(H,G)-coincidence theorems for manifolds and a topological Tverberg type theorem for any natural number r

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Abstract

Let X be a paracompact space, let G be a finite group acting freely on X and let H a cyclic subgroup of G of prime order p. Let $f : X \to M$ be a continuous map where M is a connected m-manifold (orientable if $p > 2$) and $f^*(V_k) = 0$, for $k \geq 1$, where V_k are the Wu classes of M. Suppose that $\operatorname{ind} X \geq n > (|G| - r)m$, where $r = \frac{|G|}{p}$. In this work, we estimate the cohomological dimension of the set $A(f, H, G)$ of (H, G)-coincidence points of f. Also, we estimate the index of a (H, G)-coincidence set in the case that H is a p-torus subgroup of a particular group G and as application we prove a topological Tverberg type theorem for any natural number r. Such result is a weak version of the famous topological Tverberg conjecture, which was proved recently fail for all r that are not prime powers. Moreover, we obtain a generalized Van Kampen-Flores type theorem for any integer r.

1 Introduction

Let G be a finite group which acts freely on a space X and let $f : X \to Y$ be a continuous map from X into another space Y. If H is a subgroup of G, then H

*The author was supported in part by FAPESP of Brazil Grant numbers 12/24454-8 and 2013/24845-0

†The author was supported in part by FAPESP of Brazil Grant numbers 12/24454-8

‡The author was supported by CAPES and CNPq

Received by the editors in October 2016 - In revised form in January 2017.

Communicated by K. Dekimpe, D.L. Gonçalves and P. Wong.

2010 *Mathematics Subject Classification* : Primary 55M20, 52A35; Secondary 55M35, 55S35.

Key words and phrases : (H, G)-coincidence, G-action, topological Tverberg theorem.

acts on the right on each orbit Gx of G as follows: if $y \in Gx$ and $y = gx$, with $g \in G$, then $h \cdot y = gh^{-1}x$. A point $x \in X$ is said to be a (H, G)- *coincidence point of f* (as introduced by Gonçalves and Pergher in [7]) if f sends every orbit of the action of H on the G-orbit of x to a single point. Of course, if H is the trivial subgroup, then every point of X is a (H, G)-coincidence. If $H = G$, this is the usual definition of G-coincidence, that is, $f(x) = f(gx)$, for all $g \in G$. If $G = \mathbb{Z}_p$ with p prime, then a nontrivial (H, G)-coincidence point is a G-coincidence point. Let us denote by $A(f, H, G)$ the set of all (H, G)-coincidence points. A kind of Borsuk-Ulam type theorems consists in estimating the cohomological dimension of the set $A(f, H, G)$. Two main directions for this problem are either when the target space Y is a manifold or Y is a CW complex. In the first direction are the papers of Borsuk [4] (the classical theorem of Borsuk-Ulam, for $H = G = \mathbb{Z}_2$, $X = S^n$ and $Y = R^n$), Conner and Floyd [5] (for $H = G = \mathbb{Z}_2$, $X = S^n$ and Y a n-manifold), Munkholm [13] (for $H = G = \mathbb{Z}_p$, $X = S^n$ and $Y = R^m$), Nakaoka [14] (for $H = G = \mathbb{Z}_p$, X under certain (co)homological conditions and Y a m-manifold) and the following more general version proved by Volovikov [17] using the index of a free $\mathbb{Z}_p$-space X (ind X, see Definition 2.2):

Theorem A.[17, Theorem 1.2] *Let X be a paracompact free $\mathbb{Z}_p$-space of* $\operatorname{ind} X \geq n$, *and $f : X \to M$ a continuous mapping of X into an m-dimensional connected manifold M (orientable if $p > 2$). Assume that:*

(1) $f^*(V_i) = 0$ *for $i \geq 1$, where the V_i are the Wu classes of M; and*

(2) $n > m(p - 1)$.

Then the $\operatorname{ind} A(f) \geq n - m(p - 1) > 0$.

In the second direction are the papers of Izydorek and Jaworowski [10] (for $H = G = \mathbb{Z}_2$, $X = S^n$ and Y a CW-complex), Gonçalves and Pergher [7] (for $H = G = \mathbb{Z}_p$, $X = S^n$ and Y a CW-complex) and for proper nontrivial subgroup H of G, Gonçalves, Jaworowski and Pergher [8] (for $H = \mathbb{Z}_p$ subgroup of a finite group G, X an homotopy sphere and Y a CW-complex) and Gonçalves, Jaworowski, Pergher and Volovikov [9](for $H = \mathbb{Z}_p$ subgroup of a finite group G, X under certain (co)homological assumptions and Y a CW-complex).

In this work, considering the target space $Y = M$ a manifold and H a proper nontrivial subgroup of G, we prove the following formulation of the Borsuk-Ulam theorem for manifolds in terms of (H, G)-coincidence.

