then $NP_q(B)NP_{\frac{m}{q}}(A^q) = 0$ if either $q \notin \mathcal{F}(B)$, or ($q \neq m$ and $m \notin \mathcal{F}(A)$).*

*Proof.* The first part is trivial, since if $q \notin \mathcal{F}(B)$ then $NP_q(B) = 0$. Similarly if $\frac{m}{q} \notin \mathcal{F}(F^q)$ then $NP_{\frac{m}{q}}(A^q) = 0$. The rest follows from Proposition 5.15. ∎

We use the following notation for the complement of the set described in Corollary 5.16.

**Notation 5.17.** Let $B$ be periodic, $A$ primitive of period $n$, and $m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$. We use the notation $m^{\mathcal{F}}$ to denote the set

$$m^{\mathcal{F}} := \{q|m \mid q \in \mathcal{F}(B) \text{ and } (q = m \text{ or } \tfrac{m}{q}(n,q) \in \mathcal{F}(A))\}.$$

This gives immediately the following Corollary of Theorem 5.1:-

**Corollary 5.18.** *Let $B$ be periodic and $A$ be primitive. If $m \notin \mathcal{F}(A) \otimes \mathcal{F}(B)$, then $NP_m(A \oplus B) = 0$. If $m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$ then*

$$NP_m(A \oplus B) = \textstyle\sum_{q \in m^{\mathcal{F}}} NP_q(B) NP_{\frac{m}{q}}(A^{(q,n)}).$$

**Example 5.19. Example 5.12 continued.** We explain the cases $m = 6$ and $m = 12$ in Example 5.12. We remind the reader that $\mathcal{F}(A) = \{1,4,5,10\}$, $\mathcal{F}(B) = \{1,6,10,15\}$ and $\mathcal{F}(A \oplus B) = \{1,4,5,6,10,12,15\}$. For $m = 6$ we have that $\{q|6\} = \{1,2,3,6\}$. Now neither 2 nor 3 belong to $\mathcal{F}(B)$, and so are eliminated. For $q = 1$ we have that $\frac{6}{q}(n,q) = 6 \notin \mathcal{F}(A)$, so $1 \notin 6!q$. On the other hand $6 \in \mathcal{F}(B)$ and $q = 6$ so $6^{\mathcal{F}} = \{6\}$. Thus $NP_6(A \oplus B) = NP_6(B)N(A^2) = NP_6(B)N(A)$ (since $2 \notin \mathcal{F}(A)$ - see Corollary 4.8.

For $m = 12$ we have $\{q|12\} = \{1,2,3,4,6,12\}$, but $2,3,4,12 \notin \mathcal{F}(B)$ so do not belong to $12^{\mathcal{F}}$. Also $1 \notin 12^{\mathcal{F}}$ since $\frac{12}{1}1 = 12 \notin \mathcal{F}(A)$. Now $\frac{12}{6}(20,6) = 4 \in \mathcal{F}(A)$ and $6 \in \mathcal{F}(B)$, so $12^{\mathcal{F}} = \{6\}$, and we thus confirm that $NP_{12}(A \oplus B) = NP_6(B)NP_2(A^2)$. At this point we do not know how to compute $NP_2(A^2)$. We look into this sort of computation in 5.17 in the next subsection.

We now establish some criteria that give shortcuts to determine the $m^{\mathcal{F}}$. Since if $m|n$ we have that $(m,n) = m$, then the Lemma below simply restates the definition in Notation 5.17.

**Lemma 5.20.** *Let $B$ be periodic and $A$ primitive of period $n$, and $m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$. If $m|n$, then*

$$m^{\mathcal{F}} := \{q|m \mid q \in \mathcal{F}(B) \textit{ and } (q = m \textit{ or } m \in \mathcal{F}(A))\ \}. \qquad \blacksquare$$

**Corollary 5.21.** *Let $B$ be periodic, $A$ primitive, and let $m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$ with $m|n$.*

*If $m \in \mathcal{F}(A)$, then*

$$m^{\mathcal{F}} := \{q|m \mid q \in \mathcal{F}(B)\}.$$

*In particular, if no $q|m$ lies in $\mathcal{F}(B) - \{1\}$, then $m^{\mathcal{F}} = \{1\}$ and so*

$$NP_m(A \oplus B) = N(B)NP_m(A).$$

*If $m \in \mathcal{F}(B)$ but $m \notin \mathcal{F}(A)$ then $m^{\mathcal{F}} = \{m\}$ and so*

$$NP_m(A \oplus B) = NP_m(B)N(A).$$

*Proof.* For the first part, if $m \in \mathcal{F}(A)$ the right hand condition in Lemma 5.20 is fulfilled and the statement that $m^{\mathcal{F}} = \{q|m \mid q \in \mathcal{F}(B)\}$ is simply the left hand condition of the Corollary. The "In particular" part is now obvious. For the second part, since $m|n$ but $m \notin \mathcal{F}(A)$, then the only way that the right hand condition in Lemma 5.20 can be fulfilled is if $q = m$. Since $m \in \mathcal{F}(B)$ then $m^{\mathcal{F}} = \{m\}$, and the formula for $NP_m(A \oplus B)$ follows from Corollary 5.18. ∎

**Example 5.22.** We explain first the cases $m = 4, 5$ and 10 in Example 5.12. Since $m \in \mathcal{F}(A)$ for $m = 4, 5$ and 10, the first part of Corollary 5.21 allows us to explain these cases. In particular $4^{\mathcal{F}} = \{q|4 \mid q \in \mathcal{F}(B)\} = \{1\}$, similarly $5^{\mathcal{F}} = \{q|5 \mid q \in \mathcal{F}(B)\} = \{1\}$, while $10^{\mathcal{F}} = \{q|10 \mid q \in \mathcal{F}(B)\} = \{1, 10\}$. Now we cannot use Corollary 5.21 for the case $m = 6$, because $6 \nmid n$. However since both $A$ and $B$ are primitives we can reverse their roles in the first part of the Corollary to deduce that $NP_6(A \oplus B) = NP_6(B)N(A)$.

