Fixed point index bounds for self-maps on closed surfaces

D.L. Gonçalves M.R. Kelly

Abstract

Given a surface with non-positive Euler characteristic and non-empty boundary, and a map which has the least number of fixed points possible within its homotopy class there are known bounds (both upper and lower) regarding the fixed point indices of the map. This paper gives a new proof of this result. In addition, a relative version of the method is developed, which is then used to establish the same index bounds for the case of a closed surface of negative Euler characteristic.

0 Introduction

In the classical Poincaré-Birkhoff theory, obtaining local fixed point index data has proved to be a useful tool in understanding global dynamical behavior (see for example [2] or [13]). As a result a problem that has attracted interest in both fixed point theory and in dynamical systems is the study of index behavior of maps in dimension 2. Many results on fixed point indices for surfaces mappings have appeared in the literature, including the papers [1], [4], [7], [8], [9], [10], [11], [14], [15], [16], [17] and [18]. In particular, the papers [7], [8], [9], [10] and [11] present a number of results which establish global bounds for the indices of fixed points, and also for Nielsen classes. The purpose of this paper is to extend some of these results. We present a variation of the method of proof used in [10], and

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then introduce a relative version which allows us to extend the results in that paper to include the closed surfaces with negative Euler characteristic.

Let X be a compact topological space. In particular, a polyhedron, CW-complex, or ANR so that the space admits a fixed point index. Let K denote a family of self-maps on X, together with a partition of the fixed points of each member. We say that K has the *index bounds property*, denoted by IBK, if there exist integers ℓ and u such that $\ell \leq \text{index}(f, P) \leq u$ holds for all maps $f \in K$ and each P a member of the partition on $\text{Fix}(f)$.

In this paper we will be interested in two very natural families when considering index bounds for a particular space X:

- The class of maps consists of all *fixed point minimal* maps. Those maps having the least number of fixed points possible among all maps in a given homotopy class. The partition classes are single fixed points. We denote by $MF[f]$ this minimal number for the homotopy class of f, and write IBMF for the bounds for this class.
- The class consists of all continuous maps and the Nielsen fixed point classes provide a partition. Given a map f a pair of fixed points are Nielsen related if there exists a path α joining the two points with $f(\alpha)$ homotopic to α rel endpoints. We write IBN for the bounds for these Nielsen classes. See the references [3], [6] for more information regarding Nielsen fixed point classes.

It is a result due to B. Jiang [5] that, for many polyhedra, property IBN is the same as property IBMF, simply because each homotopy class of maps has a representative where each essential Nielsen class is a single point. The exceptions to this occur either when X has local separating points or when X is a hyperbolic surface (i.e. a surface with negative Euler characteristic). In the case of a hyperbolic surface index bounds were studied in [7], [10],[11] and, in particular, the following global bounds were shown to be relevant:

$$\sum_{\text{index}(x)>1} (\text{index}(x) - 1) \leq 0\,, \text{ and } \sum_{\text{index}(x)<-1} (\text{index}(x) + 1) \geq 2\chi(F)\,.$$

In the papers cited, these bounds were shown to hold in two settings; (i) when x represents a Nielsen class in the case of all such surfaces, and (ii) for surfaces with non-empty boundary when x is an isolated fixed point of a fixed point minimal map. Note that the first inequality just says that $u = 1$ gives an upper bound for the index, while the second bound gives a global lower bound on all fixed points (or Nielsen classes) having index less than -1. In general, there is no reason to expect a global bound on the number of fixed points of index ± 1.

Motivated by these results we say that X has the *hyperbolic index bounds property*, denoted HIBK, for the class K if it satisfies the above inequalities for the given class K. (The H is in reference to the fact that the bounds hold for hyperbolic surfaces.) With this notation, HIBN holds for all hyperbolic surfaces, and HIBMF holds for hyperbolic surfaces with non-empty boundary. We remark that these bounds hold trivially for the two closed surfaces with zero Euler characteristic; the torus and the Klein bottle. Also, when the space X is a finite graph it is a routine exercise to verify the HIBK property where K consists of all self-maps which have a finite fixed point set. In particular, the HIBMF holds for graphs. It is proved in [7] that the HIBN property is also satisfied for graphs.

The structure of the remainder of the paper is as follows. In Section 1 we present a variation on the proof given in [10], where a large class K of maps is identified and shown to satisfy HIBK. This is stated in Proposition 1. It also follows from this variation that the HIBMF property holds for surfaces with non-empty boundary. Taking advantage of this variation, in Section 2 we introduce a relative version of the method. In particular, relative to a single disk we establish the HIBMF property for the closed surfaces in Theorem 1.

1 A family of maps that satisfy hyperbolic index bounds

In this section we consider surfaces with non-empty boundary and show that a large class of surface maps satisfy hyperbolic index bounds when fixed points are grouped in a careful way. The result, given in Proposition 1 below, has as a corollary the HIBMF for surfaces with non-empty boundary. The proof given here is much more direct than the original proof given in [10].

