Compact perturbations resulting in hereditarily polaroid operators

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Abstract

A Banach space operator $A \in B(\mathcal{X})$ is polaroid, $A \in (\mathcal{P})$, if the isolated points of the spectrum $\sigma(A)$ are poles of the operator; A is hereditarily polaroid, $A \in (\mathcal{HP})$, if every restriction of A to a closed invariant subspace is polaroid. It is seen that operators $A \in (\mathcal{HP})$ have SVEP - the single-valued extension property - on $\Phi_{sf}(A) = \{\lambda : A - \lambda \text{ is semi Fredholm }\}$. Hence $\Phi^+_{sf}(A) = \{\lambda \in \Phi_{sf}(A), \text{ind}(A-\lambda) > 0\} = \emptyset$ for operators $A \in (\mathcal{HP})$, and a necessary and sufficient condition for the perturbation $A+K$ of an operator $A \in B(\mathcal{X})$ by a compact operator $K \in B(\mathcal{X})$ to be hereditarily polaroid is that $\Phi^+_{sf}(A) = \emptyset$. A sufficient condition for $A \in B(\mathcal{X})$ to have SVEP on $\Phi_{sf}(A)$ is that its component $\Omega_a(A) = \{\lambda \in \Phi_{sf}(A) : \text{ind}(A-\lambda) \leq 0\}$ is connected. We prove: If $A \in B(\mathcal{H})$ is a Hilbert space operator, then a necessary and sufficient condition for there to exist a compact operator $K \in B(\mathcal{H})$ such that $A + K \in (\mathcal{HP})$ is that $\Omega_a(A)$ is connected.

1. Introduction

Let $B(\mathcal{X})$ (resp., $B(\mathcal{H})$) denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Banach (resp., Hilbert) space into itself. For an operator $A \in B(\mathcal{X})$, let iso$\sigma(A)$ denote the isolated points of the spectrum $\sigma(A)$, let asc(A) (resp., dsc(A)) denote the ascent (resp., descent) of A and let $A - \lambda$ denote $A - \lambda I$. A point $\lambda \in$ iso$\sigma(A)$ is a pole (of the resolvent)

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of A, equivalently A is polar at λ, if $\text{asc}(A-\lambda) = \text{dsc}(A-\lambda) < \infty$. The operator A is *polaroid* if it is polar at every $\lambda \in \text{iso}\sigma(A)$, and it is *hereditarily polaroid* if every restriction $A|_M$ of A to an (always closed) invariant subspace M of A is polaroid. Polaroid operators, and their perturbation by commuting compact perturbations, have been studied by a number of authors in the recent past (see [2, 3, 7, 8, 17] for a sample). For example, if $N \in B(\mathcal{X})$ is a nilpotent operator which commutes with $A \in B(\mathcal{X})$, then A is polaroid if and only if $A + N$ is polaroid [8, Theorem 2.6(b)]. This however does not extend to non-nilpotent quasinilpotent commuting operators: Consider for example the trivial operator $A = 0 \in B(\mathcal{X})$ and a non-nilpotent quasinilpotent $Q \in B(\mathcal{X})$. The perturbation of a polaroid operator by a compact operator may or may not effect the polaroid property of the operator. For example, if $U \in B(\mathcal{H})$ is the forward unilateral shift, $A = U \oplus U^*$ and K is the compact operator $K = \begin{pmatrix} 0 & 1-UU^* \\ 0 & 0 \end{pmatrix}$, then both A and $A+K$ are polaroid (for the reason that $\text{iso}\sigma(A) = \emptyset$ and $A + K$ is a unitary); trivially the identity operator 1 is polaroid, but its perturbation $1 + Q$ by a compact quasinilpotent operator is not polaroid.