Theorem 1.1. *Let X be a paracompact space of $indX \geq n$ and let G be a finite group acting freely on X and H a cyclic subgroup of G of prime order p. Let $f : X \to M$ be a continuous map where M is a connected m-manifold (orientable if $p > 2$) and $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M. Suppose that* $\operatorname{ind} X \geq n > (|G| - r)m$ *where $r = \frac{|G|}{p}$. Then* $\operatorname{ind} A(f, H, G) \geq n - (|G| - r)m$. *Consequently,*

$$\text{cohom.dim}\, A(f, H, G) \geq n - (|G| - r)m > 0.$$

Let us observe that if $H = G = \mathbb{Z}_p$, we have $(|G| - r)m = (p - 1)m$ and therefore Theorem 1.1 generalizes Theorem A above of Volovikov. For the case $n = (|G| - r)m$, p an odd prime, if we consider X a *mod p* homology n-sphere in the Theorem 1.1 (in this case, the continuous map f can be arbitrary), we obtain a version for (H, G)-coincidence points of the $\mathbb{Z}_p$-result of Nakaoka [14, Theorem 8]. Further, it considerably improves the estimative of Gonçalves, Jaworowski and Pergher (of [8]), when CW-complexes are replaced by manifolds: if $n > m(|G| - r)$ (which is better than $n > m|G|$ and, depending on r, may be much better than $n > m|G|$), then $\operatorname{ind} A(f; H; G) \geq n - m(|G| - r)$ (which again is better than $\operatorname{ind} A(f; H; G) \geq n - m|G|$ and, depending on r, may be much better than $\operatorname{ind} A(f; H; G) \geq n - m|G|$).

Also, we prove the following nonsymmetric theorem for (H, G)-coincidences which is a version for manifolds of the main theorem in [11].

Theorem 1.2. *Let X be a compact Hausdorff space, let G be a finite group acting freely on S^n and let H be a cyclic subgroup of G of order prime p. Let $\varphi : X \to S^n$ be an essential map [1] and let $f : X \to M$ be a continuous map where M is a connected m-manifold (orientable if $p > 2$) and $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M. Suppose that $n > (|G| - r)m$, then*

$$\text{cohom.dim}\, A_\varphi(f, H, G) \geq n - (|G| - r)m,$$

where $r = \frac{|G|}{p}$ and $A_\varphi(f, H, G)$ denotes the (H, G)-coincidence points of f relative to an essential map $\varphi : X \to S^n$.

In Section 4, we give a similar estimate in the case that H is a p-torus subgroup of a particular group G and as application, we prove a topological Tverberg type theorem for any natural number, which is a weak version of the famous topological Tverberg conjecture. Moreover, we obtain a generalized Van Kampen-Flores type theorem for any integer r.

2 Preliminaries

We introduce the following concept.

2.1 The $\mathbb{Z}_p$-index

We suppose that the cyclic group $\mathbb{Z}_p$ acts freely on a paracompact Hausdorff space X, where p is a prime number and we denote by $[X]^*$ the orbit space of X by the action of $\mathbb{Z}_p$. Then, $X \to [X]^*$ is a principal $\mathbb{Z}_p$-bundle and we can consider a classifying map $c : [X]^* \to B\mathbb{Z}_p$.

Remark 2.1. It is well known that if $\hat{c}$ is another classifying map for the principal $\mathbb{Z}_p$-bundle $X \to X^*$, then there is a homotopy between c and $\hat{c}$.

Definition 2.2. *We say that the $\mathbb{Z}_p$-index of X is greater than or equal to l if the homomorphism*

$$c^* : H^l(B\mathbb{Z}_p; \mathbb{Z}_p) \to H^l([X]^*; \mathbb{Z}_p)$$

[1] A map $\varphi : X \to S^n$ is said to be an essential map if φ induces nonzero homomorphism $\varphi^* : H^n(S^n; \mathbb{Z}_p) \to H^n(X; \mathbb{Z}_p)$.

is nontrivial. We say that the $\mathbb{Z}_p$*-index of* X *is equal to* l *if it is greater or equal than* l *and, furthermore,* $c^* : H^i(B\mathbb{Z}_p; \mathbb{Z}_p) \to H^i([X]^*; \mathbb{Z}_p)$ *is zero, for all* $i \geq l+1$.

We denote the $\mathbb{Z}_p$*-index of* X *by* ind X.

3 Proof of Theorem 1.1

To prove Theorem 1.1, we use the technique introduced in [8, Section 5], which had as a starting point the proof of the main theorem for $G = \mathbb{Z}_p$, made in [8, Section 3]: choose $a_1, a_2, ..., a_r$ a set of representatives of the left lateral classes of G/H, and define the map $F : X \to M^r$ of X to the r-fold product M^r by $F(x) = (f(a_1x), ..., f(a_rx))$.

