Generalized CAT(0) spaces

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Abstract

We extend the Gromov geometric definition of CAT(0) spaces to the case where the comparison triangles are not in the Euclidean plane but belong to a general Banach space. In particular, we study the case where the Banach space is ℓ_p, for $p > 2$.

1 introduction

A metric space X is said to be a CAT(0) space (the term is due to M. Gromov-see, e.g., [2], page 159) if it is geodesically connected, and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include the classical hyperbolic spaces, Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [10]), and many others. (On the other hand, if a Banach space is a CAT(κ) space for some $\kappa \in \mathbb{R}$, then it is necessarily a Hilbert space and CAT(0).) For a thorough discussion of these spaces and of the fundamental role they play in geometry, see Bridson and Haefliger [2]. Burago et al. [5] present a somewhat more elementary treatment, and Gromov [12] a deeper study.

The recent uptick in interest in CAT(0) spaces is due to Kirk's fixed point results [14] discovered in CAT(0) spaces. Since then many people working in metric fixed point theory discovered some interesting results following Kirk's work. For us, CAT(0) spaces are simply a nonlinear version of the Hilbert Banach spaces.

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In fact, it is easy to check that most of the metric fixed results obtained in CAT(0) are almost identical to the ones discovered in Hilbert spaces. This fact motivated our curiosity to investigate if it is possible to build CAT(0) spaces which can be seen as a nonlinear version of spaces like ℓ_p for example. In this work, we extend Gromov's definition and assume that the comparison triangle is in a more general Banach space other than the Euclidean plane. To the best of our knowledge, it is the first time such generalization has been offered. We obtained some interesting results when the Banach space is the classical sequence space ℓ_p, with $p > 2$.

2 Basic Definitions and Properties

Let (X, d) be a metric space. A continuous mapping from the interval $[0, 1]$ to X is called a *path*. A path $\gamma : [0, 1] \to X$ is called a *geodesic* if $d(\gamma(s), \gamma(t)) = |s - t| d(\gamma(0), \gamma(1))$, for every $s, t \in [0, 1]$. We will say that (X, d) is a *geodesic space* if every two points $x, y \in X$ are connected by a geodesic, i.e., there exists a geodesic $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. In this case, we denote such geodesic by $[x, y]$. Note that in general such geodesic is not uniquely determined by its endpoints. For a point $z \in [x, y]$, we will use the notation $z = (1 - t)x \oplus ty$, where $t = d(x, z)/d(x, y)$ assuming $x \neq y$. The metric space (X, d) is called *uniquely geodesic* if every two points of X are connected by a unique geodesic. In this case $[x, y]$ will denote the unique geodesic connecting x and y in X.

The most fundamental examples of geodesic spaces are normed vector spaces, complete Riemannian manifolds, and polyhedral complexes of piecewise constant curvature. In the last two cases the existence of geodesic paths is not so obvious; determining when such spaces are uniquely geodesic is also a non-trivial matter. The case of normed vector spaces is much easier [2].

For a real number $\kappa \in \mathbb{R}$, let M_κ^2 denote the unique simply connected surface (real 2-dimensional Riemannian manifold) with constant curvature κ. Denote D_κ the diameter of M_κ, which is $+\infty$ if $\kappa \leq 0$ and $\pi/\sqrt{\kappa}$ for $\kappa > 0$. Recall that $M_0^2 = \mathbb{R}^2$. A *geodesic triangle* $\Delta\,(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for a geodesic triangle $\Delta\,(x_1, x_2, x_3)$ in X is a triangle $\overline{\Delta}\,(x_1, x_2, x_3) := \Delta\,(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_κ^2 such that $d\left(\bar{x}_i, \bar{x}_j\right) = d\left(x_i, x_j\right)$ for $i, j \in \{1, 2, 3\}$. A point $\overline{x} \in [\overline{x_1}, \overline{x_2}]$ is called a comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = d(\overline{x_1}, \overline{x})$.