In order to illustrate the part of Corollary 5.21 with $m \notin \mathcal{F}(A)$, we go back to Example 5.2, where $A := \mathcal{C}(\Phi_{60})$ and $B = \mathcal{C}(\Phi_{90})$. We saw there that $\mathcal{F}(A) = \{1, 12, 15, 20, 30\}$ and $\mathcal{F}(B) = \{1, 10, 18, 30, 45\}$. So $10 \in \mathcal{F}(B)$ and $10|60$, but $10 \notin \mathcal{F}(A)$. So $10^{\mathcal{F}} = \{10\}$ so $NP_{10}(A \oplus B) = NP_{10}(B)N(A) = (3^4 - 1) \cdot 1 = 3^4 - 1$ which can also easily be seen from the table for $\mathcal{F}(A \oplus B)$ in that example.

There are in fact three more cases we investigate where $NP_m(A \oplus B)$ can be computed as a simple product. We give the first two here, but wait until the next subsection for the third. The first two results come into their own in our concluding example (5.28) and we wait until then to illustrate them. Proposition 5.23 below differs from Corollary 5.21 in that the hypotheses here do not include that $m|n$. The following Proposition is used in step 2 of Example 5.28, where $F_1$ plays the role of $B$ and $F_2$ the role of $A$.

**Proposition 5.23.** *Let $A$ and $B$ be periodic matrices with $A$ primitive of period $n$. If $m \in \mathcal{F}(B) \cap (\mathcal{F}(A) \otimes \mathcal{F}(B))$ is such that $m < \widehat{p_i^u}$ for all $\widehat{p_i^u} \in \mathcal{F}(A) - \{1\}$, then $m^{\mathcal{F}} = \{m\}$ and*

$$NP_m(A \oplus B) = NP_m(B)N(A).$$

*Proof.* Clearly $1^{\mathcal{F}} = \{1\}$, so the Proposition is true for $m = 1$. So we can assume, for the rest of the proof, that $m \neq 1$. In particular from the hypotheses we have that $m \notin \mathcal{F}(A) - \{1\}$. Moreover no $q$ with $1 < q \leq m$ lies in $\mathcal{F}(A)$. Recall that $m^{\mathcal{F}} = \{q|m \mid q \in \mathcal{F}(B)$ and ($q = m$ or $\frac{m}{q}(n, q) \in \mathcal{F}(A)$ $\}$. Now $m \in \mathcal{F}(B)$ so for $q = m$ we certainly have that $m \in m^{\mathcal{F}}$. So we must show that if $m \neq 1$, $q|m$ and $q \neq m$, then $q \notin m^{\mathcal{F}}$. We show for all such $q$ that $\frac{m}{q}(n, q) \notin \mathcal{F}(A)$. Now $\frac{(n,q)}{q} \leq 1$, so $\frac{m}{q}(n, q) \leq m$. So unless $\frac{m}{q}(n, q) = 1$ it does not belong to $\mathcal{F}(A)$. But $q \neq m$, so $\frac{m}{q} > 1$ and so $\frac{m}{q}(n, q) > 1$ as needed.

Now $NP_m(A \oplus B) = NP_m(B)NP_{\frac{m}{m}}(A^{(m,n)})$ from Corollary 5.18. We need to see that $N(A^{(m,n)}) = N(A)$. If $(m, n) = 1$ there is nothing to prove. On the other hand if $(m, n) \neq 1$, since $(m, n) \leq m$ then $(m, n) \notin \mathcal{F}(A)$ by hypotheses, and so $N(A^{(m,n)}) = N(A)$ by Corollaries 4.8 and 4.11, and we are done. ∎

**Proposition 5.24.** *(Used in step 4 of Example 5.28) Let $B$ be an arbitrary periodic matrix of period $n_B$ and let $A = \mathcal{C}(\Phi_{p^s})$ where $p$ is a prime that does not divide $n_B$. Then*

*$\mathcal{F}(A) \otimes \mathcal{F}(B) = \mathcal{F}(A)\mathcal{F}(B)$ as a set. Moreover for each $m_A m_B \in \mathcal{F}(A)\mathcal{F}(B)$ we have that $(m_A m_B)^{\mathcal{F}} = \{m_B\}$ and so*

$$NP_{m_A m_B}(A \oplus B) = NP_{m_B}(B) NP_{m_A}(A).$$

*Proof.* We note first that $(n_A, n_B) = 1$ implies that $(m_A, t) = 1$ for any $m_A \in \mathcal{F}(A)$ and any $t|n_B$. In particular, for any $m_A \in \mathcal{F}(A)$, and $m_B \in \mathcal{F}(B)$ we have that $lcm(m_A, m_B) = m_A m_B$. Recall next that $\mathcal{F}(A) \otimes \mathcal{F}(B) = \mathcal{LCM}(A,B)) - \mathcal{Z}(A,B)$. So if $lcm(m_A, m_B) \in \mathcal{F}(A) \otimes \mathcal{F}(B)$, then since $0 \neq lcm(m_A, m_B) = m_A m_B$, we must have $m_A \neq 0$ and $m_B \neq 0$ thus $lcm(m_A, m_B) = m_A m_B \in \mathcal{F}(A)\mathcal{F}(B)$.

Next let $m_A m_B \in \mathcal{F}(A)\mathcal{F}(B)$. Since $m_A m_B = lcm(m_A, m_B)$ we need only show that $N((A \oplus B)^{m_A, m_B}) \neq 0$. So let $n_1, \cdots n_\ell$ be all the zeros of $B$ which divide the order of $B$. Since $A = \mathcal{C}(\Phi_{p^s})$ we know the only zero that divides the order of $A$ is $p^s$. It is easy to see that $m$ is a zero of $B$ if and only if $(m, n_i) = n_i$ for some $i$, and that the zeros of $A \oplus B$ are multiples of either some $n_i$ or of $p^s$. But if $m_B \in \mathcal{F}(B)$ then $N(F^{m_B}) \neq 0$ so $(m_B, n_i) \neq n_i$ for any $i$. Also $(m_A, n_i) = 1$ for all $i$ ($t = n_i$ above) so $(m_A m_B, n_i) \neq n_i$ for any $i$, and neither is it a multiple of $p^s$. So $m_A m_B$ is not a zero of $A \oplus B$.