In [8], it was used the case $G = \mathbb{Z}_p$ for F and the restriction of the action of G to $H \cong \mathbb{Z}_p$. In our case, the starting point is Theorem A. However, to follow the lines of [8], we need first to understand the Wu classes of a cartesian product of manifolds and the effect of F^* in such classes, which will be made through Lemmas 3.1 and 3.2 below. The *total Wu class* of a manifold M is defined as the formal sum

$$v(M) = 1 + v_1(M) + v_2(M) + \cdots + v_k(M) + \cdots$$

where $v_k(M)$ is the k-th Wu class of M, $k = 1, 2, \ldots$ (see [12]). Let $p > 2$ be a prime. Using the *total reduced power*

$$P = P^0 + P^1 + P^2 + \cdots + P^k + \cdots$$

and the equation

$$\langle v_k(M) \smile x, [M] \rangle = \langle P^k(x), [M] \rangle$$

we obtain the formula

$$\langle v(M) \smile u, [M] \rangle = \langle P(u), [M] \rangle$$

for all $u \in H_c^*(M; \mathbb{Z}_p)$. For $p = 2$ we have a similar formula

$$\langle v(M) \smile u, [M] \rangle = \langle Sq(u), [M] \rangle$$

for all $u \in H_c^*(M; \mathbb{Z}_2)$, where

$$Sq = Sq^0 + Sq^1 + Sq^2 \cdots + Sq^k + \cdots$$

is the *total Steenrod square*. Let W and M be connected manifolds, both orientables if $p > 2$.

Lemma 3.1. *The total Wu class of* $W \times M$*, is given by:*

$$v(W) \otimes v(M) \tag{3.1}$$

where $v(W)$ *and* $v(M)$ *are the total Wu classes of* W *and* M *respectively.*

Proof. Let $p > 2$ be a prime number. Let $z = w \otimes u$ an element of $H_c^*(W \times M; \mathbb{Z}_p)$ then

$$\begin{aligned}\langle v(W) \otimes v(M) \smile z, [W \times M]\rangle &= \langle v(W) \smile w \otimes v(M) \smile u, [W \times M]\rangle \\ &= \langle P(w) \otimes P(u), [W \times M]\rangle \\ &= \langle P(w \otimes u), [W \times M]\rangle \\ &= \langle P(z), [W \times M]\rangle \\ &= \langle v(W \times M) \smile z, [W \times M]\rangle\end{aligned}$$

Therefore by uniqueness of the Wu class we conclude that the total Wu class of $W \times M$ is given by $v(W \times M) = v(W) \otimes v(M)$. By a similar argument the total Wu classes are obtained for $p = 2$; in this case are used the total Steenrod square. ∎

Lemma 3.2. *If $f^*(v_k(M)) = 0$, for all $k \geq 1$, where $v_k(M)$ are the Wu classes of M, then $F^*(v_k(M^r)) = 0$, for all $k \geq 1$, where $v_k(M^r)$ are the Wu classes of M^r.*

Proof. Since $F = (f_1 \times \ldots \times f_r) \circ D$, where $D : X \to X^r$ is the diagonal map and $f_i : X \to X$ is given by $f_i(x) = f(a_i x)$, $i = 1 \ldots r$, it suffices to show that $(f_1 \times \ldots \times f_r)^*(v_k(M^r)) = 0$, for $k \geq 1$. If $r = 1$, then $F = f_1$ and $f_1^*(v_k(M)) = g_1^* \circ f^*(v_k(M)) = 0$.

Let us denote by

$$p_1 : M^{r-1} \times M \to M^{r-1},\ p_2 : M^{r-1} \times M \to M$$

$$q_1 : X^{r-1} \times X \to X^{r-1},\ q_2 : X^{r-1} \times X \to X$$

the natural projections. If $r \geq 2$, we have

$$(f_1 \times \ldots \times f_{r-1}) \circ q_1 = p_1 \circ (f_1 \times \ldots \times f_r)$$

$$f_r \circ q_2 = p_2 \circ (f_1 \times \ldots \times f_r).$$

Since, by Lemma 3.1, $v_k(M^{r-1} \times M) = \sum_{s=0}^{k} v_s(M^{r-1}) \times v_{k-s}(M)$ and assuming inductively that $(f_1 \times \ldots \times f_{r-1})^*(v_s(M^{r-1})) = 0$, for $s \geq 1$, we conclude that

$$\begin{aligned}&(f_1 \times \ldots \times f_r)^*(v_k(M^{r-1} \times M)) = \\ &= (f_1 \times \ldots \times f_r)^* \left(\sum_{s=0}^{k} v_s(M^{r-1}) \times v_{k-s}(M) \right) \\ &= \sum_{s=0}^{k} (f_1 \times \ldots \times f_r)^*(p_1^*(v_s(M^{r-1}))) \smile (f_1 \times \ldots \times f_r)^*(p_2^*(v_{k-s}(M))) \\ &= \sum_{s=0}^{k} q_1^* \circ (f_1 \times \ldots \times f_{r-1})^*(v_s(M^{r-1})) \smile q_2^* \circ g_r^* \circ f^*(v_{k-s}(M)) \\ &= 0.\end{aligned}$$