Definition 2.1. [2] Let X be a metric space and let κ be a real number. Let Δ be a geodesic triangle in X and $\overline{\Delta} \subset M_\kappa^2$ be a comparison triangle for Δ. Then, Δ is said to satisfy the $CAT(\kappa)$ inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\overline{x}, \overline{y}).$$

If $\kappa \leq 0$, then X is called a $CAT(\kappa)$ space if X is a geodesic space all of whose geodesic triangles satisfy the $CAT(\kappa)$ inequality. If $\kappa > 0$, then X is called a $CAT(\kappa)$ space if all geodesic triangles in X of perimeter less than twice the diameter D_κ of M_κ^2 satisfy the $CAT(\kappa)$ inequality.

Complete $CAT\,(0)$ spaces are often called *Hadamard spaces* (see [14]). Let (X,d) be a $CAT\,(0)$ space. Let x, y_1, y_2 be in X, and $\frac{y_1 \oplus y_2}{2}$ is the midpoint of the segment $[y_1, y_2]$, then the $CAT\,(0)$ inequality implies:

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2\left(x, y_1\right) + \frac{1}{2}d^2\left(x, y_2\right) - \frac{1}{4}d^2\left(y_1, y_2\right).$$

This inequality is the (CN) inequality of Bruhat and Tits [4].

Strictly convex Banach spaces are obviously uniquely geodesic. As for a normed vector space to be a $CAT(\kappa)$ space, for some $\kappa \in \mathbb{R}$, we have the following result:

Theorem 2.1. *([2], Proposition 1.14.) If a normed real vector space $(X, \|.\|)$ is $CAT(\kappa)$, for some $\kappa \in \mathbb{R}$, then it is a pre-Hilbert space.*

3 Generalized $CAT(0)$ spaces

In the definition of a $CAT(0)$ metric space, the comparison triangle is a subset of the Euclidean vector space $\mathbb{R}^2$. One may wonder what structure and properties one may get if we allow the comparison triangles to belong to some vector normed space $\mathbb{E}$.

Definition 3.1. Let (X,d) be a geodesic metric space and $(\mathbb{E}, \|.\|)$ be a normed vector space. X is said to be a generalized $CAT(0)$ space if for any geodesic triangle Δ in X, there exists a comparison triangle $\overline{\Delta}$ in $\mathbb{E}$ such that the comparison axiom is satisfied, i.e., for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, we have

$$d(x,y) \leq \|\overline{x} - \overline{y}\|.$$

It is obvious to see that the normed vector space $(\mathbb{E}, \|.\|)$ is itself a generalized $CAT(0)$ space according to Definition 3.1. But according to Theorem 2.1 $\mathbb{E}$ is a $CAT(\kappa)$, for some $\kappa \in \mathbb{R}$, if and only if $(\mathbb{E}, \|.\|)$ is a pre-Hilbert space. In other words, our definition gives a new class of $CAT(0)$ metric spaces provided $(\mathbb{E}, \|.\|)$ is not a pre-Hilbert vector space. As an example, we take $\mathbb{E} = \ell_p$, for $p > 2$.

Definition 3.2. Let (X,d) be a geodesic metric space. X is said to be a $CAT_p(0)$ space, for $p > 2$, if for any geodesic triangle Δ in X, there exists a comparison triangle $\overline{\Delta}$ in ℓ_p such that the comparison axiom is satisfied, i.e., for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, we have

$$d(x,y) \leq \|\overline{x} - \overline{y}\|.$$

It is obvious that ℓ_p, for $p > 2$, is a $CAT_p(0)$ space. We suspect that a nonlinear example will be given by the open unit ball of ℓ_p endowed with the Kobayashi distance [15]. Next we discuss some of the properties of $CAT_p(0)$ metric spaces.

