A remark on the Chow ring of some hyperkähler fourfolds

Robert Laterveer

Abstract

Let X be a hyperkähler variety. Voisin has conjectured that the classes of Lagrangian constant cycle subvarieties in the Chow ring of X should lie in a subring injecting into cohomology. We study this conjecture for the Fano variety of lines on a very general cubic fourfold.

1 Introduction

For a smooth projective variety X over $\mathbb{C}$, let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups (i.e. the groups of codimension i algebraic cycles on X with $\mathbb{Q}$–coefficients, modulo rational equivalence). Let $A^i_{hom}(X)$ denote the subgroup of homologically trivial cycles.

As is well–known, the world of Chow groups is still largely shrouded in mystery, its map containing vast unexplored regions only vaguely sketched in by conjectures [6], [9], [10], [11], [14], [23], [15]. One region on this map that holds particular interest is that of hyperkähler varieties (i.e. projective irreducible holomorphic symplectic manifolds [3], [2]). Here, motivated by results for $K3$ surfaces and for abelian varieties, in recent years significant progress has been made in the understanding of Chow groups [4], [22], [24], [21], [18], [19], [16], [17], [7], [12], [13], [8].

Received by the editors in January 2017.
Communicated by J. Nicaise.
2010 *Mathematics Subject Classification :* Primary 14C15, 14C25, 14C30.
Key words and phrases : Algebraic cycles, Chow groups, motives, Bloch–Beilinson filtration, hyperkähler varieties, Fano variety of lines on cubic fourfold, Beauville's splitting principle, multiplicative Chow–Künneth decomposition, spread of algebraic cycles.

It is expected that for a hyperkähler variety X, the Chow groups split in a finite number of pieces

$$A^i(X) = \bigoplus_j A^i_{(j)}(X) ,$$

such that $A^*_{(*)}(X)$ is a bigraded ring and $A^*_{(0)}(X)$ injects into cohomology. This was first conjectured by Beauville [5], who conjectured more precisely that the piece $A^i_{(j)}(X)$ should be isomorphic to the graded $\mathrm{Gr}^j_F A^i(X)$ for the conjectural Bloch–Beilinson filtration.

What kind of cycles are contained in the subring $A^*_{(0)}(X)$? Certainly divisors and the Chern classes of X should be in this subring. In addition to this, Voisin has stated the following conjecture:

Conjecture 1.1 (Voisin [24]). *Let X be a hyperkähler variety of dimension $2m$.*
(i) Let $Y \subset X$ be a Lagrangian constant cycle subvariety (i.e., $\dim Y = m$ and the pushforward map $A_0(Y) \to A_0(X)$ has image of dimension 1). Then

$$Y \in A^m_{(0)}(X) .$$

(ii) The subring of $A^*(X)$ containing divisors, Chern classes and Lagrangian constant cycle subvarieties injects into cohomology.

(NB: part (ii) follows from part (i), provided the bigrading $A^*_{(*)}(X)$ has the desirable property that $A^*_{(0)}(X) \cap A^*_{hom}(X) = 0$, which is expected from the Bloch–Beilinson conjectures.)

Evidence for conjecture 1.1 is presented in [24]. The modest aim of this note is to determine how far conjecture 1.1 can be solved unconditionally in the special case where X is the Fano variety of lines on a cubic fourfold. Here, the *Fourier decomposition* of Shen–Vial [18] provides an unconditional splitting $A^*_{(*)}(X)$ of the Chow ring. The main result is as follows:

Proposition (=proposition 3.1). *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let X be the Fano variety of lines in Z. Assume $Y \subset X$ is a Lagrangian constant cycle subvariety. Then*

$$Y \in A^2_{(0)}(X)$$

(where $A^*_{(*)}(X)$ denotes the Fourier decomposition of [18]).