To show that $(m_A m_B)^{\mathcal{F}} = \{m_B\}$, note that $(m_A m_B)^{\mathcal{F}} := \{q | m_A m_B \mid q \in \mathcal{F}(B)$ and ($q = m_A m_B$ or $\frac{m_A m_B}{q}(n_A, q) \in \mathcal{F}(A))\}$ by definition. We show first that if either $m_A = 1$ or $m_B = 1$ then the result holds. If $m_A = m_B = 1$ then clearly $(1 \cdot 1)^{\mathcal{F}} = \{1\}$. If $m_A = 1$ and $m_B \neq 1$, then $NP_{\frac{m_B}{q}}(A^q) = 0$ for all $q \neq m_B$, since for any such $q$ we have that $A^q = A$ and $\frac{1 m_B}{q}(n_A, q) = \frac{m_B}{q} \cdot 1 \notin \mathcal{F}(A)$. Thus $(1 \cdot m_B)^{\mathcal{F}} = \{m_B\}$. If $m_B = 1$ and $m_A \neq 1$, then the only $q | 1 \cdot m_A$ that belonging to $\mathcal{F}(B)$ is $q = m_B$ so again $(m_A \cdot 1)^{\mathcal{F}} = \{1\}$. So without loss we may assume that $m_A \neq 1$ and $m_B \neq 1$.

Let $q \in (m_A m_B)^{\mathcal{F}}$, then $q \in \mathcal{F}(B)$, so $(m_A, q) = 1$ since $(m_A, m_B) = 1$. In particular $q \neq m_A$, and we must have that $\frac{m_A m_B}{q}(n_A, q) = \frac{m_A m_B}{q} \in \mathcal{F}(A)$. So $q$ must cancel $m_B$ and be equal to it, and we have shown that $q = m_B$ as required. ∎

### 5.3 Computing $NP_{\frac{m}{q}}(A^{(q,n)})$ for $q \in m^{\mathcal{F}}$

The main thrust of this subsection is the determination of the $NP_{\frac{m}{q}}(A^q)$ for $m \in \mathcal{F}(A) \otimes \mathcal{F}(B)$ and for $q \in m^{\mathcal{F}}$. We will use our results to continue to give simplifications of the sum product. Since $N((F^q)^{\frac{m}{q}}) = N(F^m)$, we can expect some connection between the $NP_{\frac{m}{q}}(F^q)$ and the $NP_m(F)$.

**Example 5.25.** We use two primitives $F_1 := \mathcal{C}(\Phi_{2^6})$ and $F := \mathcal{C}(\Phi_{2520})$ together with Corollary 5.9 to illustrate the fact that $NP_{\frac{m}{q}}(F^q)$ may or may not be equal to $NP_m(F)$. These computations also illustrate Proposition 5.26 below. The cases, in that Proposition, are indicated in brackets following the computations.

The determination of $\mathcal{F}(F_1)$, $\mathcal{F}(F_1^2)$ and $\mathcal{F}(F_1^{24})$ are shown in the table below.

So $NP_{\frac{2}{2}}(F_1^2) = N(F_1^2) = 2^2 \neq NP_2(F_1) = 2^2 - 2$. Similarly $NP_{\frac{24}{24}}(F_1^8) \neq NP_8(F_1)$ (both illustrate $r = 1$ and $m = q$ in Proposition 5.26). On the other

| $\mathcal{F}(F_1)$ | $m$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 |
|---|---|---|---|---|---|---|---|---|
| | $N(F_1^m)$ | 2 | $2^2$ | $2^4$ | $2^8$ | $2^{16}$ | $2^{32}$ | 0 |
| $\mathcal{F}(F_1^2)$ | $t$ | | 1 | 2 | 4 | 8 | 16 | 32 |
| | $N((F_1^2)^t)$ | | $2^2$ | $2^4$ | $2^8$ | $2^{16}$ | $2^{32}$ | 0 |
| $\mathcal{F}(F_1^{24})$ | $t$ | | | | 1 | 2 | 4 | 8 |
| $(24, 2^6) = 8$ | $N((F_1^8)^t)$ | | | | $2^8$ | $2^{16}$ | $2^{32}$ | 0 |

hand $NP_{\frac{16}{2}}(F_1^2) = NP_8(F_1^2) = 2^{16} - 2^8 = NP_{16}(F_1)$ and $NP_{\frac{32}{8}}(F_1^{24}) = NP_4(F_1^8) = 2^{32} - 2^{16} = NP_{32}(F_1)$ ($r = 1$ and $q \neq m$ in Proposition 5.26).

We now use $F := \mathcal{C}(\Phi_{2520})$, and the table in Example 5.14, to illustrate the indicated parts (in brackets) of Proposition 5.26. So

$$NP_{\frac{504}{42}}(F^{42}) = NP_{12}(F^{42}) = 5^{144} - 1 = NP_{504}(F) \text{ } (r > 1 \text{ and } u = s_i)$$

$$NP_{\frac{1260}{42}}(F^{42}) = NP_{30}(F^{42}) = 2^{576} - 2^{288} = NP_{1260}(F) \text{ } (r > 1, u < s_i \text{ and } p_i \mid \tfrac{m}{q})$$

$$NP_{\frac{630}{42}}(F^{42}) = NP_{15}(F^{42}) = 2^{288} - 1 \neq NP_{630}(F) = 2^{288} - 2^{144} \text{ } (u < s_i \text{ and } p_i \nmid \tfrac{m}{q})$$

$$\text{Similarly } NP_{\frac{840}{105}}(F^{105}) = NP_8(F^{105}) = 3^{288} - 1 \neq NP_{840}(F) = 3^{288} - 3^{96}.$$

The Proposition below gives conditions under which $NP_{\frac{m}{q}}(F^q) = NP_{\frac{m(q,n)}{q}}(F)$, and under which $NP_{\frac{m}{q}}(F^q) = NP_m(F)$.