Theorem 3.1. *Let (X,d) be a $CAT_p(0)$ metric space, with $p \geq 2$. Then for any x, y_1, y_2 in X, we have*

$$d^p\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^p(x, y_1) + \frac{1}{2}d^p(x, y_2) - \frac{1}{2^p}d^p(y_1, y_2), \tag{3.1}$$

which we will call the (CN_p) inequality.

Proof. Let x, y_1, y_2 be in X and Δ be the associated geodesic triangle in X. Since X is a $CAT_p(0)$ space, there exists a comparison geodesic triangle $\overline{\Delta}$ in ℓ_p, with $p \geq 2$. The associated comparison points in ℓ_p will be denoted by $\overline{x}, \overline{y}_1$ and $\overline{y}_2$. The comparison axiom implies:

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \left\| \overline{x} - \frac{\overline{y}_1 + \overline{y}_2}{2} \right\|,$$

which implies

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^p \leq \left\| \overline{x} - \frac{\overline{y}_1 + \overline{y}_2}{2} \right\|^p.$$

Recall what is known as the Clarkson's inequality [8] in ℓ_p:

$$\|x+y\|^p + \|x-y\|^p \leq 2^{p-1}\left(\|x\|^p + \|y\|^p\right), \tag{3.2}$$

for any x, y in l_p, for $p \geq 2$. Applying this inequality for $a = \dfrac{\overline{x} - \overline{y}_1}{2}$ and $b = \dfrac{\overline{x} - \overline{y}_2}{2}$, yields:

$$\left\| \frac{\overline{x} - \overline{y}_1}{2} + \frac{\overline{x} - \overline{y}_2}{2} \right\|^p + \left\| \frac{\overline{x} - \overline{y}_1}{2} - \frac{\overline{x} - \overline{y}_2}{2} \right\|^p \leq 2^{p-1}\left(\left\| \frac{\overline{x} - \overline{y}_1}{2} \right\|^p + \left\| \frac{\overline{x} - \overline{y}_2}{2} \right\|^p \right).$$

Or,

$$\left\| \overline{x} - \frac{\overline{y}_1 + \overline{y}_2}{2} \right\|^p \leq \frac{1}{2}\|\overline{x} - \overline{y}_1\|^p + \frac{1}{2}\|\overline{x} - \overline{y}_2\|^p - \frac{1}{2^p}\|\overline{y}_1 - \overline{y}_2\|^p.$$

Hence,

$$d^p\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}\|\overline{x} - \overline{y}_1\|^p + \frac{1}{2}\|\overline{x} - \overline{y}_2\|^p - \frac{1}{2^p}\|\overline{y}_1 - \overline{y}_2\|^p.$$

Since $\|\bar{x} - \bar{y}_j\| = d\left(x, y_j\right)$, for $j \in \{1, 2\}$, we get

$$d^p\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^p(x, y_1) + \frac{1}{2}d^p(x, y_2) - \frac{1}{2^p}d^p(y_1, y_2).$$

■

Note that the (CN_p) inequality coincides with the classical (CN) inequality if $p = 2$. One of the implications of the (CN) inequality is the uniform convexity of the distance of a $CAT(0)$ space. Next we discuss the case of uniform convexity of the $CAT_p(0)$ metric spaces.

Definition 3.3. Let (X, d) be a uniquely geodesic metric space. We say that X is uniformly convex (in short, UC) if and only if

$$\delta(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\, d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right); d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon\right\} > 0,$$

for any $a \in X$, for every $r > 0$, and for each $\varepsilon > 0$.

The definition of uniform convexity finds its origin in Banach spaces [8]. To the best of our knowledge, the first attempt to generalize this concept to metric spaces was done in [11], also see [10, 16, 20]. A direct consequence of the (CN_p) inequality is the following result:

Theorem 3.2. *Any $CAT_p(0)$ metric space, with $p \geq 2$, is uniformly convex. Moreover we have*

$$\delta(r, \varepsilon) \geq 1 - \left(1 - \frac{\varepsilon^p}{2^p}\right)^{1/p},$$

for every $r > 0$ and for each $\varepsilon > 0$.