This doesn't settle conjecture 1.1(ii) (because it is not known whether $A^2_{(0)}(X) \cap A^2_{hom}(X) = 0$). However, this at least implies some statements along the lines of conjecture 1.1(ii):

Corollary (=corollaries 4.2 and 4.1). *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let X be the Fano variety of lines in Z.*
(i) Let $a \in A^3(X)$ be a 1–cycle of the form

$$a = \sum_{i=1}^r Y_i \cdot D_i \quad \in A^3(X) ,$$

where Y_i is a Lagrangian constant cycle subvariety and $D_i \in A^1(X)$. Then a is rationally trivial if and only if a is homologically trivial.
(ii) Let $a \in A^4(X)$ be a 0–cycle of the form

$$a = \sum_{i=1}^{r} Y_i \cdot b_i \quad \in A^4(X) ,$$

where Y_i is a Lagrangian constant cycle subvariety and $b_i \in A^2(X)$. Then a is rationally trivial if and only if a is homologically trivial.

Conventions. *In this article, the word* variety *will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A* subvariety *is a (possibly reducible) reduced subscheme which is equidimensional.*

All Chow groups will be with rational coefficients*: we will denote by $A_j(X)$ the Chow group of j–dimensional cycles on X with $\mathbb{Q}$–coefficients; for X smooth of dimension n the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably.*

The notations $A^j_{hom}(X)$, $A^j_{AJ}(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles.
We use $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$.

2 Preliminaries

2.1 The Fourier decomposition

Theorem 2.1 (Shen–Vial [18])**.** *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold, and let X be the Fano variety of lines in Z. There is a decomposition*

$$A^i(X) = \bigoplus_{\substack{0\le j\le i \\ j \text{ even}}} A^i_{(j)}(X) ,$$

with the following properties:
(i) $A^i_{(j)}(X) = (\Pi^X_{2i-j})_* A^j(X)$, where $\{\Pi^X_*\}$ is a certain self–dual Chow–Künneth decomposition;
(ii) $A^i_{(j)}(X) \subset A^i_{hom}(X)$ for $j > 0$;
(iii) if Z is very general, $A^*_{(*)}(X)$ is a bigraded ring.

Proof. The decomposition is defined in terms of a Fourier transform, involving the cycle $L \in A^2(X \times X)$ representing the Beauville–Bogomolov class (cf. [18, Theorem 2]). Points (i) and (ii) follow from [18, Theorem 3.3]. Point (iii) is [18, Theorem 3]. ∎

2.2 Multiplicative structure

Theorem 2.2 (Shen–Vial [18]). *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold, and let X be the Fano variety of lines in Z. There is a distinguished class $l \in A^2_{(0)}(X)$ such that intersection induces an isomorphism*

$$\cdot l: \quad A^2_{(2)}(X) \xrightarrow{\cong} A^4_{(2)}(X)\ .$$

The inverse isomorphism is given by

$$\frac{1}{25} L_*: \quad A^4_{(2)}(X) \xrightarrow{\cong} A^2_{(2)}(X)\ ,$$

where $L \in A^2(X \times X)$ is the class defined in [18, Equation (107)].

Proof. This follows from [18, Theorems 2.2 and 2.4]. ∎

2.3 The class c

Lemma 2.3 (Voisin [21], Shen–Vial [18]). *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold, and let X be the Fano variety of lines in Z. Let $c := c_2(\mathcal{E}_2) \in A^2(X)$, where $\mathcal{E}_2$ is the restriction to X of the tautological rank 2 vector bundle on the Grassmannian of lines in $\mathbb{P}^5(\mathbb{C})$. There exists a constant cycle surface $Y_0 \subset X$ such that*

$$Y_0 = c \quad \textit{in } A^2(X)\ .$$

(In particular, $\cdot c\colon A^2_{hom}(X) \to A^4(X)$ is the zero–map.)

Moreover, if Z is very general then the class c is in $A^2_{(0)}(X)$ (where $A^*_{(*)}(X)$ is the Fourier decomposition of [18]).

Proof. This is well–known. As explained in [21, Lemma 3.2], the idea is to consider $Y \subset X$ defined as the Fano surface of lines contained in $Z \cap H$, where H is a hyperplane in $\mathbb{P}^5$. For general H, the surface Y is a smooth surface of general type which is a Lagrangian subvariety of class c in $A^2(X)$. However, if one takes H such that $Z \cap H$ acquires 5 nodes, then one obtains a singular surface Y_0 which is rational, hence $A_0(Y_0) = \mathbb{Q}$. It follows that $Y_0 \subset X$ is a constant cycle subvariety of class c in $A^2(X)$.

