$\mathbb{Z}_2^k$-actions fixing a disjoint union of odd dimensional projective spaces

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Abstract

Consider the real, complex and quaternionic n-dimensional projective spaces, $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$; to unify notation, write K_dP^n for the real ($d = 1$), complex ($d = 2$) and quaternionic ($d = 4$) n-dimensional projective space. Consider a pair (M, Φ), where M is a closed smooth manifold and Φ is a smooth action of the group $\mathbb{Z}_2^k$ on M; here, $\mathbb{Z}_2^k$ is considered as the group generated by k commuting smooth involutions $T_1, T_2, ..., T_k$. Write F for the fixed-point set of Φ. In this paper we prove the following two results: i) If F is a disjoint union $F = \mathbb{R}P^{n_1} \sqcup \mathbb{R}P^{n_2} \sqcup ... \sqcup \mathbb{R}P^{n_j}$, where $j \geq 2$, each n_i is odd and $n_i \neq n_t$ if $i \neq t$, then (M, Φ) bounds equivariantly. ii) If $F = K_dP^n \sqcup K_dP^m$, where $d = 1, 2$ and 4 and n and m are odd, then (M, Φ) bounds equivariantly. These results are found in the literature for $k = 1$.

1 Introduction

Consider a pair (M, Φ), where M is a closed smooth manifold and Φ is a smooth action of the group $\mathbb{Z}_2^k$ on M. Here, $\mathbb{Z}_2^k$ is considered as the group generated by k commuting smooth involutions $T_1, T_2, ..., T_k$. Then it is well known that the fixed-point set of Φ, F, is a finite and disjoint union of closed submanifolds of M. If $\eta \to F$ is the normal bundle of F in M, then η decomposes under Φ into the Whitney sum of the subbundles on which $\mathbb{Z}_2^k$ acts as one of the irreducible (nontrivial) real representations, which are all one-dimensional over $\mathbb{R}$ and can

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be described by homomorphisms $\rho : \mathbb{Z}_2^k \to \mathbb{Z}_2 = \{+1, -1\}$ which are onto: $\mathbb{Z}_2^k$ acts on the reals so that $g \in \mathbb{Z}_2^k$ acts as multiplication by $\rho(g)$. In other words,

$$\eta = \bigoplus_{\rho} \varepsilon_\rho \quad ,$$

where ε_ρ is the subbundle of η on which $\mathbb{Z}_2^k$ acts in the fibers as ρ; that is, where each T_j acts as multiplication by $\rho(T_j)$, and where the sum excludes the trivial homomorphism $1 \in Hom(\mathbb{Z}_2^k, \mathbb{Z}_2)$. Alternatively, ε_ρ is the normal bundle of F in the fixed-point set F_ρ of the subgroup $kernel(\rho)$. Setting $\mathcal{P} = Hom(\mathbb{Z}_2^k, \mathbb{Z}_2) - \{1\}$, we define the *fixed data* of $(M; \Phi)$ as the object $(F; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$, the fixed-point set F and a list of $2^k - 1$ vector bundles over it indexed by $\mathcal{P}$. In this setting, a natural question is the classification, up to equivariant cobordism, of the pairs (M, Φ) with a given condition on the fixed data of Φ; in particular, for a given F, such a question is the classification of the pairs (M, Φ) for which the fixed-point set is F. This is a well-established problem in the literature, and started in 1964 with the results of Conner and Floyd [3], with $F = S^n \sqcup \{point\}$ and $k = 1$, F= an n-dimensional real projective space, $\mathbb{R}P^n$, with $k = 1$ and n odd, and $F =$ a finite set of isolated points and every k. These results were applications of the equivariant cobordism theory, introduced in [3], which extended the famous cobordism theory of 1954 of René Thom. Taking into account this theory, to be plausible, this problem demands the knowledge of the real K-theory of F or, more specifically, the knowledge of all possible characteristic classes of vector bundles over F. This is the case of the projective spaces K_dP^n, $d = 1, 2$ and 4. So the case F = a disjoint union of projective spaces has an intense and still unfinished history in the literature, started, as mentioned above, with the case $F = \mathbb{R}P^n$, where $k = 1$ and n is odd, of [3]. In this case, Conner and Floyd proved that (M, Φ) bounds equivariantly, and with the same arguments it is possible to prove this $k = 1$ result for $F = K_dP^n$, $d = 2$ and 4 (n odd). Later, in [18], Stong solved the case $F = \mathbb{R}P^n$ with n even and $k = 1$, showing that in this case (M, Φ) is equivariantly cobordant to the involution $(\mathbb{R}P^n \times \mathbb{R}P^n, twist)$, where *twist* maps (x, y) into (y, x). The same is valid for $F = K_dP^n$, $d = 2$ and 4, and proofs (similar to the real case) can be found in [6]. In [2], F. Capobianco solved the case $F = \mathbb{R}P^n$ and $k > 1$ for every $n \geq 1$, and cited the fact that the same arguments work for $d = 2$ and 4. This closes the one component case. The two components case was started with the Royster paper [16], where a partial classification was obtained for the case $F = \mathbb{R}P^n \sqcup \mathbb{R}P^m$ and $k = 1$, leaving open only the case in which m and n are even and nonzero (that is, Royster solved also the case $F = \mathbb{R}P^n \sqcup \mathbb{R}P^0 = \mathbb{R}P^n \sqcup \{point\}$ for any $n \geq 1$). Among their results, Royster proved that if n and m are odd, then (M, Φ) bounds equivariantly; the same is valid for complex and quaternionic projective spaces, and proofs can be found in [5]. Continuing with the two components case and $k = 1$, in [12] and [11], P. L. Q. Pergher, A. Ramos and R. de Oliveira obtained the complex and quaternionic versions of the results of Royster not covered by [5] (m and n odd). Specifically, they solved the cases $F = K_dP^n \sqcup K_dP^m$ for $d = 2$ and 4, where $m \geq 0$ is even and $n \geq 1$ is odd, and where $m = 0$ and $n \geq 2$ is even. In addition, they solved the particular case of the problem left open by Royster, given by $F = \mathbb{R}P^n \sqcup \mathbb{R}P^m$, where $m = 2^s$, n is even and $n \geq 2^{s+1}$, which includes the case $F = \mathbb{R}P^2 \sqcup \mathbb{R}P^m$, $m \geq 4$ even, which had been proved in [11]. Also the complex and quaternionic versions of these cases were obtained in [12]. Still concerning the case $k = 1$, if the number of components of F is greater than 2, then the only known result is due to Torrence and Huo [4; Section 3]: if F is an arbitrary union of odd-dimensional real projective spaces, then (M, Φ) bounds equivariantly.