**Proposition 5.26.** *Let $F = \mathcal{C}(\Phi_n)$ where $n = p_1^{s_1} \cdots p_r^{s_r}$ and $r \geq 1$. Suppose that $m$ is an arbitrary positive integer and $q \mid m$.*

$$\text{If } q = m \text{ (and } \neq n) \text{ then } NP_{\frac{m}{q}}(F^q) = N(F^{(m,n)}) \text{ } (\neq NP_m(F) \text{ if } m \neq 1).$$

*If $q \neq m$ and $\frac{m(q,n)}{q} = \widehat{p_i^u} \in \mathcal{F}(F)$, then*

$$NP_{\frac{m}{q}}(F^q) = NP_{\frac{m(q,n)}{q}}(F) \quad \text{if} \begin{cases} r > 1 \text{ and } u = s_i, \text{ or} \\ r > 1, 0 < u < s_i \text{ and } p_i \mid \frac{m}{q}, \\ \text{or } r = 1. \end{cases}$$

*In particular, if in addition to any of the above conditions we have that $q \mid n$, then*

$$NP_{\frac{m}{q}}(F^q) = NP_m(F).$$

*Finally if $r > 1$, $u < s_i$ and $p_i \nmid \frac{m}{q}$ then*

$$NP_{\frac{m}{q}}(F^q) = N(F^m) - 1 \neq NP_m(F) = N(F^m) - N(F^{\frac{m}{p_i}}).$$

*Proof.* Let $q$ and $q|m$ be given. From Proposition 3.2 we have $N(F^m) = N(F^{\frac{m}{q}q}) = N(F^{\frac{m(q,n)}{q}})$. Now if $q = m$, then $NP_{\frac{m}{q}}(F^q) = NP_1(F^m) = N(F^m) = N(F^{(m,n)})$, and the first part is shown.

Next suppose that $q \neq m$ and $\frac{m(q,n)}{q} = \widehat{p_i^u} \in \mathcal{F}(F)$. Since $F^q$ has period $n_q := \frac{n}{(q,n)}$ then $n_q \mid n$, and we must have that $n_q = p_1^{t_1} \cdots p_r^{t_r}$ for some $t_i$ with $0 \leq u \leq t_i \leq s_i$ for $i = 1, \cdots r$. By Proposition 5.15 we have that $\frac{m(q,n)}{q} = \widehat{p_i^{s_i}} = \frac{n}{p_i^u} \in \mathcal{F}(F)$ if and only if $\frac{m}{q} = \frac{n_q}{p_i^u} \in \mathcal{F}(F^q)$. Armed with this, we consider the various cases separately.

So firstly then let $r > 1$ and $\frac{m(q,n)}{q} = \widehat{p_i^{s_i}} = \frac{n}{p_i^{s_i}} \in \mathcal{F}(F)$. By Corollary 4.12 we have that $NP_{\frac{m(q,n)}{q}}(F) = N(F^{\frac{m(q,n)}{q}}) - 1 = N(F^m) - 1$. On the other hand from above we have that $\frac{m}{q} = \frac{n_q}{p_i^{s_i}} \in \mathcal{F}(F^q)$. Clearly $t_i = s_i$ for this $i$, so again from Corollary 4.12, we have that $NP_{\frac{m}{q}}(F^q) = N((F^q)^{\frac{m}{q}}) - 1 = N(F^m) - 1$, and the first case follows.

Next let $\frac{m(q,n)}{q} = \widehat{p_i^u} = \frac{n}{p_i^u} \in \mathcal{F}(F)$ with $r > 1$, $u < s_i$ and $p_i \mid \frac{m}{q}$. Again from above we have that $\frac{m}{q} = \frac{n_q}{p_i^u} \in \mathcal{F}(F^q)$. But $p_i \mid \frac{m}{q}$ so $p_i \mid \frac{m(q,n)}{q}$, that is $\widehat{p_i^{u+1}} \in \mathcal{F}(F)$. From Corollary 4.12 again we have that $N(F^m) - N(F^{\frac{m}{p_i}})$. But also $p_i \mid \frac{n_q}{p_i^u} \in \mathcal{F}(F^q)$. In other words $\frac{n_q}{p_i^{u+1}} \in \mathcal{F}(F^q)$ so again $NP_{\frac{m}{q}}(F^q) = N(F^m) - N(F^{\frac{m}{p_i}})$, and the second case is proved.

Finally let $r = 1$ with $\frac{m(q,n)}{q} = \widehat{p_i^u} \in \mathcal{F}(F)$ and of course, $q \neq m$. Clearly neither $\frac{m}{q}$ nor $\frac{m(q,n)}{q}$ is equal to 1, and since $\mathcal{F}(F) = \{1, p_1, \cdots, p_1^{s_1-1}\}$ and $(q,n) \mid n$, we must have that $\frac{m}{q} = p_1^t$ for some $t \geq 1$. In particular $p_1 \mid \frac{m}{q}$ and the proof follows exactly as in the case $r > 1$ with $u < s_i$ and $p_i \mid \frac{m}{q}$.

If, in the previous part we also have that $q|n$, then $(q,n) = q$ and $\frac{m(q,n)}{q} = m$. The next part now follows from the previous one. The last part follows from a careful examination of both sides and Corollary 4.12. ∎

The Corollary below is useful, since many times $\mathcal{F}(A) \subset \mathcal{F}(A) \otimes \mathcal{F}(B)$.

**Corollary 5.27.** *(Used in steps 2 and 3 of Example 5.28) Let $A = \mathcal{C}(\Phi_n)$ with $n = p_1^{s_1} \cdots p_r^{s_r}$ and $r \geq 1$. Let $B$ be arbitrary periodic and $m = \widehat{p_i^u} \in \mathcal{F}(A)$. If $r > 1$ and $u = s_i$, or $r > 1$, $0 < u < s_i$ and $p \mid \widehat{p_i^{u+1}}$, or if $r = 1$ with $u \neq s_1$ then*

$$NP_{\frac{m}{q}}(A^q) = NP_m(A) \text{ for all } q \neq m.$$

*Moreover in these cases if $m \notin \mathcal{F}(B)$ then*

$$NP_m(A \oplus B) = NP_m(A)N(B^m).$$

*Proof.* If $m = \widehat{p_i^u} \in \mathcal{F}(A)$, then any $q|m$ must divide $n$. In these cases $(q,n) = q$ and of course $\frac{m(q,n)}{q} = m$. So for $q \neq m$ the first part can now be read off the formula in Proposition 5.26.