The Banach spaces ℓ_p, $p > 1$, are not only uniformly convex but they have a geometric property known as p-uniform convexity (see [1] p. 310). Theorem 3.2 implies that $CAT_p(0)$ metric spaces enjoy the p-uniform convexity as well.
Next we discuss the behavior of type functions in $CAT_p(0)$ metric spaces. It is worth mentioning that these functions are very useful when one needs to prove the existence of fixed points of mappings. A subset C of a $CAT_p(0)$ metric space (X, d) is said to be convex whenever $[x, y] \subset C$, for any $x, y \in C$. Recall that $\tau : X \to \mathbb{R}_+$ is called a type if there exists a bounded sequence $\{x_n\}$ in X such that

$$\tau(x) = \limsup_{n\to\infty} d(x, x_n).$$

Theorem 3.3. *Let (X, d) be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let C be any nonempty, closed, convex and bounded subset of X. Let τ be a type defined on C. Then any minimizing sequence of τ is convergent. Its limit z is the unique minimum of τ and satisfies*

$$\tau^p(z) + \frac{1}{2^{p-1}}\, d^p(z, x) \leq \tau^p(x), \tag{3.3}$$

for any $x \in C$.

Proof. Let $\{x_n\}$ be a sequence in C such that $\tau(x) = \limsup_{n\to\infty} d(x_n, x)$. Denote $\tau_0 = \inf\{\tau(x); x \in C\}$. Let $\{y_k\}$ be a minimizing sequence of τ. Since C is bounded, there exists $R > 0$ such that $d(x, y) \leq R$ for any $x, y \in C$. Since (X, d) is a $CAT_p(0)$ metric space, Theorem 3.1 implies

$$d^p\left(x_n, \frac{y_m \oplus y_k}{2}\right) \leq \frac{1}{2}d^p(x_n, y_m) + \frac{1}{2}d^p(x_n, y_k) - \frac{1}{2^p}d^p(y_m, y_k),$$

for any $n, m, k \in \mathbb{N}$. If we let n goes to infinity, we get

$$\tau^p\left(\frac{1}{2}y_m \oplus \frac{1}{2}y_k\right) \leq \frac{1}{2}\tau^p(y_k) + \frac{1}{2}\tau^p(y_m) - \frac{1}{2^p}d^p(y_m, y_k),$$

which implies

$$\tau_0^p \leq \frac{1}{2}\tau^p(y_k) + \frac{1}{2}\tau^p(y_m) - \frac{1}{2^p}d^p(y_m, y_k),$$

or

$$\frac{1}{2^p}d^p(y_m, y_k) \leq \frac{1}{2}\tau^p(y_k) + \frac{1}{2}\tau^p(y_m) - \tau_0^p,$$

for any $k, m \geq 1$. Since $\{y_n\}$ is a minimizing sequence of τ, we conclude that

$$\lim_{k,m\to\infty} d(y_m, y_k) = 0,$$

i.e., the sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete, $\{y_n\}$ converges to some point $z \in C$. Since τ is continuous, we get $\tau_0 = \tau(z)$. Next we show the inequality (3.3). Let $x \in C$. The (CN_p) inequality implies

$$d^p\left(\frac{1}{2}x \oplus \frac{1}{2}z, x_n\right) \leq \frac{1}{2}d^p(x, x_n) + \frac{1}{2}d^p(z, x_n) - \frac{1}{2^p}d^p(x, z),$$

for any n. Hence

$$\limsup_{n\to\infty} d^p\left(\frac{1}{2}x \oplus \frac{1}{2}z, x_n\right) \leq \frac{1}{2}\limsup_{n\to\infty} d^p(x, x_n) + \frac{1}{2}\limsup_{n\to\infty} d^p(z, x_n) - \frac{1}{2^p}d^p(x, z).$$

The definition of z implies that

$$\limsup_{n\to\infty} d^p(z, x_n) \leq \limsup_{n\to\infty} d^p\left(\frac{1}{2}x \oplus \frac{1}{2}z, x_n\right).$$

Hence

$$\frac{1}{2}\limsup_{n\to\infty} d^p(z, x_n) \leq \frac{1}{2}\limsup_{n\to\infty} d^p(x, x_n) - \frac{1}{2^p}d^p(x, z),$$

which implies the desired inequality. ■

Note that the inequality (3.3) is similar to the Opial condition for Banach spaces [19].