Summarizing, one has the one component case completely solved and, with exception of $(m, n) = (even, even)$, $m, n > 0$ and $m \neq 2^s$, $n \neq 2^s$, the two components case solved for $k = 1$. Then there is the project of considering the two components case for $k > 1$. In this direction, we cite the results of P. Pergher of [13] ($F = \mathbb{R}P^n \sqcup \{point\}$, n odd and $k > 1$), [7] ($F = \mathbb{R}P^n \sqcup \{point\}$, n even and $k = 2$) and [15] ($F = \mathbb{R}P^n \sqcup \{point\}$, n even and $k > 1$). Also one has the results of P. Pergher, A. Ramos and R. Oliveira, obtained by joining [11], [12] and [8] ($F = K_dP^n \sqcup K_dP^m$, $d = 1, 2$ and 4, $n > 0$ even and $m = 2^s$, and $k > 1$). We emphasize that the methods used in these papers ([13], [7], [15], [12], [11] and [8]) are not suitable for the case where $m > 0$, $n > 0$ and $(m, n) \neq (even, even)$. Therefore, our contribution to this case will be to introduce a new technique and to prove the following two correlated theorems:

Theorem 1. *Let (M, Φ) be a $\mathbb{Z}_2^k$-action, $k > 1$, with fixed-point set $F = \mathbb{R}P^{n_1} \sqcup \mathbb{R}P^{n_2} \sqcup ... \sqcup \mathbb{R}P^{n_j}$, where $j \geq 2$, each n_i is odd and $n_i \neq n_t$ if $i \neq t$. Then (M, Φ) bounds equivariantly.*

Theorem 2. *Let (M, Φ) be a $\mathbb{Z}_2^k$-action, $k > 1$, with fixed-point set $F = K_dP^n \sqcup K_dP^m$, where $d = 1, 2$ and 4 and n and m are odd. Then (M, Φ) bounds equivariantly.*

Theorem 1 extends partially, for $k > 1$, the result of Torrence and Huo [4; Section 3], and Theorem 2 extends, for $k > 1$, the similar $k = 1$ result of [16] and [5]. The paper is organized as follows: Section 2 is devoted to some preliminary technical stuff. In Section 2, we prove Theorem 1 and the $k = 2$ case of Theorem 2. In Section 3 we prove Theorem 2, by joining its previous $k = 2$ case with a result of P. Pergher and R. Oliveira of [9] (which is the subtle point of the proof; in fact, this technique is a novelty in the related literature, as mentioned above).