Under these same conditions if $m \notin \mathcal{F}(B)$, then $q = m$ is excluded from $m^{\mathcal{F}}$. It follows from Lemma 5.20 that $m^{\mathcal{F}} := \{q|m \mid q \in \mathcal{F}(B)\}$ for this $m$. Since $NP_q(B) = 0$ if $q \notin \mathcal{F}(B)$ we have that $\sum_{q\in m^{\mathcal{F}}} NP_q(B) = \sum_{q|m} NP_q(B) = N\Phi_m(B) = N(B^m)$, with the last two steps from Theorem 2.3. Using this we have that

$$\begin{aligned} NP_m(A \oplus B) &= \textstyle\sum_{q\in m^{\mathcal{F}}} NP_q(B) NP_{\frac{m}{q}}(A^q) \\ &= NP_m(A) \textstyle\sum_{q\in m^{\mathcal{F}}} NP_q(B) \\ &= NP_m(A) N(B^m). \end{aligned}$$

■

### 5.4 Computing a four stage example

In this final subsection we demonstrate the induction process by working a four stage example. In doing this we will indicate at each step, which matrix plays the role of $A$ (the primitive) in our earlier results, and which plays the role of $B$ the matrix to which we are "adding" $A$.

**Example 5.28. Four stage induction example.** Let $F$ be as given below, we determine $N(F^m)$ for all $m$, and indicate shortcuts to determine $N(F^m)$ and $N\Phi_m(F)$ for all $m$ for this $F$.

$$F = \mathcal{C}(\Phi_{63}) \oplus \mathcal{C}(\Phi_{210})^{[3]} \oplus \mathcal{C}(\Phi_{2520}) \oplus \mathcal{C}(\Phi_{1331})^{[2]}$$

has period $lcm(63, 210, 2520, 1331) = 3,354,120$. For convenience we assign the following names to each of the primitives (or powers thereof):- $F_1 := \mathcal{C}(\Phi_{63})$, $F_2 := \mathcal{C}(\Phi_{210})^{[3]}$, $F_3 := \mathcal{C}(\Phi_{2520})$ and $F_4 := \mathcal{C}(\Phi_{1331})^{[2]}$. The primitive $F_3$ is of course familiar from Example 1.1.

Step 1 $F_1 = \mathcal{C}(\Phi_{63})$. We use the methods of section 4 to compute the table for $F_1$. The "hat" notation here is of course, with respect to the period $n_1 = 63 = 3^2 \cdot 7$, of $F_1$.

| $\mathcal{F}(F_1)$ | $m$ | 1 | 7 | 9 | 21 |
|---|---|---|---|---|---|
| | $m$ | 1 | $\widehat{3^2}$ | $\widehat{7}$ | $\widehat{3}$ |
| | $N(F_1^m)$ | 1 | $3^6$ | $7^6$ | $3^{18}$ |
| | $NP_m(F_1)$ | 1 | $3^6-1$ | $7^6-1$ | $3^{18}-3^6$ |

. (2)

As mentioned earlier, we defer the computation of the $N\Phi_m(F)$ until the end.

Step 2 Adding $F_2 := \mathcal{C}(\Phi_{210})^{[3]}$ to obtain $\oplus_{i=1}^{2} F_i$. Note that the period of $\oplus_{i=1}^{2} F_i$ is $lcm(63, 210) = 630$. Now $\mathcal{F}(F_2) = \mathcal{F}(\mathcal{C}(\Phi_{210}))$ by Proposition 5.8, but note that $N(F_2^m) \neq N((\mathcal{C}(\Phi_{210})^m)$ since $F_2 = \mathcal{C}(\Phi_{210})^{[3]}$ (not $\mathcal{C}(\Phi_{210})$). We use Corollary 5.9 to construct the following table. Here of course the "hat" notation is with respect to $F_2$.

| $\mathcal{F}(F_2)$ | $m$ | 1 | 30 | 42 | 70 | 105 |
|---|---|---|---|---|---|---|
| | $m$ | 1 | $\widehat{7}$ | $\widehat{5}$ | $\widehat{3}$ | $\widehat{2}$ |
| | $N(F^m)$ | 1 | $(7^8)^3$ | $(5^{12})^3$ | $(3^{24})^3$ | $(2^{48})^3$ |
| | $NP_m(F)$ | 1 | $7^{24}-1$ | $5^{36}-1$ | $3^{72}-1$ | $2^{144}-1$ |

. (3)

In this step $F_1$ plays the role of $B$ and $F_2$ the role of $A$ in our earlier results. So we need to find $\mathcal{F}(F_2) \otimes \mathcal{F}(F_1)$ (Definition 5.3), from $\mathcal{F}(F_1) = \{1,7,9,21\}$ and $\mathcal{F}(F_2) = \{1,30,42,70,105\}$. Obviously, since we exclude any lcm that is a zero, we must exclude any multiple of either 63 or 210. In fact $\mathcal{F}(F_2) \otimes \mathcal{F}(F_1) = \mathcal{F}(F_2 \oplus F_1) = \mathcal{F}(F_2) \cup \mathcal{F}(F_1) \cup \{90\}$. The values of $N((\oplus_{i=1}^2 F_i)^m)$ in the table come from the fact that $N((\oplus_{i=1}^2 F_i)^m) = N(F_1^m)N(F_2^m) = N(F_1^{(63,m)})N(F_2^{(210,m)})$ from Proposition 5.11.

| $\mathcal{F}(\oplus_{i=1}^2 F_i)$ | $m$ | 1 | 7 | 9 | 21 | 30 | 42 | 70 | 90 | 105 |
|---|---|---|---|---|---|---|---|---|---|---|
| | $N((\oplus_{i=1}^2 F_i)^m)$ | 1 | $3^6$ | $7^6$ | $3^{18}$ | $7^{24}$ | $3^{18}5^{36}$ | $3^{78}$ | $7^{30}$ | $2^{144}3^{18}$ |

.