∎

Proof. Now we return to the proof of Theorem 1.1. We have

$$A(f,H,G) \supset A_F = \{x \in X : F(x) = F(hx), \forall h \in H\}.$$

In fact, let x be a point in the set A_F, then

$$(f(a_1x), \ldots, f(a_rx)) = (f(a_1hx), \ldots, f(a_rhx)),$$

for all $h \in H$. Thus, $f(a_ix) = f(a_ihx)$, for all $h \in H$ and $i = 1, \ldots, r$. According to the definition of the action of H on the orbit Gx, $h^{-1} \cdot a_ix := a_i(h^{-1})^{-1}x = a_ihx \in a_iHx$, for $i = 1, \ldots, r$. Thus, f collapses each orbit a_iHx determined by the action of H on a_ix, for $i = 1, \ldots, r$, therefore $x \in A(f,H,G)$.

Now we observe that $H \cong \mathbb{Z}_p$ acts freely on X by restriction and by hypothesis $\operatorname{ind} X \geq n > n-(p-1)rm$. By Lemma 3.2, $F^*(v_k) = 0$, for all $k \geq 1$, where v_k are the Wu classes of M^r. Thus, according to Theorem A,

$$\operatorname{ind} A_F \geq n-(p-1)rm = n-(|G|-r)m.$$

Let us consider the inclusion $i : A_F \to A(f,H,G)$, which is an equivariant map, and so it induces $\bar{i} : [A_F]^* \to [A(f,H,G)]^*$ a map between the orbit spaces. Therefore, if $c : [A(f,H,G)]^* \to B\mathbb{Z}_p$ is any classifying map, we have that $c \circ \bar{i} : [A_F]^* \to B\mathbb{Z}_p$ is a classifying map. Thus,

$$\operatorname{ind} A(f,H,G) \geq \operatorname{ind} A_F \geq n-(|G|-r)m. \qquad \blacksquare$$

Corollary 3.3. *Let X be a paracompact space and let G be a finite group acting freely on X. Let M be a orientable m-manifold, and p a prime number that divide $|G|$. Suppose that ind$X \geq n > (|G|-r)m$, where $r = \frac{|G|}{p}$. Then, for a continuous map $f : X \to M$ such that $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M, there exists a non-trivial subgroup H of G, such that*

$$\text{cohom.dim}\, A(f,H,G) \geq n-(|G|-r)m.$$

Proof. Let p be a prime number such that divide $|G|$. By Cauchy Theorem, there is a cyclic of order p subgroup H of G . Then, we apply Theorem 1.1. $\blacksquare$

Remark 3.4. Let us observe that, if $f^* : H^i(M;\mathbb{Z}_p) \to H^i(X;\mathbb{Z}_p)$ is trivial, for $i \geq 1$, and p is the smallest prime number dividing $|G|$, then $r = \frac{|G|}{p} \geq \frac{|G|}{q}$, where q can be any other prime number dividing $|G|$. Thus, $n > (|G| - \frac{|G|}{q})m$, therefore for each prime number q dividing $|G|$, there exists a cyclic subgroup of order q, H_q of G such that $\operatorname{ind} A(f,H_q,G) \geq n-(|G|-r)m$.

The following theorem is a version for manifolds of the main result in [8].

Theorem 3.5. *Let G be a finite group which acts freely on n-sphere S^n and let H be a cyclic subgroup of G of prime order p. Let $f : S^n \to M$ be a continuous map where M be a m-manifold (orientable if $p > 2$). If $n > (|G|-r)m$ where $r = \frac{|G|}{p}$, then*

$$\text{cohom.dim}(A(f,H,G)) \geq n-(|G|-r)m.$$

Proof. Since $n > (|G|-r)m \geq m$, $f^*(V_k) = 0$, for all $k \geq 1$. Moreover, $\operatorname{ind} S^n = n$ and thus we apply the Theorem 1.1. $\blacksquare$