Let F be a compact, connected surface with non-empty boundary ∂F and non-positive Euler characteristic. Exactly as in [10] we fix a handle structure consisting of one 0-handle, which is a disk denoted by D, and $1-\chi(F)$ 1-handles. These are disks that are glued to D along a pair of disjoint arcs, called attaching arcs. Let A denote the union of all the attaching arcs for the 1-handles, a total of $2(1-\chi(F))$ pairwise disjoint arcs on the boundary of D.

In order to define our family of maps we first give some terminology. We recall that a *proper 1-manifold in F* is a submanifold embedded in $(F, \partial F)$ by a proper map. A self-map $f : F \to F$ is said to be *A-transverse* if it has a finite fixed point set which is disjoint from A, and with the property that the preimage of A is a proper 1-manifold C meeting A in a finite set of points not in ∂A and the image of any open set in F meeting C meets at least two components of $F \setminus A$. Given an A-transverse map f, a component R of $F \setminus (A \cup f^{-1}(A))$ is called a *region*. If R and $f(R)$ are contained in the same component of $F \setminus A$ the region is *critical*. Let $\bar{R}$ denote the closure of the region R. By a *segment* we mean a component of $f^{-1}(A) \cap \bar{R}$. The *border* of R is $\bar{R} \setminus R$ and consists of segments, as defined just above, and subarcs from $A \cup \partial F$.

Lemma 1. *Given a fixed point minimal map f there is an A-transverse map homotopic to f with the same number of fixed points and the same fixed point indices.*

Proof: Let f be a fixed point minimal map, and without loss A is chosen so that it does not contain any fixed points. By general position we arrange that $f^{-1}(A)$ is a 1-dimensional complex without changing the fixed point set, and further, that the transversality condition given in the definition holds. Consider a non-manifold point q of $f^{-1}(A)$ and a regular neighborhood N. Then $N \cap f^{-1}(A)$ consists of $2k$ arcs, $k \geq 2$, meeting q, and as one traverses the boundary of N

the $2k$ complimentary domains map to opposite sides of the component of A containing $f(q)$. Apply a small deformation of f near q to produce a new map (by abuse of notation we call f) where now $N \cap f^{-1}(A)$ consists of k arcs with no non-manifold points. Apply to each such q to obtain the desired A-transverse map. ∎

We define a graph G_f associated to f whose vertices are the critical regions, and two vertices are joined by an edge if and only if there is a point $p \in A$ common to both boundaries such that p and $f(p)$ lie in the same component A_0 of A. Note that the two vertices lie on opposite sides of A_0 and the edge simply crosses A_0. So the graph may be embedded in F. Since f has no fixed points on A, critical regions have a well-defined fixed point index. The index of a component of G_f is the sum of the indices of its vertices. The utility of considering the index of components is stated in the following lemma, which is taken from [10, Prop 2.1]. The method of proof is illustrated in Example 2 below.

Lemma 2. *Let $f : F \to F$ be an A-transverse map and let M be the number of components of G_f that have a non-zero index. Then there is an A-transverse map g homotopic to f with G_g isomorphic to G_f and g has exactly M fixed points. In addition, the graph isomorphism preserves indices of components.*

Given a homotopy class of maps γ let $\mathcal{I}$ denote the collection of A-transverse maps f in γ such that G_f has exactly MF[γ] components with non-zero index. We now have a class different from N and MF. The maps are those in $\mathcal{I}$ and the fixed points are placed into collections determined by the components of the graph constructed above. We denote this class by CRG.

Proposition 1. *For any compact, connected surface with non-empty boundary and non-positive Euler characteristic, and for any $f \in \mathcal{I}$ the HIBCRG hold. That is*

$$\sum_{\text{index}(C) \geq 1} (\text{index}(C) - 1) \ \leq \ 0 \tag{1}$$

and

$$\sum_{\text{index}(C) \leq -1} (\text{index}(C) + 1) \ \geq \ 2\chi(F)\,, \tag{2}$$

where C denotes a component of G_f.

Immediate from the above lemmas and Proposition 1 we have the following corollary. This result was implicitly proved in [10], but Equation (2) was only stated for a single fixed point, as opposed to a global result stated in the HIB.

Corollary 1. *The HIBMF property holds for all compact surfaces with non-empty boundary.*

Remark: The proof of Proposition 1, given in this section is much more direct than the proof found in [10]. The argument in [10] was motivated by the second authors earlier work in attempting to understand the structure and dynamics of fixed point minimal maps. This proof shows that most of that structure is not needed to obtain results about fixed point indices for fixed point minimal maps.

Also, the argument here clearly highlights (see Lemma 5) the need for MF in defining CRG above.

The following terminology will be used in the proof of Proposition 1. Let ∂A denote the set consisting of the endpoints of the arcs in A. Consider a handle H (either a $0-$ or $1-$handle), which is bordered by the simple closed curve ∂H. Let η be a segment that lies in H and let U_1, U_2 be the two components of ∂H with the endpoints of η removed. Define the *length of* η to be the cardinality of the smaller of the sets $U_1 \cap \partial A$ and $U_2 \cap \partial A$.