To find the $NP_m(\oplus_{i=1}^2 F_i)$ note that for each $m$ in $\mathcal{F}(F_1) = \{1,7,9,21\} \subset \mathcal{F}(\oplus_{i=1}^2 F_i)$ that $m < \widehat{p_i^u}$ for all $\widehat{p_i^u} \in \mathcal{F}(F_2) - \{1\}$, so $NP_m(\oplus_{i=1}^2 F_i) = NP_m(F_1)N(F_2) = NP_m(F_1)$ for each such $m$, by Proposition 5.23. This gives table (4) below (we show later that $NP_m(\oplus_{i=1}^2 F_i) = NP_m(\oplus_{i=1}^3 F_i)$ for these $m$).

| $m$ | 1 | 7 | 9 | 21 |
|---|---|---|---|---|
| $NP_m(\oplus_{i=1}^2 F_i) = NP_m(\oplus_{i=1}^3 F_i)$ | 1 | $3^6-1$ | $7^6-1$ | $3^{18}-3^6$ |

. (4)

Table (5) below continues table (4) (including the claim that $NP_m(\oplus_{i=1}^2 F_i) = NP_m(\oplus_{i=1}^3 F_i)$).

| 30 | 42 | 70 | 90 | 105 |
|---|---|---|---|---|
| $7^{24}-1$ | $(5^{36}-1)3^{18}$ | $(3^{72}-1)3^6$ | $7^{30}-7^{24}-7^6+1$ | $(2^{144}-1)3^{18}$ |

. (5)

To see this note firstly that for $m = 30, 42, 70$, and 105 we have that $m = \widehat{p_1^{s_i}} \in \mathcal{F}(F)$ and $NP_m(\oplus_{i=1}^2 F_i) = NP_m(F_2)N(F_1^m)$ by Corollary 5.27. The values of the $N(F_1^m)$ for these $m$, come from Corollary 4.8. So $NP_{30}(\oplus_{i=1}^2 F_i) = NP_{30}(F_2) \cdot 1 = 7^{24} - 1$ while $NP_{42}(\oplus_{i=1}^2 F_i) = NP_{42}(F_2)N(F_1^{21}) = (5^{36}-1)3^{18}$. Next $NP_{70}(F_2 \oplus F_1) = NP_{70}(F_2)N(F_1^7) = (3^{72}-1)3^6$, and $NP_{105}(F_2 \oplus F_1) = NP_{105}(F_2)N(F_1^{21}) = (2^{144}-1)3^{18}$. Lastly for $m = 90$, since $N((F_2 \oplus F_1)^{90}) \neq 0$, and $90!\mathcal{F}(F_2 \oplus F_1) - \{90\} = \{1,9,30\}$, then from Theorem 3.7 we have that $NP_{90}(F_2 \oplus F_1) = 7^{30} - 7^{24} + 1 - 7^6 + 1 - 1 = 7^{30} - 7^{24} - 7^6 + 1$.

The justification that $\mathcal{F}(F_2) \otimes \mathcal{F}(F_1) = \mathcal{F}(F_2 \oplus F_1)$ comes from Proposition 3.4 because $N((F_2 \oplus F_1)^q) \neq N((F_2 \oplus F_1)^{90}) = 7^{30}$ for any $q < 90$.

Step 3 Adding $F_3 := \mathcal{C}(\Phi_{2520})$ (Example 1.1) to obtain $\oplus_{i=1}^3 F_i$. Note that the period of $\oplus_{i=1}^3 F_i$ is $lcm(630, 2520) = 2,520$. In this step $\oplus_{i=1}^2 F_i$ plays the role of $B$ and $F_3$ the role of $A$. So we need to find $\mathcal{F}(F_1 \oplus F_2) \otimes \mathcal{F}(F_3)$ from $\mathcal{F}(\oplus_{i=1}^2 F_i)$ $= \{1, 7, 9, 21, 30, 42, 70, 90, 105\}$ and $\mathcal{F}(F_3) = \{1, 280, 315, 360, 504, 630, 840, 1260\}$. Obviously we still exclude any lcm that is a multiple of either 63 or 210 since they are zeros. This automatically excludes $315, 504, 630, 840$ and $1260$ from $\mathcal{F}(F_1 \oplus F_2) \otimes \mathcal{F}(F_3)$. In fact we can compute this as $\{280, 360\} \otimes \{1, 7, 9, 21, 30, 42, 70, 90, 105\}$. So

$$\mathcal{F}(F_1 \oplus F_2) \otimes \mathcal{F}(F_3) = \{1, 7, 9, 21, 30, 42, 70, 90, 105, 280, 360\} = \mathcal{F}(\oplus_{i=1}^3 F_i).$$

We confirm first the equation $NP_m(\oplus_{i=1}^3 F_i) = NP_m(\oplus_{i=1}^2 F_i)$ claimed for each $m$ in tables (4) and (5). To see this note, for each $m \in \{7, 9, 21, 30, 42, 70, 90, 105\}$, that $m|2520$, the period of $F_3$, and that $m \in \mathcal{F}(\oplus_{i=1}^2 F_i)$, but $m \notin \mathcal{F}(F_3)$. So then $NP_m(\oplus_{i=1}^3 F_i) = NP_m(\oplus_{i=1}^2 F_i) N(F_3)$ by Corollary 5.21. But $N(F_3) = 1$, and since the formula also holds for $m = 1$ trivially, the claim is established.

It remains to compute $NP_m(F_3)$ and $NP_m(\oplus_{i=1}^3 F_i)$ for $m \in \{280, 360\}$. We combine the results in the table below with the explanation to follow.

| $m$ | 1 | 280 | 360 |
|---|---|---|---|
| $N(F_3^m)$ | 1 | $3^{96}$ | $7^{96}$ |
| $NP_m(F_3)$ | 1 | $3^{96} - 1$ | $7^{96} - 1$ |
| $NP_m(\oplus_{i=1}^3 F_i)$ | 1 | $(3^{96} - 1)3^{84}$ | $(7^{96} - 1)7^{30}$ |

. (6)

The computations for $N(F_3^m)$ and $NP_m(F_3)$ are familiar from table (1) in Example 1.1 in the introduction. Note for $m = 280$, and 360 we have that $m \in \mathcal{F}(F_3)$, but $m \notin \mathcal{F}(\oplus_{i=1}^2 F_i)$. So for $m = 280, 360$ we have, from Corollary 5.27, that $NP_m(\oplus_{i=1}^3 F_i)) = NP_m(F_3) N((\oplus_{i=1}^2 F_i)^m)$. From Proposition 5.11 we have that $N((\oplus_{i=1}^2 F_i)^{280}) = N(F_1^7)(N(F_2^{70}) = 3^6 3^{78} = 3^{84}$ and similarly $N((\oplus_{i=1}^2 F_i)^{360}) = 7^{30}$. Thus $NP_{280}(\oplus_{i=1}^3 F_i)$ and $NP_{360}(\oplus_{i=1}^3 F_i)$ are as shown in table (6).