3.1 Proof of Theorem 1.2

Now, let us consider X a compact Hausdorff space and an essential map $\varphi : X \to S^n$. Suppose G be a finite group de order s which acts freely on S^n and H be a subgroup of order p of G. Let $G = \{g_1, ..., g_s\}$ be a fixed enumeration of elements of G, where g_1 is the identity of G. A nonempty space X_φ can be associated with the essential map $\varphi : X \to S^n$ as follows:

$$X_\varphi = \{(x_1, ..., x_s) \in X^s : g_i\varphi(x_1) = \varphi(x_i), i = 1, ..., s\},$$

where X^s denotes the s-fold cartesian product of X. The set X_φ is a closed subset of X^s and so it is compact. We define a G-action on X_φ as follows: for each $g_i \in G$ and for each $(x_1, ..., x_s) \in X_\varphi$,

$$g_i(x_1, ..., x_s) = (x_{\sigma_{g_i}(1)}, ..., x_{\sigma_{g_i}(s)}),$$

where the permutation σ_{g_i}, is defined by $\sigma_{g_i}(k) = j$, $g_k g_i = g_j$. We observe that if $x = (x_1, ..., x_s) \in X_\varphi$ then $x_i \neq x_j$, for any $i \neq j$ and therefore G acts freely on X_φ.

Let us consider a continuous map $f : X \to M$, where M is a topological space and $\widetilde{f} : X_\varphi \to M$ given by $\widetilde{f}(x_1, ..., x_s) = f(x_1)$,

Definition 3.6. *The set $A_\varphi(f, H, G)$ of (H, G)-coincidence points of f relative to φ is defined by*

$$A_\varphi(f, H, G) = A(\widetilde{f}, H, G).$$

Proof of Theorem 1.2. Let $\widetilde{f} : X_\varphi \to M$ given by $\widetilde{f}(x_1, ..., x_r) = f(x_1)$, that is, $\widetilde{f} = f \circ \pi_1$, where π_1 is the natural projection on the 1-th coordinate. By hypothesis, $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M, then we have $\widetilde{f}^*(V_k) = 0$, for all $k \geq 1$. Moreover, the $\mathbb{Z}_p$-index of X_φ is equal to n by [11] Theorem 3.1. In this way, X_φ and $\widetilde{f}$ satisfy the hypothesis of Theorem 1.1 which implies that the $\mathbb{Z}_p$-index of the set $A(\widetilde{f}, H, G)$ is greater than or equal to $n - (|G| - r)m$. By definition, $A_\varphi(f, H, G) = A(\widetilde{f}, H, G)$, and then

$$\text{cohom.dim}\, A_\varphi(f, H, G) \geq n - (|G| - r)m. \qquad \blacksquare$$

By a similar argument to that used in the proof of Corollary 3.3 we have the following corollary of Theorem 1.2

Corollary 3.7. *Let X be a compact Hausdorff space and let G be a finite group acting freely on S^n. Let M be a orientable m-manifold and p a prime number dividing $|G|$. Suppose that $n > (|G| - r)m$, where $r = \frac{|G|}{p}$. Then, for a continuous map $f : X \to M$, with $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M, there exists a non-trivial subgroup H of G, such that*

$$\text{cohom.dim}\, A_\varphi(f, H, G) \geq n - (|G| - r)m.$$

4 Topological Tverberg type theorem

The history of Tverberg theorem begins with a Birch's paper (see [2]) which contained the following conjecture

"Any $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned in N subsets whose convex hulls have a common point ".

The Birch's conjecture was proved by Helge Tverberg (see [16]) and since then is known as Tverberg theorem.

We note that the convex hull of $l+1$ points in $\mathbb{R}^d$ is the image of the linear map $\Delta_l \to \mathbb{R}^d$ that maps the $l+1$ vertices of Δ_l to these $l+1$ points. Thus the Tverberg theorem can be reformulated as follows:

Tverberg Theorem. *Let f be a linear map from the N-dimensional simplex Δ_N to $\mathbb{R}^d$. If $N=(d+1)(r-1)$ then there are r disjoint faces of Δ_N whose images have a common point.*

The following conjecture is a generalization of Tverberg Theorem to arbitrary continuous maps.

The topological Tverberg conjecture. *Let f be a continuous map from the N-dimensional simplex Δ_N to $\mathbb{R}^d$. If $N=(d+1)(r-1)$ then there are r disjoint faces of Δ_N whose images have a common point.*

The topological Tverberg conjecture was considered a central unsolved problem of topological combinatorics. For a prime number r the conjecture was proved by Bárány, Shlosman and Szűcs ([1]) and it was extended for a prime power r by Özaydin (unpublished) ([15]) and Volovikov ([19]). This result is known as the *topological Tverberg theorem*. Recently, in [6], Frick presents surprising counterexamples to the topological Tverberg conjecture for any r that is not a power of a prime and dimensions $d \geq 3r+1$ (see also [3]). Although, the conjecture is not true for an integer $r \geq 6$ that is not a prime power, it is possible to prove a weak version of the topological Tverberg conjecture, more precisely, in this paper we show that if r is a natural number with prime factorization $r = p_1^{n_1}\cdots p_k^{n_k}$ then there is, for each $j=1,\ldots,k$, a set with r closed sides mutually disjoint of Δ_N which can be divided into $\dfrac{r}{p_j^{n_j}}$ subsets, each one having $p_j^{n_j}$ elements, whose images have a common point. Specifically, we prove the following *Topological Tverberg type theorem for manifolds and for any integer number r.*