A region is *large* if either it has a segment on its border with length at least 2 or the closure of the region meets all components of $(\partial H \setminus A)$, where H is the handle containing the region. Otherwise, the region is *small*. Two large critical regions are said to be *adjacent* if they meet a common component A_0 of A, and are in the same component of G_f, joined by a sequence of edges and vertices where each vertex corresponds to a small critical region, each meeting A_0. We will also use the notion of large and small segments, which is more subtle. Note that a segment with length one meets exactly one component of A. The segment η is *large* if either its length is greater than 1 or its length is exactly 1 and the following holds: $f(\eta)$ lies in the same component of A as the one that meets η, and η is on the border of a large critical region that is adjacent to another large critical region. A segment is *small* if it is not large.

The following lemma is a variation on [12, Lemma 5]. This will be used for computing fixed point indices for critical regions.

Lemma 3. *Let f be A-transverse and let R be a critical region for f. Let $\alpha_1, \ldots, \alpha_\ell$ denote the segments on the boundary of R. Let X denote the closure of the component of $F \setminus A$ for which $R \subset X$. Suppose that for each i, $f(\alpha_i)$ is a single point. Set $x_i = 0$ if $f(\alpha_i)$ and $R \setminus \alpha_i$ lie in the same component of $X \setminus \alpha_i$, otherwise set $x_i = 1$. Then*

$$\operatorname{index}(R) = 1 - \sum_{i=1}^{\ell} x_i \, .$$

Proof. We first observe that if R is not a disk, then the border of R will contain the α_i together with a finite number of simple closed curves that are mapped into A. But each of these simple closed curves contributes 0 to the value of the fixed point index. So we may assume that R is a disk.

Consider a tree T in X which has one vertex v_0 in the interior of R, an edge joining v_0 to a vertex on each α_i, and edge in each component of $X \setminus R$ ending at vertex at $f(\alpha_i)$ in the case $x_i = 1$.

By a homotopy relative to $\cup\alpha_i$ we deform f restricted to R to a map τ which has image in T. Furthermore, the only fixed point of τ is the vertex v_0. Thus, $\operatorname{index}(R)$ is the same as the fixed point index of the graph map $\tau : (T \cap \bar{R}) \to T$ at v_0. Consider a valence one vertex v of $T \cap \bar{R}$, which is a point on α_j. Then by construction τ is expanding on the edge $v_o v$ if and only if $x_j = 1$. The result now follows from the formula for the fixed point index of graph maps. ∎

Remark: The assumption that $f(\alpha_i)$ is a single point is restrictive. Certainly it rules out any index larger than one. In general, up to index invariance, one can arrange that the image is either a point or an arc that contains the endpoints of α_i. The later contribute to positive index. See Example 2.

The following examples illustrates some of the terms given above, and index calculations using Lemma 3 and the remark above.

Example 1: Figure 1 illustrates a component A_p of A and four segments $\alpha_1, \ldots, \alpha_4$ with $\alpha_2 \cap \alpha_3 = p$, meeting critical regions R_1, R_2. Suppose that $f(p)$ is on A_p located "above" the point p. Then in the index calculation of R_1 using Lemma 3, the segment α_2 contributes an $x_i = 0$, while segment α_3 contributes an $x_i = 1$ to the index of R_2. Assuming no other segments, and that both α_1, α_4 map downward relative to the critical region, then segment α_1 contributes an $x_i = 0$ and segment α_4 contributes an $x_i = 1$ to index calculations, and so index$(R_1) = 1$ and index$(R_2) = -1$.

If the map f is deformed as in the adjustment in the proof of Proposition 1 so that $f'(p)$ is now located below p on A_p, then the index of R_1 decreases by 1 and R_2 increases by 1. There is no change to the index of the components of G_f.

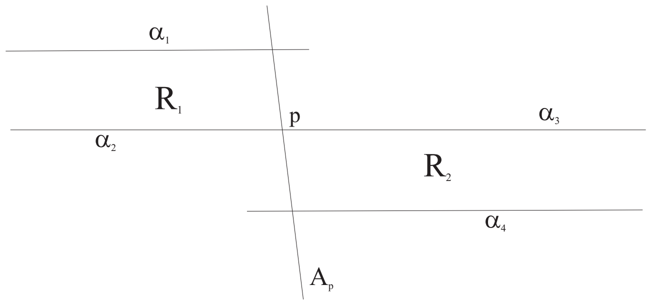

Figure 1: Fixed point index in the CRG

Example 2: We continue with Example 1, giving an example that illustrates the result given in Lemma 2.