The last statement is [18, Theorem 21.9(iii)]. ∎

2.4 A result in cohomology

Definition 2.4 (Voisin [24]). *Let X be a hyperkähler variety of dimension $2m$. A Hodge class $a \in H^{2m}(X) \cap F^m$ is* coisotropic *if*

$$\cup a: \quad H^{2,0}(X) \to H^{m+2,m}(X)$$

is the zero–map.

(This is [24, Definition 1.5], where coisotropic cohomology classes are defined in any degree $2i$.)

Proposition 2.5. *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let X be the Fano variety of lines in Z. Assume $a \in H^4(X)$ is coisotropic. Then*

$$a = \lambda \cdot c \quad \textit{in } H^4(X)\ ,$$

where $\lambda \in \mathbb{Q}$ and $c \in A^2(X)$ is as in lemma 2.3.

Proof. For very general Z, it is known that $N^2H^4(X)$ (which is the subspace of Hodge classes, as the Hodge conjecture is known for X) has dimension 2. This is all that we need for the proof.

For any ample class $g \in A^1(X)$, the $\mathbb{Q}$–vector space $N^2H^4(X)$ is generated by g^2 and c. (These two elements cannot be proportional, as cupping with g^2 induces an isomorphism $H^{2,0}(X) \cong H^{4,2}(X)$ by hard Lefschetz, whereas cupping with c is the zero–map $H^{2,0}(X) \to H^{4,2}(X)$.) Let us write

$$a = \lambda_1 c + \lambda_2 g^2 \quad \text{in } N^2H^4(X)\ .$$

The coisotropic condition forces λ_2 to be 0, and we are done. ■

Remark 2.6. *In particular, proposition 2.5 implies that any Lagrangian subvariety $Y \subset X$ is proportional to c in cohomology:*

$$Y = \lambda \cdot c \quad \textit{in } H^4(X)\ .$$

This was first observed by Amerik [1, Remark 9].

3 Main result

Proposition 3.1. *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let X be the Fano variety of lines in Z. Assume $Y \subset X$ is a constant cycle subvariety of codimension 2. Then*

$$Y \in A^2_{(0)}(X)\ .$$

Proof. We assume there is a decomposition

$$Y = b_0 + b_2 \quad \text{in } A^2_{(0)}(X) \oplus A^2_{(2)}(X)\ ,$$

with $b_i \in A^2_{(i)}(X)$. We will show that b_2 must be 0.

First, we claim that

$$Y \cdot a \quad \in A^4_{(0)}(X) \quad \forall a \in A^2(X)\ . \tag{1}$$

Indeed, the subvectorspace $Y \cdot A^2(X) \subset A^4(X)$ has dimension 1, as $Y \subset X$ is a constant cycle subvariety. To prove (1), it remains to exclude the possibility that

$$\big(Y \cdot A^2(X)\big) \cap A^4_{(0)}(X) = 0\ .$$

But we know (proposition 2.5) that

$$Y = \lambda c \quad \text{in } H^4(X) ,$$

for some $\lambda \in \mathbb{Q}^*$. Since $c \in A^2_{(0)}(X)$, this implies there is a further decomposition

$$Y = \lambda c + b_0' + b_2 \quad \text{in } A^2(X) ,$$

with $b_0' \in A^2_{(0)}(X) \cap A^2_{hom}(X)$ (which is conjecturally, but not provably, zero). Consider the intersection

$$Y \cdot c = \lambda c^2 + b_0' \cdot c + b_2 \cdot c = \lambda c^2 \quad \text{in } A^4(X) .$$

(Here we have used that $c \cdot A^2_{hom}(X) = 0$ in $A^4(X)$, which is lemma 2.3 or [18, Lemma A.3(iii)].) Since $c^2 = 27\mathfrak{o}_X$ where $\mathfrak{o}_X$ is a certain distinguished generator of $A^4_{(0)}(X)$ [18, Lemma A.3(i)], the intersection $Y \cdot c$ defines a non–zero element in $A^4_{(0)}(X)$. This proves the claim.