2 Preliminaries

Keeping the notation of the previous section, let $(M; \Phi)$, $\Phi = (T_1, ..., T_k)$, a smooth $\mathbb{Z}_2^k$-action with fixed data $(F; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$. Each s-dimensional component of $(F; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$ can be considered as an element of $\mathcal{N}_s(\prod_{\rho \in \mathcal{P}} BO(n_\rho))$, the bordism of s-dimensional manifolds with a map into a product of classifying spaces $BO(n_\rho)$ for n_ρ-dimensional vector bundles, where n_ρ denotes the dimension of ε_ρ over the component (this is the *simultaneous cobordism* between lists of vector bundles: if $\mathcal{P}$ is any finite set, two lists (indexed by $\mathcal{P}$) of vector bundles over closed n-dimensional manifolds, $(F^n; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$ and $(V^n; \{\mu_\rho\}_{\rho \in \mathcal{P}})$, are *simultaneously cobordant* if there exists a $(n + 1)$-dimensional manifold W^{n+1} with boundary $\partial(W^{n+1}) = F^n \sqcup V^n$ (disjoint union) and a list of vector bundles over W^{n+1}, $(W^{n+1}; \{\eta_\rho\}_{\rho \in \mathcal{P}})$, so that each η_ρ restricted to $F^n \sqcup V^n$ is equivalent to $\varepsilon_\rho \sqcup \mu_\rho$). According to [17], the equivariant cobordism class of $(M; \Phi)$ is determined by the simultaneous cobordism class of $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$. On the other hand, the simultaneous cobordism class of $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$ is determined by its *characteristic numbers* : write $W(F) = 1 + w_1 + w_2 + ...$ and, for each $\rho \in \mathcal{P}$, $W(\varepsilon_\rho) = 1 + v_1^\rho + v_2^\rho + ...$, for the Stiefel-Whitney classes of the tangent bundle of F and of the bundle ε_ρ, respectively. Then a characteristic number of $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$ is an evaluation of the form $K[F]$, where K is a product of w_i's and v_j^ρ's, $\rho \in \mathcal{P}$, and $[F]$ is the fundamental $\mathbb{Z}_2$-homology class of F. $K[F]$ must be understood as a sum $\sum_s K_s[F^s]$, where F^s is the union of the $s-$ *dimensional* components of F, and K_s is the part of K with degree s. A class $X \in H^*(F)$ is called a *characteristic term* if it is a sum of products of characteristic classes of F and of the ε_ρ, $\rho \in \mathcal{P}$; that is, each homogeneous part of X can participate as a term of a product of classes yielding a characteristic number of $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$.

3 Proofs

As announced in Section 1, we first prove Theorem 1. Let $\alpha_d \in H^d(K_dP^n, \mathbb{Z}_2)$ be the generator, $d = 1, 2$ and 4. It is well known that, if $\eta \to K_dP^n$ is a vector bundle, from the structure of the Grothendieck ring of vector bundles over projectives spaces, there exists a natural number $p \geq 0$ so that the Stiefel-Whitney class $W(\eta)$ has the form $(1 + \alpha_d)^p$. The number p is unique modulo 2^s, where s is the smallest number with $n < 2^s$. For example, if η is the tangent bundle, then $W(\eta) = W(K_dP^n) = (1 + \alpha_d)^{n+1}$.

Lemma 3.1. *Suppose $\eta \to K_dP^n$ a vector bundle, where n is odd. Then η bounds if, and only if, $W(\eta) = (1 + \alpha_d)^p$ with p even.*

Proof. One has $W(K_dP^n) = (1 + \alpha_d)^{n+1}$, with $n + 1$ even. So, if p is even, $w_{di}(\eta) = 0$ and $w_{di}(K_dP^n) = 0$ if i is odd. Since $dim(K_dP^n) = dn$, every characteristic number of η comes from a cohomology class which necessarily contains a term $w_{di}(\eta)$ (or $w_{di}(K_dP^n)$) with i odd. Hence η bounds. If p is odd, $w_d(\eta) = \alpha_d$ and the characteristic number $(w_d(\eta))^n[K_dP^n] = (\alpha_d)^n[K_dP^n]$ is nonzero. ■

Corollary 3.1. *Consider a list of vector bundles over K_dP^n, $(K_dP^n; \eta_1, \eta_2, ..., \eta_t)$, where n is odd and each η_i bounds. Then this list bounds simultaneously.*