An interesting problem, recently considered by Li and Zhou [17], is the following: Given an operator $A \in B(\mathcal{H})$, do there exist compact operators $K_0, K \in B(\mathcal{H})$ such that (i) $A + K_0$ is polaroid and (ii) $A + K$ is not polaroid. The answer to both these problems is an emphatic "yes" (see [17, Theorems 1.4 and 1.5]). The argument used to prove these results ties up with the work of Herrero and his co-authors [11, 12, 13, 4], Ji [15], and Zhu and Li [19]. A natural extension of this problem is the question of whether there exist compact operators $K_0, K \in B(\mathcal{H})$ such that (i)′ $A + K_0$ is hereditarily polaroid and (ii)′ $A + K$ is not hereditarily polaroid. Here the answer to (ii)′ is a "yes" [17, Theorem 5.2], but there is caveat to the answer to (i)′ - the answer is "yes if the set $\Phi^+_{sf}(A) = \{\lambda \in \sigma(A) : A - \lambda$ is semi- Fredholm and $\text{ind}(A-\lambda) > 0\} = \emptyset$". The authors of [17] leave the problem of a straight "yes or no" answer to (i)′ open. This note considers this problem to prove that if $A, K \in B(\mathcal{X})$ with K compact, then $A + K$ hereditarily polaroid implies $\Phi^+_{sf}(A) = \emptyset$. Indeed, we prove that if $A \in B(\mathcal{H})$, then there exists a compact $K \in B(\mathcal{H})$ such that $A + K$ is hereditarily polaroid if and only if A has SVEP, *the single-valued extension property*, on $\Phi_{sf}(A)$. A sufficient condition for operators $A \in B(\mathcal{X})$ to have SVEP on $\Phi_{sf}(A)$ is that the component $\Omega_a(A) = \{\lambda \in \Phi_{sf}(A) : \text{ind}(A-\lambda) \leq 0\}$ is connected. We prove that for an operator $A \in B(\mathcal{H})$, a necessary and sufficient condition for there to exist a compact operator $K \in B(\mathcal{H})$ such that $A + K \in (\mathcal{HP})$ is that $\Omega_a(A)$ is connected.

2. Complementary results

We start by introducing our notation and terminology. We shall denote the class of polaroid operators by $(\mathcal{P})$ and the subclass of hereditarily polaroid operators by $(\mathcal{HP})$. The boundary of a subset S of the set $\mathbb{C}$ of complex numbers will be denoted by ∂S. An operator $A \in B(\mathcal{X})$ has SVEP, *the single-valued extension property*, at a point $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \longrightarrow \mathcal{X}$ satisfying $(A-\lambda)f(\lambda) = 0$ is the function $f \equiv 0$. (Here, as before, we have shortened $A - \lambda I$, equivalently $A - \lambda 1$, to $A - \lambda$.) Evidently, every A has SVEP at points in the resolvent $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. *We say that T has SVEP on a set S if it has SVEP at every $\lambda \in S$.* The *ascent of A*, $\operatorname{asc}(A)$ (resp. *descent of A*, $\operatorname{dsc}(A)$), is the least non-negative integer n such that $A^{-n}(0) = A^{-(n+1)}(0)$ (resp., $A^n(\mathcal{X}) = A^{n+1}(\mathcal{X})$): If no such integer exists, then $\operatorname{asc}(A)$, resp. $\operatorname{dsc}(A)$, $= \infty$. It is well known that $\operatorname{asc}(A) < \infty$ implies A has SVEP at 0, $\operatorname{dsc}(A) < \infty$ implies A^* (= the dual operator) has SVEP at 0, finite ascent and descent for an operator implies their equality, and that a point $\lambda \in \sigma(A)$ is a pole (of the resolvent) of A if and only if $\operatorname{asc}(A-\lambda) = \operatorname{dsc}(A-\lambda) < \infty$ (see [1, 14, 16]).