4 Application: a fixed point theorem

In this Section we discuss the existence of fixed points of uniformly Lipschitzian mappings defined on a $CAT_p(0)$ metric space.

Definition 4.1. Let (X, d) be a metric space. Let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *Lipschitzian* if there exists a non-negative number k such that $d(T(x), T(y)) \leq k\, d(x, y)$ for all x and y in C. The smallest such k is called *Lipschitz constant* and will be denoted by $Lip(T)$. The mapping T is called *uniformly Lipschitzian* if $\sup_{n\geq 1} Lip(T^n) < \infty$. A point $x \in C$ is said to be a fixed point of T whenever $T(x) = x$. The set of fixed points of T will be denoted by $Fix(T)$.

It is well-known that if a map is uniformly Lipschitzian, then one may find an equivalent distance for which the mapping is nonexpansive (see [10, pp. 34–38]). Indeed, let $T : C \to C$ be uniformly Lipschitzian. Denote

$$\rho(x, y) = \sup\{d(T^n(x), T^n(y)),\ n = 0, 1, \cdots\}$$

for all $x, y \in C$, one can obtain a metric ρ on C which is equivalent to the metric d and relative to which T is nonexpansive. In this context, the following question

naturally arises: if a set C has the fixed point property (fpp) for nonexpansive mappings with respect to the metric d, then does C also have (fpp) for mappings which are nonexpansive relative to an equivalent metric? This is known as the stability of (fpp). The first result in this direction is due to Goebel and Kirk [9]. Motivated by such questions, we investigate the fixed point property of uniformly Lipschitzian mappings in $CAT_P(0)$, for $p \geq 2$.
Let (X, d) be a $CAT_p(0)$, $p \geq 2$. Define the normal structure coefficient $N(X)$ (see [7]) by :

$$N(X) = \inf \frac{\operatorname{diam}(C)}{R(C)},$$

where the infimum is taken over all C bounded convex nonempty subset of X not reduced to one point. Recall that $\operatorname{diam}(C) = \sup\{d(x, y);\ x, y \in C\}$ is the diameter of C, and $R(C) = \inf\left\{ \sup_{y \in C} d(x, y);\ x \in C \right\}$ is the Chebyshev radius of C. Note that since X satisfies the property (R) (see [13]), then for any C bounded convex closed nonempty subset of X, there exists $x \in C$ such that $R(C) = \sup_{y \in C} d(x, y)$. It is easy to check that $N(X) \leq 2$. Using the (CN_p) inequality, we can show that

$$N(X) \geq \left(1 - \frac{1}{2^p}\right)^{-1/p} > 1.$$

The main result of this section is similar to [21, Theorem 3].

Theorem 4.1. *Let (X, d) be complete $CAT_p(0)$, $p \geq 2$, metric space. Let C be a nonempty, closed, convex and bounded subset of X. Let $T : C \to C$ be uniformly Lipschitzian with*

$$\lambda(T) = \sup_{n \geq 1}\ Lip(T^n) < \left(\frac{1 + \sqrt{1 + 8(N(X)/2)^p}}{2}\right)^{1/p}.$$

Then T has a fixed point.