Theorem 4.1. *Let $d \geq 1$ a natural number. Consider a natural number r with prime factorization $r = p_1^{n_1}\cdots p_k^{n_k}$ and set $N=(r-1)(d+1)$. Let $f:\partial\Delta_N \to M$ be a continuous map into a compact d-dimensional topological manifold. If $r=2$, suppose additionally that the modulo 2 degree of the map $f:\partial\Delta_{d+1}\to M$ is equal to zero. Then, for each $j=1,\ldots,k$, among the sides of Δ_N there are $r=q_jr_j$, where $r_j=p_j^{n_j}$, and $q_j=\dfrac{r}{r_j}$, mutually disjoint closed sides $\sigma_{1_1},\ldots,\sigma_{1_{r_j}};\ldots;\sigma_{i_1},\ldots,\sigma_{i_{r_j}};\ldots;\sigma_{q_{j_1}},\ldots,\sigma_{q_{j_{r_j}}}$, such that*

$$f(\sigma_{i_1})\cap\cdots\cap f(\sigma_{i_{r_j}})\neq\emptyset,\ \textit{for each } i=1,\ldots,q_j.$$

Definition 4.2 (Index). *Let p be a prime. We suppose the p-torus $H = \mathbb{Z}_p^k = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (k factors) acting freely on a paracompact space X. The covering $X \to X/H$ is induced from the universal covering $EH \to BH$ by means of a classifying map $c : X/H \to BH$, defined uniquely up to homotopy. We say that the* **index** *of X is greater than or equal to N (abbreviated by $\operatorname{ind} X \geq N$) if $c^* : H^N(BH; \mathbb{Z}_p) \to H^N(X/H; \mathbb{Z}_p)$ is a monomorphism.*

Consider $G = \mathbb{Z}_{p_1}^{n_1} \times \ldots \times \mathbb{Z}_{p_k}^{n_k}$, where $\mathbb{Z}_{p_j}^{n_j} = \mathbb{Z}_{p_j} \times \ldots \times \mathbb{Z}_{p_j}$ (n_j factors), $j = 1, \ldots, k$. We suppose that G acts freely on a paracompact space X.

Lemma 4.3. *Let $f : X \to M$ be a continuous map into a compact d-dimensional topological manifold (orientable for $p_j > 2$). Suppose that the homomorphism $f^* : H^i(M; \mathbb{Z}_{p_j}) \to H^i(X; \mathbb{Z}_{p_j})$ is trivial for $i \geq 1$ and $\operatorname{ind} X \geq N \geq d(r - q_j)$, where $q_j = r/p_j^{n_j}$. Then*

$$\operatorname{ind} A\left(f, \mathbb{Z}_{p_j}^{n_j}, G\right) \geq N - d(r - q_j).$$

Proof. We denote by $a_1, \ldots, a_{q_j}$ a set of representatives of the left lateral classes of $G/\mathbb{Z}_{p_j}^{n_j}$. Consider the map $F : X \to M^{q_j}$ defined by

$$F = (f_1 \times \ldots \times f_{q_j}) \circ D,$$

where $D : X \to X^{q_j}$ is the diagonal map and $f_i : X \to X$ is given by $f_i(x) = f(a_i x)$, $i = 1, \ldots, q_j$.

We have $F^* : H^i(M^{q_j}; \mathbb{Z}_{p_j}) \to H^i(X; \mathbb{Z}_{p_j})$ trivial for $i \geq 1$, therefore the index of $A(F) = \{x \in X : F(x) = F(gx) \, \forall g \in \mathbb{Z}_{p_j}^{n_j}\}$ is greater than or equal to $N - q_j d\left(p_j^{n_j} - 1\right)$ (see [18, Theorem 1]). Since $A(F) \subset A\left(f, \mathbb{Z}_{p_j}^{n_j}, G\right)$ and the inclusion $A(F) \hookrightarrow A\left(f, \mathbb{Z}_{p_j}^{n_j}, G\right)$ is an equivariant map we have $\operatorname{ind} A\left(f, \mathbb{Z}_{p_j}^{n_j}, G\right) \geq A(F)$. Then

$$\operatorname{ind} A\left(f, \mathbb{Z}_{p_j}^{n_j}, G\right) \geq N - d(r - q_j).$$ ∎