Proof. The characteristic number argument is the same of Lemma 3.1. ■

Now we prove Theorem 1. Let $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$ be the fixed data of (M, Φ), and take $\rho \in \mathcal{P}$ and a component $\mathbb{R}P^{n_i}$ of F. As in Section 1, set F_ρ for the fixed-point set of the subgroup $kernel(\rho) \subset \mathbb{Z}_2^k$, and $(F_\rho)_i$ for the component of F_ρ that contains $\mathbb{R}P^{n_i}$. Choosing $T \in \mathbb{Z}_2^k$ with $T \notin kernel(\rho)$, the involution $((F_\rho)_i, T)$ fixes $\mathbb{R}P^{n_i}$ and possibly some other components of F. By Torrence and Huo , $((F_\rho)_i, T)$ bounds equivariantly, and so its fixed data bounds. Since $n_l \neq n_s$ if $l \neq s$, the normal bundle of $\mathbb{R}P^{n_i}$ in $(F_\rho)_i$, which is $\varepsilon_\rho \to \mathbb{R}P^{n_i}$, bounds. By Corollary 3.1, the list $(\mathbb{R}P^{n_i};\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$ bounds simultaneously. It follows that $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$ bounds simultaneously, which proves Theorem 1.

The next step is to prove the $k = 2$ case of Theorem 2. Let (M, Φ), $\Phi = (T_1, T_2)$, be a $\mathbb{Z}_2^2$-action. Set $T_3 = T_1T_2$. The fixed data of (M, Φ) can be written, in this case, as $(F; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $\varepsilon_i \to F$ is the normal bundle of F in F_{T_i} = the fixed-point set of T_i. Take $P \subset F_{T_1}$ any component of F_{T_1}, and set $B \subset P$ for the union of components of F contained in P. Consider $\mathbb{R}P(\varepsilon_1) \to F$ the real projective space bundle associated to $\varepsilon_1 \to F$, and let $\lambda \to \mathbb{R}P(\varepsilon_1)$ be the usual Hopf line bundle over $\mathbb{R}P(\varepsilon_1)$.

Lemma 3.2. *Over B, the list of two bundles $(\mathbb{R}P(\varepsilon_1); \lambda, \varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda))$ bounds simultaneously (here we are suppressing bundle maps).*
Proof. See [10] (or [14]). ∎

Taking into account that any pair (S_1, S_2), where $S_1, S_2 \in \{T_1, T_2, T_3\}$ and $S_1 \neq S_2$, determines a $\mathbb{Z}_2^2$-action, that is, any $\{T_i, T_j, T_k\}$, $i \neq j$, $j \neq k$, $i \neq k$, plays the role of $\{T_1, T_2, T_3\}$, with same fixed data up to permutation, Lemma 3.2 can be re-written as: over B, the list $(\mathbb{R}P(\varepsilon_i); \lambda, \varepsilon_j \oplus (\varepsilon_k \otimes \lambda))$ bounds simultaneously.

Take then a $\mathbb{Z}_2^2$-action (M, Φ) with fixed-point set $F = K_dP^n \sqcup K_dP^m$, where $d = 1, 2$ and 4 and n and m are odd; we want to show that (M, Φ) bounds equivariantly. Suppose first $n \neq m$. Then the case $d = 1$ is covered by Theorem 1, and if $d = 2$ or 4 the same approach of Theorem 1 together with the classification for $k = 1$ of [5] works. Therefore we can suppose that F consists of two copies of K_dP^n, which we call $F = F_1 \sqcup F_2$. Denote the fixed data of (M, Φ) by $(F_1; \epsilon_1, \epsilon_2, \epsilon_3) \sqcup (F_2; \mu_1, \mu_2, \mu_3)$, and by P_i and Q_i, $i = 1, 2, 3$, the component of F_{T_i} that contains F_1 and F_2, respectively. From [5], one has the following result.

Lemma 3.3. *Let (M, T) be an involution fixing two odd dimensional copies of K_dP^n, with normal bundles η and μ. Write $W(\eta) = (1 + \alpha_d)^p$, $W(\mu) = (1 + \beta_d)^q$. Then either p and q are even, or p and q are odd and $p = q$.*