Suppose now that other segments meet R_1 and R_2 so that index$(R_1) = -1$ and index$(R_2) = -2$. Deforming f to f' as in Example 1 simply interchanges the values of the indices. With the goal of combining these two values we deform f so that $\alpha_2 \cup \alpha_3$ appears as in Figure 2. Assuming the three intersection points alternate direction the small segment bordering R_2 contributes -1 to its index using Lemma 3. As the index of $R_1 \cup R_2$ is independent of the choice of directions for the intersection points, it follows that under our assumption we now have index$(R_1) = -1 + 1$ and index$(R_2) = -2 - 1$. As a result, all of the index is concentrated in R_2.

The argument used in this example gives the main idea behind proof of Lemma 2. Given large segments α_2, α_3 meeting component A_p, and mapping by f to A_p we can add a suitable number of small segments and alternate images as above so that by Lemma 3 the index of one of the regions is changed by a value $-k$, for some positive k. As only two critical regions are impacted, it follows that the index of the second region changes by $+k$. Hence, after making suitable choices one of the two indices can be changed to zero.

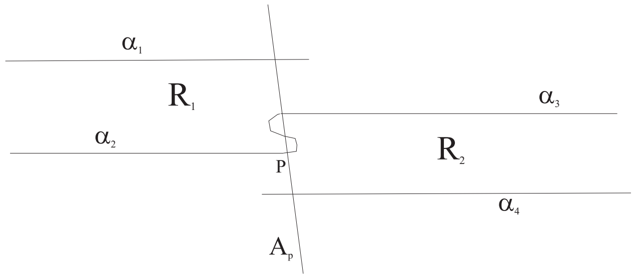

Figure 2: concentration of fixed point index

Proof of Proposition 1. Let $f \in \mathcal{I}$ and let G_f be as defined above.

In order to verify the index bounds we will first need to make some adjustments to a given A-transverse map.

The first adjustment that we make extends that used in Example 2 as follows. Suppose that $p \in A$ is such that p and $f(p)$ lie on the same component of A. If we deform f with support on a prescribed neighborhood of A so that $f^{-1}(A)$ remains unchanged as a set, and also all critical regions remain unchanged, then we get a new map f' which is in $\mathcal{I}$ as long as $f'(p) \neq p$. Moreover, $G(f') = G(f)$ and the index of each component is unchanged during the homotopy. As an illustration, we arrange that a small segment β which meets a point p as above, is mapped to a single point on $A \setminus I_\beta$ by choosing a neighborhood which contains β. More generally, consider a connected finite union of small segments as above. Assuming that there is a point q on the component of A that is not contained in the union of the I_β we adjust so that the entire union of the small segments is mapped to q.

The second adjustment used is the notion of *coalescing* of segments. For this consider a pair of points p, q in $A \cap f^{-1}(A)$ that are adjacent along A_i and both are mapped by f to A_j. Let β_p, β_q be segments ending respectively at p, q, contained in the same component of $F \setminus A$, and where we assume these segments are not equal. This determines three regions; R_0 between the segments, and R_1, R_2 outside. Let δ be an arc in R_0 parallel to the subarc of A_i joining p, q with endpoints in β_p, β_q.

For our application we will also require that if R_0 is critical and $i = j$, then in G_f the vertex R_0 only joins across A_i a vertex of valence 1 corresponding to a small region.

Now apply a homotopy with support on a small neighborhood of δ which has the effect of joining the two segments. That is, viewing δ as vertical and β_p, β_q as horizontal, remove small arcs from each of β_p, β_q and replace with vertical arcs parallel to δ, and contained in R_0.

A variation on the coalescing move is to have δ as above, but now joining a segment to the boundary of F.

Let g denote the end of the coalescing homotopy. We now consider the difference between the graphs G_f and G_g. When R_0 is critical, then the assumption made above will imply that G_f and G_g differ by at most a single component which corresponds to small critical regions. By the adjustment above this component will have index zero. If R_0 is not critical, then this move will join R_1 and R_2 into a single region R'. If these regions are critical, then the index of R' is the sum of the two indices, and moreover, the index of the component of G_g containing R' is the sum of the indices of the components containing R_1 and R_2. Hence, by Lemma 2 if f is fixed point minimal, then one of these two components must have zero index.

Thus, the coalescing move may change the graph, but assuming fixed point minimal the number of fixed points and their indices are unchanged.

Two applications of these adjustments are the following lemmas. The first gives control on the location of fixed point index, while the second will be used to obtain the bounds on the index in the components of G_f.

Lemma 4. *Given a fixed point minimal A-transverse map there is another A-transverse map f in its homotopy class, which has the same fixed point data and such that each small critical region has a unique segment on its border that contributes $x_i = 1$ to its index calculation. Consequently, small critical regions have index equal to zero.*

Proof. Given a small segment bordering a small critical region whose endpoints map to the component A_0 of A containing these points, consider the arc η that is a maximal connected union of small segments containing the given segment. Let I_η denote the union of the I_β corresponding to η. If I_η is not all of A_0, then apply the adjustment given above to obtain the conclusion for the small regions corresponding to η.