To prove the proposition, consider the intersection

$$Y \cdot \ell = b_0 \cdot \ell + b_2 \cdot \ell \quad \text{in } A^4(X) ,$$

where ℓ is the class of theorem 2.2. Since $\ell \in A^2_{(0)}(X)$ and $A^*_{(*)}(X)$ is a bigraded ring, we have that $b_i \cdot \ell \in A^4_{(i)}(X)$. It follows from (1) that $Y \cdot \ell \in A^4_{(0)}(X)$ and so

$$b_2 \cdot \ell = 0 \quad \text{in } A^4_{(2)}(X) .$$

But then, applying theorem 2.2, we find that $b_2 = 0$ and we are done. ■

Remark 3.2. *Let X be the Fano variety of a very general cubic fourfold. We have seen (proposition 2.5) that any Lagrangian constant cycle subvariety Y is proportional to the class c in cohomology. Proposition 3.1 suggests that the same should be true modulo rational equivalence: indeed, Y is proportional to c in $A^2(X)$ modulo the "troublesome part" $A^2_{(0)}(X) \cap A^2_{hom}(X)$ (which is conjecturally zero).*

4 Corollaries

We present three corollaries that provide weak versions of conjecture 1.1(ii).

Corollary 4.1. *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let X be the Fano variety of lines in Z. Let $a \in A^4(X)$ be a 0–cycle of the form*

$$a = \sum_{i=1}^{r} Y_i \cdot b_i \quad \in A^4(X) ,$$

where Y_i is a Lagrangian constant cycle subvariety and $b_i \in A^2(X)$. Then a is rationally trivial if and only if a is homologically trivial.

Proof. We know from claim (1) that a is in $A^4_{(0)}(X)$. But $A^4_{(0)}(X) \cong \mathbb{Q}$ injects into cohomology. ■

Corollary 4.2. *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let X be the Fano variety of lines in Z. Let $a \in A^3(X)$ be a 1–cycle of the form*

$$a = \sum_{i=1}^{r} Y_i \cdot D_i \quad \in A^3(X) ,$$

where Y_i is a Lagrangian constant cycle subvariety and $D_i \in A^1(X)$. Then a is rationally trivial if and only if a is homologically trivial.

Proof. We know from proposition 3.1 that each Y_i is in $A^2_{(0)}(X)$. Since $D_i \in A^1(X) = A^1_{(0)}(X)$, it follows that a is in $A^3_{(0)}(X)$. But we know [18] that

$$A^3_{(0)}(X) \cap A^3_{hom}(X) = 0 .$$

■

Corollary 4.3. *Let $Z \subset \mathbb{P}^5(\mathbb{C})$ be a very general smooth cubic fourfold, and let X be the Fano variety of lines in Z. Let $\phi \colon X \dashrightarrow X$ be the degree 16 rational map defined in [20]. Let $a \in A^2(X)$ be a 2–cycle of the form*

$$a = \phi^*(b) - 4b \quad \in A^2(X) ,$$

where b is a linear combination of Lagrangian constant cycle subvarieties and intersections of 2 divisors. Then a is rationally trivial if and only if a is homologically trivial.

Proof. We know from proposition 3.1 that b is in $A^2_{(0)}(X)$. Let V^2_λ denote the eigenspace

$$V^2_\lambda := \{\alpha \in A^2(X) \mid \phi^*(\alpha) = \lambda \cdot \alpha\} .$$

Shen–Vial have proven that there is a decomposition

$$A^2_{(0)}(X) = V^2_{31} \oplus V^2_{-14} \oplus V^2_4$$

[18, Theorem 21.9]. The "troublesome part" $A^2_{(0)}(X) \cap A^2_{hom}(X)$ is contained in V^2_4 [18, Lemma 21.12]. This implies that

$$(\phi^* - 4(\Delta_X)^*) A^2_{(0)}(X) = V^2_{31} \oplus V^2_{-14}$$

injects into cohomology. ■

Acknowledgements. *Thanks to Jiji and Baba for kindly receiving me in quiet Sayama, where this note started its life. Thanks to Charles Vial for helpful email exchanges.*