Write $W(\epsilon_i) = (1 + \alpha_d)^{p_i}$, $W(\mu_i) = (1 + \beta_d)^{q_i}$, $k_i = dim(\epsilon_i)$ and $l_i = dim(\mu_i)$, $i = 1, 2, 3$. One has that either $P_i = Q_i$, or $P_i \cap Q_i = \emptyset$. Take $T_j \neq T_i$. If $P_i = Q_i$, (P_i, T_j) is an involution with fixed data $(\epsilon_i \to F_1) \sqcup (\mu_i \to F_2)$, and if $P_i \cap Q_i = \emptyset$, (P_i, T_j) and (Q_i, T_j) are involutions with fixed data $\epsilon_i \to F_1$ and $\mu_i \to F_2$, respectively. Taking into account Lemma 3.3 and the one component case, we conclude that, for $i = 1, 2, 3$, either p_i and q_i are even, or p_i and q_i are odd and $p_i = q_i$. If p_1, p_2 and p_3 (and so q_1, q_2 and q_3) are even, then Corollary 3.1 says that both the lists $(F_1; \epsilon_1, \epsilon_2, \epsilon_3)$ and $(F_2; \mu_1, \mu_2, \mu_3)$ bound simultaneously, which implies that (M, Φ) bounds equivariantly. So, without loss of generality, we can suppose that p_1 is odd. We assure that $k_i = l_i$, $i = 1, 2, 3$. Since p_1 is odd, $P_1 = Q_1$ and thus $k_1 = l_1$. Then $k_2 + k_3 = l_2 + l_3$. So, if p_2 or p_3 are odd, $k_2 = l_2$ and $k_3 = l_3$, and thus we can suppose p_2 and p_3 even. By contradiction, suppose $k_2 \neq l_2$. Then $P_2 \cap Q_2 = \emptyset$ and, by Lemma 3.2, over F_1, the list $(\mathbb{R}P(\epsilon_2); \lambda, \epsilon_3 \oplus (\epsilon_1 \otimes \lambda))$ bounds simultaneously, where $\lambda \to \mathbb{R}P(\epsilon_2)$ is the Hopf line bundle. Write $W(\lambda) = 1 + c$. Then it is known that

$$W(\epsilon_3 \oplus (\epsilon_1 \otimes \lambda)) = (1 + \alpha_d)^{p_3}.(\sum_{i=0}^{k_1} \binom{p_1}{i} \binom{k_1}{i} c^{k_1 - i} \alpha_d^i).$$

After some routine computations, we conclude that

$$W(\epsilon_3 \oplus (\epsilon_1 \otimes \lambda)) = (1 + \alpha_d)^{p_3}.(1 + \alpha_d + c^d).(1 + c^d)^{k_1 - p_1 d}.$$

Since $W(\epsilon_3 \oplus (\epsilon_1 \otimes \lambda))$ and c are characteristic terms (see the definition in Section 2) of the list $(\mathbb{R}P(\epsilon_2); \lambda, \epsilon_3 \oplus (\epsilon_1 \otimes \lambda))$, $(1 + c)^r.W(\epsilon_3 \oplus (\epsilon_1 \otimes \lambda))$ also is, for any $r \geq 0$. In particular, for any natural number S, $(1+c)^{2^S-(k_1-p_1d)}.W(\epsilon_3 \oplus (\epsilon_1 \otimes \lambda))$ is a characteristic term. For S sufficiently large, $(1+c)^{2^S} = (1+c^{2^S}) = 1$, and thus

$$(1+c)^{2^S-(k_1-p_1d)}.W(\epsilon_3 \oplus (\epsilon_1 \otimes \lambda)) = (1+\alpha_d)^{p_3}.(1+\alpha_d+c^d)$$

is a characteristic term. Because p_3 is even and p_1 is odd, the homogeneous part of degree d of this characteristic term is $\alpha_d + c^d$. It follows that α_d is a characteristic term of the list in question, which yields the characteristic number $\alpha_d^n c^{k_2-1}[\mathbb{R}P(\epsilon_2)]$. Since $H^*(\mathbb{R}P(\epsilon_2), \mathbb{Z}_2)$ is the free $H^*(F_1, \mathbb{Z}_2)$-module on $1, c, c^2, \ldots, c^{k_2-1}$ (see [1]), we have that $\alpha_d^n c^{k_2-1}[\mathbb{R}P(\epsilon_2)] = 1$, which contradicts the fact that $(\mathbb{R}P(\epsilon_2); \lambda, \epsilon_3 \oplus (\epsilon_1 \otimes \lambda))$ bounds simultaneously. Therefore $k_2 = l_2$, $k_3 = l_3$ and $P_2 = Q_2$. We shall prove that $p_i = q_i$ for $i = 2$ and 3; since $p_1 = q_1$, this will give that the lists $(F_1; \epsilon_1, \epsilon_2, \epsilon_3)$ and $(F_2; \mu_1, \mu_2, \mu_3)$ have the same characteristic numbers, that is, are simultaneously cobordant. This will give that $(F_1; \epsilon_1, \epsilon_2, \epsilon_3) \sqcup (F_2; \mu_1, \mu_2, \mu_3)$ bounds simultaneously, which means that (M, Φ) bounds equivariantly, thus ending the proof.

If p_2 and p_3 are odd, by Lemma 3.3, $p_2 = q_2$ and $p_3 = q_3$, and the desired result follows. So we can suppose p_2 or p_3 even. Suppose first p_3 even. Set $\xi \to \mathbb{R}P(\mu_2)$ for the Hopf line bundle, and write $W(\xi) = 1 + e$. Since $P_2 = Q_2$, by Lemma 3.2 the lists $\theta_1 = (\mathbb{R}P(\epsilon_2); \lambda, \epsilon_3 \oplus (\epsilon_1 \otimes \lambda))$ and $\theta_2 = (\mathbb{R}P(\mu_2); \xi, \mu_3 \oplus (\mu_1 \otimes \xi))$ are simultaneously cobordant, which means that their corresponding characteristic numbers are the same.