Now suppose that I_η is all of A_0. This occurs when η is a component of $f^{-1}(A)$, and moreover η meets no other components of A. Hence, η is isotopic to A_0 as proper arcs in F. In this case apply the coalescing move to replace η with a proper arc that is disjoint from A together with a number of simple closed curves consisting of small segments. ∎

Lemma 5. *Let g be a fixed point minimal A-transverse map. Then g is homotopic to an A-transverse map f that has the same fixed point data as g such that associated to each pair of adjacent large critical regions there are two large segments, one contributes $x_i = 0$ to the index calculation of the region it borders and the other contributes $x_i = 1$ to its corresponding region.*

Proof. Consider a pair R_1, R_2 of adjacent large critical regions meeting A_0. Since they are adjacent the two regions are joined by a connected union δ of small segments which maps to A_0. We consider two cases: (1) R_1, R_2 lie on opposite sides of A_0 or (2) they lie on the same side of A_0.

In case (2) since R_1, R_2 are large it follows that there are large segments meeting A_0 between the regions. Moreover, the existence of δ implies an even number of such segments and they are naturally paired. It is possible that none of these large segments maps to A_0 and a failure to get the conclusion of the lemma. Instead, using an innermost pair of large segments we have the set-up for a coalescing move. Repeatedly apply the coalescing move until R_1 and R_2 are joined into a single critical region.

The proof now reduces to all adjacent pairs being in case (1), and on opposite sides of A_0. If δ consists of an even number (possibly zero) of small segments the two regions are configured as in Figure 1, with δ connected to two large segments, one on the border of each R_i. If δ consists of an odd number of segments, then δ is connected to two large segments which are on the border of the same R_i. It follows directly from the definition that one large segment has $x_i = 1$ and one has $x_i = 0$. ∎

We are now ready to verify the two inequalities in Proposition 1. Without loss we may assume that $f \in \mathcal{I}$ satisfies the conclusion of Lemmas 4 and 5. Let C be a component of G_f with v vertices, v_l of which correspond to large critical regions.

Equation (1) is equivalent to showing that the index of a component of G_f is at most one. By Lemma 4 each small region contributes a segment with $x_i = 1$. By Lemma 5 there are at least $v_l - 1$ segments with $x_i = 1$ corresponding to the large regions in C. Hence there are at least $v - 1$ segments with $x_i = 1$ corresponding to C. Thus, following Lemma 3

$$\mathrm{index}(C) = \sum(1 - \sum x_i)\,,$$

where the outer sum is over the vertices in C and the inner sum is over the segments on the border of a critical region. This is equal to $v - \sum x_i$, with the sum now over all x_i corresponding to C. But this is less than or equal to $v - (v - 1)$, which establishes equation (1).

We now verify equation (2). If R is a small region in C, then by Lemma 4 its index is zero and so does not contribute to the index calculation. We ignore these regions. In this case $\mathrm{index}(C) = \sum(1 - x_i)$, where the sum is taken over the large critical regions in C. Also, if R is large, then each small segment on its border contributes $x_i = 0$ to the index calculation. So the conclusion of Lemma 3 can be stated as

$$\mathrm{index}(R) = 1 - l_1(R)\,,$$

where $l_1(R)$ counts the number of large segments with $x_i = 1$. Let $l_0(R)$ denote the number of large segments with $x_i = 0$ and $l(R)$ the total number of large segments on the border of R. Summing over the large regions corresponding to C we have

$$\mathrm{index}(C) = \sum_R (\,1 - l_1(R)\,) = \sum_R (\,1 + l_0(R) - l(R)\,)$$

$$= v_l + \sum_R l_0(R) - \sum_R l(R) \;\geq v_l + (v_l - 1) - \sum_R l(R)\,,$$

where the inequality above is a result of Lemma 5. Hence,

$$\mathrm{index}(C) + 1 \geq \sum_R (\,2 - l(R)\,)\,.$$

Now let $\mathcal{C}$ be any collection of components of G_f. Then,

$$\sum_{C \in \mathcal{C}} (\mathrm{index}(C) + 1) \;\geq\; \sum_R (\,2 - l(R)\,)\,,$$

where the summation on the right is over all large regions corresponding to $\mathcal{C}$.

We now establish the topological inequality

$$\sum_R (l(R) - 2) \leq -2\chi(F) \ . \tag{3}$$

To do so first consider the case where all large regions in D which meet the boundary of F are critical.

Now remove all small segments from the collection of critical regions and segments corresponding to $\mathcal{C}$. Also note that associated to each point of $A \cap \partial F$ there is at most one length 1 large segment on the border of a large critical region. To simplify counting we move any such segment that lies in D into the 1-handle on the other side of A.

In a given 1-handle H, if there are no large segments of length 2, then there is only one region and $(l(R) - 2) \leq 2$, corresponding to the length 1 large segments in this region. Otherwise all the large segments of length 2 are parallel in H in that they all have endpoints in the same two components of $\partial H \setminus \partial A$, which consists of four components. This implies that the complimentary regions each satisfy $l(R) - 2 = 0$ with two exceptions. Each of the two end regions meeting ∂A will satisfy $(l(R) - 2) \leq 1$ as a result of the length 1 large segments. As before $\sum(l(R) - 2) \leq 2$.