References

[1] K. Amerik, A computation of invariants of a rational self–map, Ann. Fac. Sci. Toulouse 18 no. 3 (2009), 481—493,

[2] A. Beauville, Some remarks on Kähler manifolds with $c_1 = 0$, in: Classification of algebraic and analytic manifolds (Katata, 1982), Birkhäuser Boston, Boston 1983,

[3] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Differential Geom. 18 no. 4 (1983), 755—782,

[4] A. Beauville and C. Voisin, On the Chow ring of a $K3$ surface, J. Alg. Geom. 13 (2004), 417—426,

[5] A. Beauville, On the splitting of the Bloch–Beilinson filtration, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), London Math. Soc. Lecture Notes 344, Cambridge University Press 2007,

[6] S. Bloch, Lectures on algebraic cycles, Duke Univ. Press Durham 1980,

[7] L. Fu, On the action of symplectic automorphisms on the CH_0–groups of some hyper-Kähler fourfolds, Math. Z. 280 (2015), 307—334,

[8] L. Fu, Z. Tian and C. Vial, Motivic hyperkähler resolution conjecture for generalized Kummer varieties, arXiv:1608.04968,

[9] U. Jannsen, Motivic sheaves and filtrations on Chow groups, in: Motives (U. Jannsen et alii, eds.), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1,

[10] U. Jannsen, On finite–dimensional motives and Murre's conjecture, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), Cambridge University Press, Cambridge 2007,

[11] S. Kimura, Chow groups are finite dimensional, in some sense, Math. Ann. 331 (2005), 173—201,

[12] H. Lin, On the Chow groups of zero–cycles of a generalized Kummer variety, Advances in Mathematics 298 (2016), 448—472,

[13] H. Lin, Lagrangian constant cycle subvarieties in Lagrangian fibrations, arXiv:1510.01437,

[14] J. Murre, On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II, Indag. Math. 4 (1993), 177—201,

[15] J. Murre, J. Nagel and C. Peters, Lectures on the theory of pure motives, Amer. Math. Soc. University Lecture Series 61, Providence 2013,

[16] U. Rieß, On the Chow ring of birational irreducible symplectic varieties, Manuscripta Math. 145 (2014), 473—501,

[17] U. Rieß, On Beauville's conjectural weak splitting property, to appear in Int. Math. Res. Notes,

[18] M. Shen and C. Vial, The Fourier transform for certain hyperKähler fourfolds, Memoirs of the AMS 240 (2016), no.1139,

[19] C. Vial, On the motive of some hyperkähler varieties, to appear in J. für Reine u. Angew. Math.,

[20] C. Voisin, Intrinsic pseudo-volume forms and K-correspondences, in: The Fano Conference (A.Collino et al., eds.), Univ. Torino, Turin 2004,

[21] C. Voisin, On the Chow ring of certain algebraic hyperkähler manifolds, Pure Appl. Math. Q. 4 no. 3 part 2 (2008), 613—649,

[22] C. Voisin, Chow rings and decomposition theorems for $K3$ surfaces and Calabi–Yau hypersurfaces, Geom. Topol. 16 (2012), 433—473,

[23] C. Voisin, Chow Rings, Decomposition of the Diagonal, and the Topology of Families, Princeton University Press, Princeton and Oxford, 2014,

[24] C. Voisin, Remarks and questions on coisotropic subvarieties and 0–cycles of hyper–Kähler varieties, in: K3 Surfaces and Their Moduli, Proceedings of the Schiermonnikoog conference 2014 (C. Faber et al., eds.), Progress in Maths 315, Birkhäuser 2016,

Institut de Recherche Mathématique Avancée, CNRS
Université de Strasbourg,
7 Rue René Descartes, 67084 Strasbourg CEDEX, FRANCE.
email: robert.laterveer@math.unistra.fr