As above, one has

$$W(\epsilon_3 \oplus (\epsilon_1 \otimes \lambda)) = (1+\alpha_d)^{p_3}(1+\alpha_d+c^d)^{p_1}(1+c^d)^{k_1-p_1}, \quad (1)$$

$$W(\mu_3 \oplus (\mu_1 \otimes \xi)) = (1+\beta_d)^{q_3}(1+\beta_d+e^d)^{q_1}(1+e^d)^{l_1-q_1}. \quad (2)$$

Since $k_1 - p_1 = l_1 - q_1$, the same trick with characteristic numbers used above cancels the terms $(1+c^d)^{k_1-p_1}$ and $(1+e^d)^{l_1-q_1}$ in (2) and (3), thus showing that $(1+\alpha_d)^{p_3}(1+\alpha_d+c^d)^{p_1}$ and $(1+\beta_d)^{q_3}(1+\beta_d+e^d)^{q_1}$ are corresponding characteristic terms of the lists θ_1 and θ_2. Because p_3 (and so q_3) is even and $p_1 = q_1$ is odd, the homogeneous part of degree d of these characteristic terms are $\alpha_d + c^d$ and $\beta_d + e^d$; in particular, α_d and β_d are corresponding characteristic terms of θ_1 and θ_2. It follows that $(1+\alpha_d+c^d)^r$ and $(1+\beta_d+e^d)^r$ are corresponding characteristic terms of θ_1 and θ_2, for any $r \geq 0$. In particular, for S sufficiently large, $(1+\alpha_d+c^d)^{2^S-p_1}.(1+\alpha_d)^{p_3}(1+\alpha_d+c^d)^{p_1} = (1+\alpha_d)^{p_3}$ and $(1+\beta_d+e^d)^{2^S-q_1}.(1+\beta_d)^{q_3}(1+\beta_d+e^d)^{q_1} = (1+\beta_d)^{q_3}$ are corresponding characteristic terms of θ_1 and θ_2, noting that these terms are exactly $W(\epsilon_3)$ and $W(\mu_3)$. Since ϵ_3 and μ_3 are bundles over copies of K_dP^n, $w_{di}(\epsilon_3) = 0$ and $w_{di}(\mu_3) = 0$ if $i > n$. Take then $i \leq n$. Because $W(\epsilon_3)$ and $W(\mu_3)$ are corresponding characteristic terms of θ_1 and θ_2, $w_{di}(\epsilon_3) = \binom{p_3}{i}\alpha_d^i$ and $w_{di}(\mu_3) = \binom{q_3}{i}\beta_d^i$ also are. Together with the fact that $\alpha_d^i \in H^{di}(\mathbb{R}P(\epsilon_2), \mathbb{Z}_2)$ over F_1 and $\beta_d^i \in H^{di}(\mathbb{R}P(\epsilon_2), \mathbb{Z}_2)$ over F_2 are nonzero and corresponding characteristic terms of θ_1 and θ_2, this gives that $\binom{p_3}{i} = \binom{q_3}{i}$. Then $w_{di}(\epsilon_3) = \alpha_d^i$ if, and only if, $w_{di}(\mu_3) = \beta_d^i$. It follows that $p_3 = q_3$, as desired. Note that, in the above arguments, after we suppose that p_1 is odd and p_3 is even, we concluded that $P_2 = Q_2$ and $p_3 = q_3$ with no involvement of p_2 in all the characteristic number arguments. That is, in both cases, p_2 odd or even, one has $p_1 = q_1$ and $p_3 = q_3$. Thus, if in addition p_2 is odd, by Lemma 3.3 $p_2 = q_2$ and the result follows. On the other hand, if p_2 is even, we repeat all the above procedure interchanging the roles of p_2 and p_3 to also conclude that $p_2 = q_2$. Since the (remaining) case $(p_1, p_2, p_3) = (odd, even, odd)$ is a permutation of the case $(p_1, p_2, p_3) = (odd, odd, even)$, the result follows.