Taking the sum over all of the 1-handles we get

$$\sum (l(R) - 2) \leq 2(1 - \chi(F)) \ .$$

We now focus on D. We first include all large regions in D that are between those regions in $\mathcal{C}$. Since each region in this collection has at least two large sides we note that if the inequality (3) holds for this new collection, then it is valid for the regions in $\mathcal{C}$.

Consider two regions in this collection that meet along a segment. By removing this segment we get a new configuration with one less region and a smaller count by two towards $\sum l(R)$. Thus, the left hand side of (3) remains unchanged in the process. By our assumption on the large critical regions meeting the boundary we reduce to the case where D contains exactly one critical region R_0 with no large segments on the border. Hence, $\sum(l(R) - 2) \leq -2$ for the original collection of critical regions in D.

Combine regions of $\mathcal{C}$ in D with the 1-handles to get

$$\sum (l(R) - 2) \leq 2(1 - \chi(F)) - 2 = -2\chi(F) \ .$$

This establishes the desired inequality in the case where all large regions in D meeting ∂F are critical.

Now suppose that one or more of the large regions in D meeting the boundary are not critical. Consider one such region R', bordered by the large segment α. Since α is large, the boundary of R' contains at least two points of $A \cap \partial F$. By definition of length 1 large, it follows that there is no length 1 large segment corresponding to each of these k points.

This has the following effect on the sum in the first case. On the 1-handles the right hand side is reduced by k. In D, since $l(R') - 2 = -1$ and is not critical, we increase the right hand side of the inequality by $+1$ when summing over critical regions. A similar reasoning applies to a finite number of non-critical regions.

Finally, choosing $\mathcal{C}$ corresponding to the set of vertices of all components of G_f that have a negative index we obtain inequality (2) of the proposition. ■

Remark: In the next section of the paper we will give an adaptation of the construction given in this section. One feature that this adaptation will use is that if W is a finite set of points in F, the entire construction can be done relative to W. That is, given a map f we choose a handle structure so that $(W \cup f(W)) \cap A = \emptyset$. Then all homotopies applied in the construction can be made relative to the set W. This was not the case with the methods from [10] where segments were often moved across A by homotopies.

To conclude this section we now suppose that $g : F \to F$ has the additional property that there exists a boundary component ∂_0 of F such that $g(\partial_0)$ is null homotopic. We can then arrange for $f \in \mathcal{I}$ such that $f^{-1}(A) \cap \partial_0 = \emptyset$ and $f^{-1}(A)$ does not contain any simple closed curves isotopic to ∂_0. It then follows that there is a region in D which contains two components of $\partial_0 \cap D$. This in turn detects a difference of 2 in the inequality (3) above, resulting in the following variation of inequality (2). See also [10, Proposition 6.1] for a slightly different proof of this result.

Proposition 2. *With $f \in \mathcal{I}$ as above* $\sum_{\mathrm{index}(C) \leq -1} (\mathrm{index}(C) + 1) \geq 2\chi(F) + 2$, *where C denotes a component of G_f.*

2 Closed surfaces

In this section we introduce a relative version of the method presented in the previous section. The main application of this will be to verify the HIBMF property for surfaces without boundary. The result appears at the end of this section in Theorem 1. Although we are primarily interested in closed surfaces in this section, the method applies to bounded surfaces as well.

Let F be a compact, connected surface with non-positive Euler characteristic. Consider a prescribed compact subsurface F_0 in F. It is bordered by a collection of simple closed curves which we denote by λ. Let $f : F \to F$ be given and suppose that there are no fixed points on λ and that $f^{-1}(\lambda)$ is a 1-manifold meeting λ transversally in a finite set of points. Throughout this section we assume that all maps have this property.

Let S denote the closure of $F \setminus F_0$. Consider a handle structure for S consisting of one 0-handle for each connected component of S, and a total of $n - \chi(S)$ 1-handles, where n is the number of components of S. We use the symbol D to denote a 0-handle, and A will denote the union of all the attaching arcs for the various 1-handles. Note that the border of a given D consists of a number of arcs that are part of 1-handles and an equal number of arcs that are subsets of either ∂F or λ.

Given the assumptions on λ above we can consider the analogous notion of an A-transverse map. Here we have that (1) the preimage of λ is a proper

1-manifold C_λ and (2) the preimage of A is a 1-manifold with endpoints in $\partial F \cup C_\lambda$ which meets A and λ transversally. Here transverse is as in the previous section; the sets intersect transversally in a finite set of points, none of which are fixed points, and the image of any open set meeting the preimage of A meets at least two components of $S \setminus A$. All homotopies are constant along F_0.

The following lemma is the analog of Lemma 1 for the relative setting. The proof is exactly the same.