Step 4 Adding $F_4 := \mathcal{C}(\Phi_{1331})^{[2]}$ to obtain $F = \oplus_{i=1}^4 F_i$ (= $F$). In this step $\oplus_{i=1}^3 F_i$ plays the role of $B$ and $F_4$ the role of $A$. Now the table for $F_4$ is

| $\mathcal{F}(F_4)$ | $m$ | 1 | 11 | 121 |
|---|---|---|---|---|
| | $N(F_4^m))$ | $(11)^2$ | $(11^{11})^2$ | $(11^{121})^2$ |
| | $NP_m(F_4)$ | $11^2$ | $11^{22} - 11^2$ | $11^{242} - 11^{22}$ |

. (7)

The period of $\oplus_{i=1}^4 F_i$ is $lcm(2520, 1331) = 3,354,120$, and since $(2520, 1331) = 1$, then by Proposition 5.24 we have that $\mathcal{F}(F_4) \otimes \mathcal{F}(\oplus_{i=1}^3 F_i) = \mathcal{F}(F_4)\mathcal{F}(\oplus_{i=1}^3 F_i)$ with elements which we write as $11^u m$ for $u = 0, 1, 2$ and $m \in \mathcal{F}(\oplus_{i=1}^3 F_i)$. Using this formulation, we record the 33 elements of the set $\mathcal{F}(F_4)\mathcal{F}(\oplus_{i=1}^3 F_i)$ in the table below.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $11^0 m$ | 1 | 7 | 9 | 21 | 30 | 42 | 70 | 90 | 105 | 280 | 360 |
| $11^1 m$ | 11 | 77 | 99 | 231 | 330 | 462 | 770 | 990 | 1155 | 3080 | 3960 |
| $11^2 m$ | 121 | 847 | 1099 | 2541 | 3630 | 5082 | 8470 | 10890 | 12705 | 33880 | 43560 |

.

By Proposition 5.24 we have that $NP_{11^u m}(\oplus_{i=1}^4 F_i) = NP_m(\oplus_{i=1}^3 F_i) NP_{11^u}(F_4)$. Thus for each $m \in \mathcal{F}(\oplus_{i=1}^3 F_i)$ we have from table (7) that

$$\begin{array}{lcl} NP_m(\oplus_{i=1}^4 F_i) & = & 11^2 NP_m(\oplus_{i=1}^3 F_i) \\ NP_{11m}(\oplus_{i=1}^4 F_i) & = & (11^{22} - 11^2) NP_m(\oplus_{i=1}^3 F_i) \\ NP_{121m}(\oplus_{i=1}^4 F_i) & = & (11^{242} - 11^{22}) NP_m(\oplus_{i=1}^3 F_i). \end{array} \tag{8}$$

We now construct the table for the $NP_m(F)$. We do this using this and tables (4), (5), (6), (7) and (8). The constructed table also allows us, by Proposition 5.6, to deduce that $\mathcal{F}(F_4) \otimes \mathcal{F}(\oplus_{i=1}^3 F_i) = \mathcal{F}(\oplus_{i=1}^4 F_i) = \mathcal{F}(F_4) \otimes \mathcal{F}(\oplus_{i=1}^3 F_i) = \mathcal{F}(F)$.

| $m$ | $NP_m(F)$ | $NP_{11m}(F)$ | $NP_{121m}(F)$ |
|---|---|---|---|
| 1 | $11^2$ | $11^{22} - 11^2$ | $11^{242} - 11^{22}$ |
| 7 | $11^2(3^6 - 1)$ | $(11^{22} - 11^2)(3^6 - 1)$ | $(11^{242} - 11^{22})(3^6 - 1)$ |
| 9 | $11^2(7^6 - 1)$ | $(11^{22} - 11^2)(7^6 - 1)$ | $(11^{242} - 11^{22})(7^6 - 1)$ |
| 21 | $11^2(3^{18} - 3^6)$ | $(11^{22} - 11^2)(3^{18} - 3^6)$ | $(11^{242} - 11^{22})(3^{18} - 3^6)$ |
| 30 | $11^2(7^{24} - 1)$ | $(11^{22} - 11^2)(7^{24} - 1)$ | $(11^{242} - 11^{22})(7^{24} - 1)$ |
| 42 | $11^2(5^{36} - 1)3^{18}$ | $(11^{22} - 11^2)(5^{36} - 1)3^{18}$ | $(11^{242} - 11^{22})(5^{36} - 1)3^{18}$ |
| 70 | $11^2(3^{72} - 1)3^6$ | $(11^{22} - 11^2)(3^{72} - 1)3^6$ | $(11^{242} - 11^{22})(3^{72} - 1)3^6$ |
| 90 | $11^2(7^{30} - 7^{24} - 7^6 + 1)$ | $(11^{22} - 11^2)(7^{30} - 7^{24} - 7^6 + 1)$ | $(11^{242} - 11^{22})(7^{30} - 7^{24} - 7^6 + 1)$ |
| 105 | $11^2(2^{144} - 1)3^{18}$ | $(11^{22} - 11^2)(2^{144} - 1)3^{18}$ | $(11^{242} - 11^{22})(2^{144} - 1)3^{18}$ |
| 280 | $11^2(3^{96} - 1)3^{84}$ | $(11^{22} - 11^2)(3^{96} - 1)3^{84}$ | $(11^{242} - 11^{22})(3^{96} - 1)3^{84}$ |
| 360 | $11^2(7^{96} - 1)7^{30}$ | $(11^{22} - 11^2)(7^{96} - 1)7^{30}$ | $(11^{242} - 11^{22})(7^{96} - 1)7^{30}$ |

**The numbers $N(F^m)$ and $N\Phi_m(F)$ for all $m$.**

So the big table gives the values of $NP_{11^u m}(F)$ for each $11^u m \in \mathcal{F}(F)$. And of course $NP_q(F) = 0$ if $q \notin \mathcal{F}(F)$. As mentioned in the introduction it is convenient to wait until the end of the induction process to do the complete identification of the $N(F^m)$ and $N\Phi_m(F)$ for all $m$.