4 The case $F = K_dP^n \sqcup K_dP^n$ with n odd and $k \geq 2$

This final section will be devoted to the above case. As mentioned in Section 1, the crucial point will be a result from [9], which combined with the previous $k = 2$ case will give the desired result. As said there, this technique is a novelty in the related literature. The result from [9] in question can be described as follows. Let $(M; \Phi)$ be a $\mathbb{Z}_2^k$-action with fixed data $(F; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$, and let Ω be a subgroup of $Hom(\mathbb{Z}_2^k, \mathbb{Z}_2)$. Then the part of the fixed data of $(M; \Phi)$ given by $(F; \{\varepsilon_\rho\}_{\rho \in \Omega \cap \mathcal{P}})$ can be realized as the fixed data of a subgroup $G \subset \mathbb{Z}_2^k$ acting (by restriction) on the fixed-point set of the restriction of Φ to some appropriate subgroup $H \subset \mathbb{Z}_2^k$. In fact, in [9] it is proved that there exists a subgroup $G \subset \mathbb{Z}_2^k$ so that the restriction $Hom(\mathbb{Z}_2^k, \mathbb{Z}_2) \to Hom(G, \mathbb{Z}_2)$ maps Ω isomorphically onto $Hom(G, \mathbb{Z}_2)$. If, as a $\mathbb{Z}_2$-vector space, $dim(G) = r < k$, one can take a $(k-r)$-dimensional subgroup $H \subset \mathbb{Z}_2^k$ so that $\mathbb{Z}_2^k = G \oplus H$. Set $F_H =$ the fixed-point set of H, and Ψ= the restriction of Φ to $G \times F_H$. One then has the following

Lemma 4.1. *The fixed data of the G-action $(F_H; \Psi)$ is $(F; \{\mu_{\rho'}\}_{\rho' \in \mathcal{P}'})$, where for each $\rho' \in \mathcal{P}' = Hom(G, \mathbb{Z}_2) - \{1\}$ one has $\mu_{\rho'} = \varepsilon_\rho$, where ρ is the unique element of $\Omega \cap \mathcal{P}$ with $\rho_{|G} = \rho'$. In other words, the fixed data of G acting on the fixed-point set of H is F with the subbundles ε_ρ, $\rho \in \Omega \cap \mathcal{P}$, and in terms of $\mathcal{P}'$, these subbundles are indexed under the restriction $\Omega \cap \mathcal{P} \to \mathcal{P}'$.*
Proof. See Lemma 3.1 of [9]. ∎

Take now a $\mathbb{Z}_2^k$-action (M, Φ) with the fixed-point set F consisting of two copies of K_dP^n (F_1 and F_2), where $d = 1, 2$ and 4 and n is odd. Write $(F_1; \{\epsilon_\rho\}_{\rho \in \mathcal{P}}) \sqcup (F_2; \{\mu_\rho\}_{\rho \in \mathcal{P}})$ for the fixed data of (M, Φ). As before, write F_ρ for the fixed-point set of the subgroup $kernel(\rho) \subset \mathbb{Z}_2^k$, and P_ρ and Q_ρ for the components of F_ρ that contains F_1 and F_2, respectively. Choose $T \in \mathbb{Z}_2^k$ with $T \notin kernel(\rho)$. If $P_\rho = Q_\rho$, (P_ρ, T) is an involution fixing $F_1 \sqcup F_2$, and if $P_\rho \cap Q_\rho = \emptyset$, (P_ρ, T) and (Q_ρ, T) are involutions fixing F_1 and F_2, respectively. Write $W(\epsilon_\rho) = (1 + \alpha_d)^{p_\rho}$ and $W(\mu_\rho) = (1 + \beta_d)^{q_\rho}$. As before, for any $\rho \in \mathcal{P}$, either p_ρ and q_ρ are even, or p_ρ and q_ρ are odd and $p_\rho = q_\rho$. If p_ρ (and so q_ρ) is even for every $\rho \in \mathcal{P}$, by Corollary 3.1 the lists $(F_1; \{\epsilon_\rho\}_{\rho \in \mathcal{P}})$ and $(F_2; \{\mu_\rho\}_{\rho \in \mathcal{P}})$ bound simultaneously, and thus (M, Φ) bounds equivariantly. Therefore we can assume that at least for one $\rho_0 \in \mathcal{P}$, p_{ρ_0} (and so q_{ρ_0}) is odd (and so $p_{\rho_0} = q_{\rho_0}$). Choose an arbitrary $\rho \in \mathcal{P}$. If $\rho = \rho_0$, $p_\rho = q_\rho$. If $\rho \neq \rho_0$, we consider the subgroup $\Omega = \{1, \rho, \rho_0, \rho.\rho_0\} \cong \mathbb{Z}_2^2 \subset Hom(\mathbb{Z}_2^k, \mathbb{Z}_2)$. Then, there exists a subgroup $G \cong \mathbb{Z}_2^2 \subset \mathbb{Z}_2^k$ so that the restriction $Hom(\mathbb{Z}_2^k, \mathbb{Z}_2) \to Hom(G, \mathbb{Z}_2)$ maps Ω isomorphically onto $Hom(G, \mathbb{Z}_2)$. Choose a $(k-2)$-dimensional subgroup $H \subset \mathbb{Z}_2^k$ so that $\mathbb{Z}_2^k = G \oplus H$. Set $F_H =$