Lemma 6. *Let F_0 be as above. Given a fixed point minimal map $f : F \to F$, relative to F_0, there is an A-transverse map homotopic to f rel F_0 with the same number of fixed points and the same fixed point indices.*

Given an A-transverse map f a component R of $S \setminus (A \cup f^{-1}(A) \cup f^{-1}(\lambda))$ is called a *region*. It is said to be *critical* if R and $f(R)$ are contained in the same component of $S \setminus A$. By a *segment* we will mean a component of $(f^{-1}(A) \cup f^{-1}(\lambda))$ on the border of a region. We define the graph G_f just as in the absolute case. Vertices are the critical regions and two regions are joined by an edge if and only if there is a point $p \in A$ common to both boundaries such that p and $f(p)$ lie in the same component of A.

Note that since we are only using A to connect vertices we should not expect that G_f is going to always obtain useful index bounds related only to the surface S. Moreover, fixed points in F_0 could also effect bounds.

Lemma 2 can be applied directly in this setting, but our main issue is going to be with Lemma 3. To compute the index of a critical region we will need a result similar to, and which generalizes, Lemma 3. When considering segments in $f^{-1}(\lambda)$ which are on the border of a critical region, those that are proper arcs (say in D) present no problem. They behave just as segments from $f^{-1}(A)$. The difficulty occurs with segments formed from combinations of arcs from both of $f^{-1}(\lambda)$ and $f^{-1}(A)$. In this case it is not immediately clear how to read off the value of the index as was done in Lemma 3. We will refer to such segments as being *singular segments*. The notions of *large* and *small* for regions and segments is unchanged.

In order to avoid these potential problems, and also with fixed points in F_0, we will restrict our attention to situations where F_0 has little topology. In particular, F_0 might be a neighborhood of a single simple closed curve, or as we consider below, the extreme case where F_0 is a disk and λ is an inessential curve in F.

In the following lemma we will use the fact that F_0 is a disk to gain some control over the singular segments for the purpose of our index calculations. A critical region is said to be *null* if it is small, has exactly one segment on its border and this segment is singular.

Lemma 7. *Let F be a closed surface with $\chi(F) \leq 0$. Given a homotopy class of self-maps there is a fixed point minimal representative f and a disk F_0 in F such that $f^{-1}(F_0)$ is a finite set of disks, each disjoint from F_0. Moreover, if $f^{-1}(F_0)$ is non-empty there is an associated handle structure of S such that*

(1) singular segments have length zero and all have endpoints contained in a single component of A,

(2) if an edge in G_f corresponds to a point that is an endpoint of a singular segment, then

the singular segment borders a null critical region,
(3) singular segments never intersect small segments.

Proof: After a suitable triangulation of F we may assume that the map f is simplicial. If f is not onto, then F_0 is simply a small neighborhood of a point not in the image of f. Otherwise, the disk F_0 is just a small neighborhood of a point in the interior of a 2-simplex that is not fixed by f. We also arrange that each simple closed curve on the border of a disk in $f^{-1}(F_0)$ is mapped to λ by a monotone map determined by the local degree, which we may assume to be nonzero.

We now produce the desired handle structure when $f^{-1}(F_0)$ is non-empty. Let τ denote a collection of $2 - 2\chi(S)$ arcs in S each with one endpoint on λ. When the arcs in τ are suitably small we arrange that $f^{-1}(\tau)$ is also a pairwise disjoint collection of arcs, each arc having one endpoint on $f^{-1}(\lambda)$.

The components of τ are paired as follows. Let $w_1, \dots, w_{1-\chi(S)}$ be a pairwise disjoint collection of proper arcs in S such that a regular neighborhood of the arcs gives a handle structure for S and each w_i contains two components of τ.

We construct an arc η inductively as follows. Set η to be a component t_0 of τ on w_1. Now choose a component E of $f^{-1}(\lambda \cup \tau)$ and consider an (oriented) arc which starts at the free endpoint of η and then traverses each of the arcs of $E \cap f^{-1}(\tau)$ in exactly one point. Furthermore, we keep η disjoint from $f^{-1}(F_0) \cup F_0 \cup \tau$, and also we assume (**) a further condition on the choice of η relative to E which is given below. This is our new arc η. Proceed building the oriented arc η by choosing another component of $f^{-1}(\lambda \cup \tau)$ and extending the old η to this component. Continue until all components of $f^{-1}(\lambda \cup \tau)$ are used.

Now join η to t_0' to form a proper arc in S which we will assume to be isotopic to w_1. The remaining pairs of arcs in τ are joined by arcs, each isotopic to the corresponding w_i. The 1-handles for a handle structure are to be thin neighborhoods of this collection of proper arcs. Now that we have 1-handles we explain the extra condition (**). Let A_1 denote one of the attaching arcs for the handle corresponding to w_1. Choose η so that A_1 is the component nearest to $E \cap f^{-1}(\lambda)$ and also that the first and last arcs (of $f^{-1}(A)$) traversed near $E \cap f^{-1}(\lambda)$ do not belong to $f^{-1}(A_1)$. Since τ has at least 4 components (so at least two 1-handles) this is always possible. The same attaching arc A_1 is to be used for each of the components E considered.