Proof of Theorem 4.1. We consider the CW-complex $Y_{N,r}$ that consists of points $(y_1, \ldots, y_r)$, y_i in the boundary $\partial\Delta_N$ of the simplex Δ_N, that have mutually disjoint closed faces. It is known that for all natural numbers r and N, where $N \geq r + 1$, $Y_{N,r}$ is $(N - r)$-connected (see [1]). Let $G = \{g_1, \ldots, g_r\}$ be a fixed enumeration of elements of G. We define a G-action on $Y_{N,r} \subset (\Delta_N)^r$ as follows: for each $g_i \in G$ and for each $(y_1, \ldots, y_r) \in Y_{N,r}$

$$g_i(y_1, ..., y_r) = (y_{\phi_{g_i}(1)}, ..., y_{\phi_{g_i}(r)}),$$

where the permutation ϕ_{g_i}, is defined by $\phi_{g_i}(k) = j$, $g_k g_i = g_j$. Then G acts freely on $Y_{N,r}$, since $Y_{N,r}$ consists of points $(y_1, \ldots, y_r)$, $y_i \in \partial\Delta_N$ that have mutually disjoint closed faces.

Let $\tilde{f} : Y_{N,r} \to M$ given by $\tilde{f}(y_1, \ldots, y_r) = f(y_1)$, that is, $\tilde{f} = f \circ \pi_1$ where $\pi_1 : Y_{N,r} \to \partial\Delta^N$ is the projection on the 1-th coordinate. Since $N = (r-1)(d+1)$

and $Y_{N,r}$ is $(N-r)$-connected, it follows that $\tilde{f}^* : H^i(M;\mathbb{Z}_{p_j}) \to H^i(Y_{N,r};\mathbb{Z}_{p_j})$ is trivial for $i \geq 1$ and $\mathrm{ind}Y_{N,r} \geq (N-r)+1 = d(r-1) > d(r-q_j)$ (if M is non-orientable, we consider the lifting of the map $f : \partial\Delta_N \to M$ to the universal covering space). Then, according to Lemma 4.3, the set $A\left(\tilde{f}, \mathbb{Z}_{p_j}^{n_j}, G\right)$ is not empty, for $j = 1, \ldots, k$.

Let $H = \mathbb{Z}_{p_j}^{n_j} = \{h_1, \ldots, h_{r_j}\}$ be a fixed enumeration of elements of $H = \mathbb{Z}_{p_j}^{n_j} \subset G$. We denote by $a_1, \ldots, a_{q_j}$ a set of representatives of the left lateral classes of $G/\mathbb{Z}_{p_j}^{n_j}$. Then, for each $i = 1, \cdots, q_j$, $a_i h_1^{-1} = g_{i_1}, \ldots, a_i h_{r_j}^{-1} = g_{i_{r_j}}$ are elements of G. Thus, if $y = (y_1, \ldots, y_r) \in A\left(\tilde{f}, \mathbb{Z}_{p_j}^{n_j}, G\right)$,

$$\tilde{f}(g_{i_1} \cdot (y_1, \ldots, y_r)) = \cdots = \tilde{f}(g_{i_{r_j}} \cdot (y_1, \ldots, y_r)),$$

that is,

$$f(y_{\phi_{g_{i_1}}(1)}) = \cdots = f(y_{\phi_{g_{i_{r_j}}}(1)}).$$

Therefore, for each $j = 1, \ldots, k$, among the sides of Δ_N there are $r = q_j r_j$ mutually disjoint closed sides $\{\sigma_{i_1}, ..., \sigma_{i_{r_j}}\}_{i=1}^{q_j}$, such that

$$f(\sigma_{i_1}) \cap \cdots \cap f(\sigma_{i_{r_j}}) \neq \emptyset,$$

for each $i = 1, \ldots, q_j$. ∎

Let us observe that since the d-dimensional Euclidean space $\mathbb{R}^d$ is homeomorphic to the interior of the closed d-dimensional ball, Theorem 4.1 holds also for maps into $\mathbb{R}^d$, and we have the following *weak version of the topological Tverberg conjecture* or *topological Tverberg type theorem for any integer r*.

Theorem 4.4 (Topological Tverberg type theorem for any integer r)**.** *Let $r \geq 2$, $d \geq 1$ be integers and $N = (r-1)(d+1)$. Consider $r = r_1 \ldots r_k$ the prime factorization of r and denote $q_j = r/r_j$, $j = 1, \ldots, k$. Then for any continuous map $f : \Delta_N \to \mathbb{R}^d$, for each $j = 1, \ldots, k$, there are $r = q_j r_j$ pairwise disjoint faces $\{\sigma_{i_1}, ..., \sigma_{i_{r_j}}\}_{i=1}^{q_j}$ such that*

$$f(\sigma_{i_1}) \cap \cdots \cap f(\sigma_{i_{r_j}}) \neq \emptyset, \text{ for each } i = 1, \ldots, q_j.$$

Let us note that if we consider r a prime power in Theorem 4.4, we obtain the topological Tverberg theorem for prime powers.