the fixed-point set of H, and Ψ= the restriction of Φ to $G \times F_H$. Then, by Lemma 4.1, $(F_H;\Psi)$ is a $\mathbb{Z}_2^2$-action with fixed data $(F_1;\epsilon_\rho,\epsilon_{\rho_0},\epsilon_{\rho.\rho_0}) \sqcup (F_2;\mu_\rho,\mu_{\rho_0},\mu_{\rho.\rho_0})$, and with at least one subbundle of the fixed data over F_1 and its corresponding subbundle over F_2 having Stiefel classes of the form $(1+\alpha_d)^p$ and $(1+\beta_d)^p$ with p odd. This is the situation found in Section 3, where it was proved that $p_\rho = q_\rho$. Since this is valid for every $\rho \in \mathcal{P}$, the lists $(F_1;\{\epsilon_\rho\}_{\rho\in\mathcal{P}})$ and $(F_2;\{\mu_\rho\}_{\rho\in\mathcal{P}})$ have the same characteristic numbers, and (M,Φ) bounds equivariantly.

Remark. Let F be a disjoint (finite) union of connected, smooth and closed manifolds such that each component of F bounds (which is the case of $F = K_dP^n \sqcup K_dP^m$ with n and m odd and $F = \mathbb{R}P^{n_1} \sqcup \mathbb{R}P^{n_2} \sqcup ... \sqcup \mathbb{R}P^{n_j}$ of Theorem 1). We will discuss on the existence of bounding $\mathbb{Z}_2^k$-actions fixing F. Write $F = \bigsqcup_{j=0}^{n} F^j$, where F^j denotes the union of those components of F having dimension j, and thus n is the dimension of the components of F of largest dimension. Choose a natural number $m > n$, and for each $0 \le j \le n$ a bounding vector bundle $\eta^{m-j} \to F^j$ of dimension $m-j$ (for example, a trivial $(m-j)$-dimensional vector bundle over F^j). Consider $\partial : \bigoplus_{j=0}^{m} \mathcal{N}_j(BO(m-j)) \to \mathcal{N}_{m-1}(BO(1))$ the homomorphism of the Conner and Floyd short exact sequence of [3]. Choosing a trivial vector bundle as a representative of $[\eta^{m-j}] \in \mathcal{N}_j(BO(m-j))$, it is easy to see that $\partial([\eta^{m-j}]) = 0$. Thus, for each $0 \le j \le n$, there is a bounding involution (M_j^m, T_j) with fixed data $\eta^{m-j} \to F^j$; here, each M_j^m is a closed smooth m-dimensional manifold. Then $(M^m,T) = \bigsqcup_{j=0}^{n}(M_j^m,T_j)$ is a bounding involution fixing F. Now, for each $k \ge 2$, we can construct a special $\mathbb{Z}_2^k$-action fixing F, denoted by $\Gamma_k^k(M^m,T)$, having (M^m,T) as a starting point. Because (M^m,T) bounds, $\Gamma_k^k(M^m,T)$ also bounds (for details on $\Gamma_k^k(M^m,T)$, see [8; Section 1]).

We remark that there are nonbounding $\mathbb{Z}_2^k$-actions so that its fixed-point set bounds. For example, consider the Dold manifold

$$P(1,n) = \frac{\mathbb{S}^1 \times \mathbb{C}P^n}{-1 \times (conjugation)}.$$

The mod 2 cohomology of $P(1,n)$ is $H^*(P(1,n),\mathbb{Z}_2) = \mathbb{Z}_2[c,d]/(c^2 = 0, d^{n+1} = 0)$, where $c \in H^1(P(1,n),\mathbb{Z}_2)$ and $d \in H^2(P(1,n),\mathbb{Z}_2)$. The tangential Stiefel-Whitney class of $P(1,n)$ is $W(P(1,n)) = (1+c)(1+c+d)^{n+1}$. Over $P(1,n)$ one has a 2-dimensional vector bundle $\eta \to P(1,n)$ with $W(\eta) = 1+c+d$; as a reference for these facts, see [19; Section 1]. With this data in hand, a routine characteristic number calculation shows that, if n is odd, then $P(1,n)$ bounds, η does not bound and $\partial(\eta) = 0$. Then there exists a nonbounding involution (M^{2n+3},T) with fixed data $\eta \to P(1,n)$. Again, for $k \ge 2$, $\Gamma_k^k(M^{2n+3},T)$ are nonbounding $\mathbb{Z}_2^k$-actions fixing $P(1,n)$.

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