After making local adjustments as needed so that $f^{-1}(A)$ is a 1-manifold the choice of η ensures that (1) holds. In addition, the condition (**) implies that (2) holds. By the construction of the arc η each singular segment has the property that an edge from $f^{-1}(\lambda)$ to A_1 in the segment must extend into the 1-handle as a large segment, one that intersects next the other attaching arc for the handle. Hence, we have property (3). ■

We now show how the handle structure produced in the above lemma fits with the three lemmas used to derive the index bounds for the bounded surface case.

Each singular segment bordering a region constructed in the proof above can be decomposed into three arcs. Two come from $f^{-1}(A)$ and the third is the preimage of one of the arcs in $\lambda \setminus A$. Moreover, f maps the singular segment onto this arc in $\lambda \setminus A$ together with points on A that are very close to the arc. As a result

index calculations involving singular segments occur as follows:
(i) Null critical regions are quite simple. The bordering singular segment maps to an arc outside the region and so the region has index zero. Also, in G_f the null critical region corresponds to a vertex that has valence at most one.
(ii) For all other critical regions. Since singular segments for these regions map to an arc which does not intersect the segment the contribution to index is the same as $x_i = 0$ for a segment in Lemma 3. So we can ignore the singular segments for index calculations.

In order for Lemma 4 to apply for index calculations we must be able to arrange that each connected union of small segments is mapped to one side of the union of the regions bounding the segments. Property (3) ensures this for the singular segments, and so Lemma 4 applies as before. Properties (1) and (2) imply that no singular segments are used to join large regions in G_f. Hence the use of Lemma 5 and the inequalities in the proof of 1 are exact;y the same.

In summary, starting with the CRG G_f given by Lemma 7, we remove the vertices (and adjacent edges) corresponding to null regions to obtain a new graph G_0 which carries the same index data. We then proceed with the proof of Proposition 1 where we ignore singular segments on the border of critical regions. This leads to a relative version of the index bounds for certain critical region graphs.

Proposition 3. *Let F be a closed surface with Euler characteristic $\chi(F) \leq 0$. Then for any $g : F \to F$ that satisfies the conclusion of Lemma 7 above, the HIBCRG hold. That is*

$$\sum_{\text{index}(C)\geq 1} (\text{index}(C) - 1) \ \leq\ 0 \tag{4}$$

and

$$\sum_{\text{index}(C)\leq -1} (\text{index}(C) + 1) \ \geq\ 2\chi(S)\,, \tag{5}$$

where C denotes a component of G_g.

Remark: For the two closed surfaces having positive Euler characteristic the hyperbolic index bounds do not hold. We point out where the above construction breaks down in these cases. For real projective space the surface S is a Möbvious band. In this case it is not possible to arrange condition (**). Consequently, the graph G_f does not reduce to the bounded surface case. For the 2-sphere, the surface S is a disk, and so there is no handle structure to start the construction. One may proceed with a generalization of the method using two (or more) disks for F_0, but failure again occurs in that (**) is not possible to arrange. The presence of edges that join singular segments to large regions in D has the effect of taking the graph G_0, which satisfies HIBCRG, and joining components with no control on the indices.

We now return to fixed point minimal maps on closed surfaces. As mentioned in the introduction, the index bounds result was established for homeomorphisms of hyperbolic surfaces [8]. For surfaces with Euler characteristic zero this result is classical. As a consequence of Proposition 3 we obtain the following generalization.

Theorem 1. *The HIBMF property holds for all closed surfaces with non-positive Euler characteristic.*

Proof. Let F be a closed surface with non-positive Euler characteristic and let $f : F \to F$ be a fixed point minimal self-map. Without loss we may assume that f satisfies the conclusion of Lemma 7 for some disk F_0. Since f is minimal, just as in Lemma 2, each component of G_f has one fixed point if its index is nonzero and no fixed points if its index is zero. Equation (4) of Proposition 3 automatically gives the desired upper bound. Equation (5) of the proposition gives a lower bound of $2\chi(S)$, where S was obtained by removing the interior of the disk F_0. Since the image of λ is clearly null homotopic, by Proposition 2 we obtain a lower bound of $2\chi(S) + 2$. But this is equal to $2\chi(F)$. ∎

This theorem when combined with Corollary 1 gives the main result of this section.

Corollary 2. *The HIBMF property holds for all compact surfaces with non-positive Euler characteristic.*

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Departamento de Matemática - IME-USP
Caixa Postal 66281 - Ag. Cidade de São Paulo
CEP: 05315-970 - São Paulo - SP - Brasil
e-mail: dlgoncal@ime.usp.br

Department of Mathematical Sciences
Loyola University, 6363 St Charles Avenue
New Orleans, LA 70118 - USA
e-mail: kelly@loyno.edu