Now, by Theorem 4.4 and using similar method as in [3], we have the following *Generalized Van Kampen-Flores type theorem for any integer r* or *a weak version of the Generalized Van Kampen-Flores theorem*. In [3, Theorem 4.2], Blagojevic, Frick and Ziegler proved that the Generalized Van Kampen-Flores theorem does not hold in general.

Theorem 4.5 (Generalized Van Kampen-Flores type theorem for any r)**.** *Let $d \geq 1$ a natural number. Consider a natural number r with prime factorization $r = r_1 \cdots r_k$,*

$r_1 < \cdots < r_k$, set $N = (r-1)(d+2)$ and let $l \geq [\frac{r-1}{r_k}d + \frac{2(r-r_k)}{r_k}]$. Let $f : \Delta_N \to \mathbb{R}^d$ be a continuous mapping. Then, there are $r = q_k r_k$ pairwise disjoint faces $\{\sigma_{i_1}, ..., \sigma_{i_{r_k}}\}_{i=1}^{q_k}$ of the l-th skeleton $\Delta_N^{(l)}$, such that

$$f(\sigma_{i_1}) \cap \cdots \cap f(\sigma_{i_{r_k}}) \neq \emptyset, \text{ for each } i = 1, \ldots, q_k.$$

Proof. Let $g : \Delta_N \to \mathbb{R}^{d+1}$ be a continuous function defined by $g(x) = (f(x), \text{dist}(x, \Delta_N^{(l)}))$. Then, we can apply Theorem 4.4 to function g which results in a collection of points

$$x_{1_1}, ..., x_{1_{r_k}}; \ldots; x_{i_1}, ..., x_{i_{r_k}}; \ldots; x_{q_{k1}}, ..., x_{q_{kr_k}},$$

such that $\{x_{i_1}, ..., x_{i_{r_k}}\}_{i=1}^{q_k}$ are points in the pairwise disjoint faces $\{\sigma_{i_1}, ..., \sigma_{i_{r_j}}\}_{i=1}^{q_k}$ with $f(x_{i_1}) = \cdots = f(x_{i_{r_k}})$ and $\text{dist}(x_{i_1}, \Delta_N^{(l)}) = \cdots = \text{dist}(x_{i_{r_k}}, \Delta_N^{(l)})$, for each $i = 1, \ldots, q_k$. We can suppose that all σ_{i_s}'s are inclusion-minimal with the property that $x_{i_s} \in \sigma_{i_s}$, that is, σ_{i_s} is the unique face with x_{i_s} in its relative interior.

Now, for each $i = 1, \ldots, q_k$ fixed, suppose that one of the faces $\sigma_{i_1}, \ldots, \sigma_{i_{r_k}}$ is in $\Delta_N^{(l)}$, e.g. σ_{i_1}. Then $\text{dist}(x_{i_1}, \Delta_N^{(l)}) = 0$, which implies that $\text{dist}(x_{i_1}, \Delta_N^{(l)}) = \cdots = \text{dist}(x_{i_{r_k}}, \Delta_N^{(l)}) = 0$, and consequently, all faces $\sigma_{i_1}, \ldots, \sigma_{i_{r_k}}$ are in $\Delta_N^{(l)}$.

Let us suppose the contrary, that no σ_{i_s} is in $\Delta_N^{(l)}$, i.e., $\dim \sigma_{i_1} \geq l+1, \ldots, \dim \sigma_{i_{r_k}} \geq l+1$. Since the faces $\sigma_{i_1}, \ldots, \sigma_{i_{r_k}}$ are pairwise disjoint we have

$$\begin{aligned} N+1 = |\Delta_N| &\geq |\sigma_{i_1}| + \cdots + |\sigma_{i_{r_k}}| \\ &\geq r_k(l+2) \\ &\geq r_k(\left[\frac{r-1}{r_k}d + \frac{2(r-r_k)}{r_k}\right] + 2) \geq (r-1)(d+2) + 2 = N+2, \end{aligned}$$

which is a contradiction and thus one of the faces $\sigma_{i_1}, \ldots, \sigma_{i_{r_k}}$ is in $\Delta_N^{(l)}$ and consequently all faces $\sigma_{i_1}, \ldots, \sigma_{i_{r_k}}$ are in $\Delta_N^{(l)}$. ∎

Remark 4.6. Let us observe that if we consider r a prime power in Theorem 4.5, we obtain the Generalized Van Kampen-Flores theorem for prime powers proved by Blagojevic, Frick and Ziegler in [3, Theorem 3.2].

Acknowledgement. We would like to express our gratitude to the referee for a careful reading and valuable suggestions which have considerably improved the paper.

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