We start with the $N(F^m)$. Perhaps the best way to proceed is to determine first if $N(F^m) = 0$, and this occurs exactly when $m$ is a multiple of 63, 210, 2520 or 1331. For the rest, the reader will have noticed that we have not given the table for the $N(F^m)$ for $m \in \mathcal{F}(F)$. This is because these values are already present, slightly hidden, in the big table for the $NP_m(F)$. The point is that for all such $m$ we have from Theorem 3.7 that $NP_m(F) = N(F^m) - \sum_{q \in m! \mathcal{F}(F) - \{m\}} NP_q(F)$, and $N(F^m)$ can be identified as the largest term in this expression (see Proposition 2.4). So for example $N(F^{90}) = 11^2 7^{30}$, and $N(F^{3080}) = 11^{22} 3^{177}$ since these are respectively the largest terms in the expressions $11^2(7^{30} - 7^{24} - 7^6 + 1)$ and $(11^{22} - 11^2)(3^{96} - 1)3^{84}$. Since the big table is the table of all firsts, then it contains all of the values of the $N(F^m)$ in this way. The task then for any $m$, when $N(F^m) \neq 0$, is to identify which of the $q \in \mathcal{F}(F)$ has $N(F^m) = N(F^q)$.

We have several tools. The first uses the equation $N(F^m) = N(F^{(m,n)})$ from Proposition 3.2. So for example $N(F^{1400}) = N(F^{(1400,3354120)}) = N(F^{280}) = 11^2 3^{177}$ since $280 \in \mathcal{F}(F)$. On the other hand $(140, 3354120) = 140 \notin \mathcal{F}(F)$, so we need something else. Here we can use the equation $N(F^m) = \sum_{q \in m!\mathcal{F}(F)} NP_q(F)$ in both directions (Theorem 3.7). So note, since $140 = 2^2 \cdot 5 \cdot 7$, that $140!\mathcal{F}(F) = \{1, 7, 70\} = 70!\mathcal{F}(F)$. So $N(F^{140}) = \sum_{q \in 140!\mathcal{F}(F)} NP_q(F) = \sum_{q \in 70!\mathcal{F}(F)} NP_q(F) = N(F^{70}) = 11^2 3^{72}$ (the last equality again from the table). Finally our last tool iterating Proposition 5.11 gives that $N(F^m) = N(F_1^{(m,63)}) N(F_2^{(m,210)}) N(F_3^{(m,2520)})$ $N(F_4^{(m,1331)})$ for any $m$. In particular $N(F^{1980}) = N(F_1^{9}) N(F_2^{30}) N(F_3^{180}) N(F_4^{11}) = 7^6 7^{24} \cdot 1 \cdot 11^{22} = 7^{30} 11^{22}$. For $N(F_3^{180})$ we needed to go back to table (1) in Example 1.1 and observe that $180 = (180, 2520) \notin \mathcal{F}(F_3)$ ($F_3$ is simply $F$ in that example), so $N(F_3^{180}) = 1$ by Corollary 4.8.

For the $N\Phi_m(F)$, when $N(F^m) \neq 0$, we can use the equation $N\Phi_m(F) = N(F^m)$ from Theorem 2.3. So in particular we have that $N\Phi_{1400}(F) = 11^2 3^{177}$, $N\Phi_{140}(F) = 11^2 3^{72}$ and $N\Phi_{1980}(F) = 7^{30} 11^{22}$.

When $N(F^m) = 0$, then we need to use the formula $N\Phi_m(F) = \sum_{q \in m!\mathcal{F}(F)} NP_q(F)$ from Theorem 3.7, but we can also combine this with the equation $N\Phi_m(F) = N\Phi_{(m,n)}(F)$, from Corollary 3.9.

We look at $72,072$. Now $(72072, 3354120) = 5544$ which is divisible by 63 so $N(F^{72072}) = N(F^{5544}) = 0$. Also $5544 = 2^3 3^2 \cdot 7 \cdot 11$ and we compute $5544!\mathcal{F}(F) = \{1, 7, 9, 21, 42, 11, 77, 99, 231, 462\} = \{1, 7, 21, 42, 11, 77, 231, 462\} \cup \{9, 99\} = 462!\mathcal{F}(F) \cup \{9, 99\}$. So $N\Phi_{72072}(F) = N\Phi_{5544}(F) = \sum_{q \in 462!\mathcal{F}(F)} NP_q(F) + NP_9(F) + NP_{99}(F) = N(F^{462}) + NP_9(F) + NP_{99}(F)$ By Theorem 3.7.

So $N\Phi_{72072}(F) = 11^{22} 5^{36} 3^{18} + 11^{22}(7^6 - 1) = 11^{22}(5^{36} 3^{18} + 7^6 - 1)$.

We can also do this in a slightly different way and I include both ways as in some circumstances one way may be more efficient than the other. The different way involves writing $5544!\mathcal{F}(F)$ as $\{1, 7, 9, 21, 42\} \cup 11\{1, 7, 9, 21, 42\}$. Now writing elements of $5544!\mathcal{F}(F)$ as $11^u m$ in the obvious way, we have from (8) and Theorem 3.7 that $N\Phi_{5544}(F)$

$$
\begin{aligned}
&= \textstyle\sum_{m \in \{1,7,9,21,42\}} 11^2 NP_m(\oplus_{i=1}^3 F_i) + \sum_{m \in \{1,7,9,21,42\}} (11^{22} - 11^2) NP_m(\oplus_{i=1}^3 F_i) \\
&= 11^{22} \textstyle\sum_{m \in \{1,7,9,21,42\}} NP_m(\oplus_{i=1}^3 F_i) \\
&= 11^{22}(1 + 3^6 - 1 + 7^6 - 1 + 3^{18} - 3^6 + (5^{36} - 1) 3^{18}) \text{ from tables (4) and (5)} \\
&= 11^{22}(5^{36} 3^{18} + 7^6 - 1).
\end{aligned}
$$

We leave it to the reader to deduce similarly that $N\Phi_{60984}(F) = 11^{242}(5^{36} 3^{18} + 7^6 - 1)$.

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