Proof. Fix $x_0 \in C$. Using an induction argument, we will construct a sequence $\{x_m\}$ in C such that x_{m+1} is the point z found in Theorem 3.3 associated with the sequence $\{T^n(x_m)\}$, for any $m \geq 0$. For any $m \geq 0$, denote

$$r_m = \limsup_{n \to \infty}\ d(x_{m+1}, T^n(x_m)) \text{ and } R_m = \sup_{n \geq 1}\ d(x_m, T^n(x_m)).$$

Set $C^* = \overline{conv}(\{T^n(x_m); n \geq 1\}$. From the properties of $CAT_p(0)$ metric spaces, there exists $z \in C^*$ such that $R(C^*) = \sup_{x \in C^*} d(x, z)$. In particular, we have

$$\sup_{n \geq n_0} d(z, T^n(x_m)) \leq \frac{1}{N(X)} \operatorname{diam}(C^*) = \frac{1}{N(X)} \operatorname{diam}(\{T^n(x_m); n \geq 1\}),$$

for any $n_0 \geq 1$. Since $r_m \leq \limsup_{n \to \infty}\ d(z, T^n(x_m))$ and

$$\operatorname{diam}(\{T^n(x_m); n \geq 1\}) \leq \lambda(T)\ \sup_{n \geq 1} d(x_m, T^n(x_m)),$$

we get

$$r_m \leq \frac{\lambda(T)}{N(X)}\, R_m,\; m = 1, \cdots .$$

This result is similar to Theorem 1 of [17]. Using Theorem 3.3, we get

$$r_m^p + \frac{1}{2^p}\, d^p(x_{m+1}, T^s(x_{m+1})) \leq \frac{1}{2} r_m^p + \frac{1}{2} \limsup_{n\to\infty}\; d^p(T^s(x_{m+1}), T^n(x_m)),$$

which implies that

$$r_m^p + \frac{1}{2^p}\, d^p(x_{m+1}, T^s(x_{m+1})) \leq \frac{1}{2} r_m^p + \frac{\lambda(T)^p}{2} \limsup_{n\to\infty}\; d^p(x_{m+1}, T^{n-s}(x_m)),$$

or

$$r_m^p + \frac{1}{2^p}\, d^p(x_{m+1}, T^s(x_{m+1})) \leq \frac{1}{2} r_m^p + \frac{\lambda(T)^p}{2} r_m^p.$$

Hence

$$\frac{1}{2^p} R_{m+1}^p = \frac{1}{2^p}\; \sup_{s\geq 1} d^p(x_{m+1}, T^s(x_{m+1})) \leq \frac{\lambda(T)^p - 1}{2} r_m^p \leq \frac{(\lambda(T)^p - 1)}{2} \frac{\lambda(T)^p}{N(X)^p} R_m^p,$$

which implies that $R_{m+1} \leq A\; R_m$, for any $m \geq 1$, where

$$A = \left(\frac{(\lambda(T)^p - 1)\lambda(T)^p}{2(N(X)/2)^p}\right)^{1/p}.$$

Our assumption on $\lambda(T)$ leads to $A < 1$. Since $R_m \leq A^{m-1}\, R_1$, for any $m \geq 1$, we conclude that $\sum_{m\geq 1} R_m$ is convergent. Since $d(x_m, x_{m+1}) \leq r_m + R_m \leq 2R_m$, for any $m \geq 1$, the series $\sum d(x_m, x_{m+1})$ is also convergent, and therefore $\{x_m\}$ is Cauchy. Since X is complete, $\{x_m\}$ converges to some point $z \in C$. Since

$$d(z, T(z)) \leq d(z, x_m) + d(x_m, T(x_m)) + d(T(x_m), T(z)),$$

we get $d(z, T(z)) \leq (1 + Lip(T))d(z, x_m) + R_m$, for any $m \geq 1$. If we let $m \to \infty$, then we get $d(z, T(z)) = 0$, i.e., $T(z) = z$. ∎

As a corollary, we obtain the following result:

Theorem 4.2. *Let (X, d) be complete $CAT_p(0)$, $p \geq 2$, metric space. Let C be a nonempty, closed, convex and bounded subset of X. Let $T : C \to C$ be a nonexpansive mapping. Then T has a fixed point.*

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