An operator $A \in B(\mathcal{X})$ is: *upper semi–Fredholm at $\lambda \in \mathbb{C}$, $\lambda \in \Phi_{uf}(A)$* or $A - \lambda \in \Phi_{uf}(\mathcal{X})$, if $(A-\lambda)(\mathcal{X})$ is closed and the deficiency index $\alpha(A-\lambda) = \dim((A-\breve{\ })^{-1}(0)) < \infty$; *lower semi–Fredholm at $\lambda \in \mathbb{C}$, $\lambda \in \Phi_{lf}(A)$* or $A - \lambda \in \Phi_{lf}(\mathcal{X})$, if $\beta(A-\lambda) = \dim(\mathcal{X}/(A-\breve{\ })(\mathcal{X})) < \infty$. A is semi–Fredholm, $\lambda \in \Phi_{sf}(A)$ or $A - \lambda \in \Phi_{sf}(\mathcal{X})$, if $A-\lambda$ is either upper or lower semi–Fredholm, and A is Fredholm, $\lambda \in \Phi(A)$ or $A - \lambda \in \Phi(\mathcal{X})$, if $A - \lambda$ is both upper and lower semi–Fredholm. The index of a semi–Fredholm operator is the integer, possibly infinite, $\operatorname{ind}(A) = \alpha(A) - \beta(A)$. Corresponding to these classes of one sided Fredholm operators, we have the following spectra: The *upper Fredholm spectrum* $\sigma_{uf}(A)$ of A defined by $\sigma_{uf}(A) = \{\lambda \in \sigma(A) : A - \lambda \notin \Phi_{uf}(\mathcal{X})\}$, and the *lower Fredholm spectrum* $\sigma_{lf}(A)$ of A defined by $\sigma_{lf}(A) = \{\lambda \in \sigma(A) : A - \lambda \notin \Phi_{lf}(\mathcal{X})\}$. The *Fredholm spectrum* $\sigma_f(A)$ of A is the set $\sigma_f(A) = \sigma_{uf}(A) \cup \sigma_{lf}(A)$, and the *Wolf spectrum* $\sigma_{ulf}(A)$ of A is the set $\sigma_{ulf}(A) = \sigma_{uf}(A) \cap \sigma_{lf}(A)$. $A \in B(\mathcal{X})$ is Weyl (at 0) if it is Fredholm with $\operatorname{ind}(A) = 0$. It is well known that a semi- Fredholm operator A (resp., its conjugate operator A^*) has SVEP at a point λ if and only if $\operatorname{asc}(A-\lambda) < \infty$ (resp., $\operatorname{dsc}(A-\lambda) < \infty$) [1, Theorems 3.16, 3.17]; furthermore, if $A-\lambda$ is Weyl , i.e., if $\lambda \in \Phi(A)$ and $\operatorname{ind}(A-\lambda) = 0$, then A has SVEP at λ implies $\lambda \in \operatorname{iso}\sigma(A)$ with $\operatorname{asc}(A-\lambda) = \operatorname{dsc}(A-\lambda) < \infty$. The Weyl (resp., the upper or approximate Weyl) spectrum of A is the set

$$\sigma_w(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_f(A) \text{ or } \operatorname{ind}(A-\lambda) \neq 0\}$$
$$(\sigma_{aw}(A) = \{\lambda \in \sigma_a(A) : \lambda \in \sigma_{uf}(A) \text{ or } \operatorname{ind}(A-\lambda) > 0\}).$$

The Browder (resp., the upper or approximate Browder) spectrum of A is the set

$$\sigma_b(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_f(A) \text{ or } \operatorname{asc}(A-\lambda) \neq \operatorname{des}(A-\breve{\ })\}$$
$$(\sigma_{ab}(A) = \{\lambda \in \sigma_a(A) : \lambda \in \sigma_{uf}(A) \text{ or } \operatorname{asc}(A-\lambda) = \infty\}.$$

Clearly, $\sigma_f(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A) \subseteq \sigma(A)$ and $\sigma_{uf}(A) \subseteq \sigma_{aw}(A) \subseteq \sigma_{ab}(A) \subseteq \sigma(A)$.

An operator $A \in B(\mathcal{X})$ is B-Fredholm (resp., upper B-Fredholm), $A \in \Phi_{Bf}(\mathcal{X})$ (resp., $\Phi_{uBf}(\mathcal{X})$), if there exists an integer $n \geq 1$ such that $A^n(\mathcal{X})$ is closed and the induced operator $A_{[n]} = A|_{A^n(\mathcal{X})}$, $A_{[0]} = A$, is Fredholm (resp., upper semi Fredholm) in the usual sense. It is seen that if $A_{[n]} \in \Phi_{sf}(\mathcal{X})$ for an integer $n \geq 1$, then $A_{[m]} \in \Phi_{sf}(\mathcal{X})$ for all integers $m \geq n$: One may thus define unambiguously the index of A by $\text{ind}(A) = \alpha(A) - \beta(A)$ (see [6, 3, 5]). The B-Fredholm (resp., the upper B-Fredholm) spectrum of A is the set

$$\sigma_{Bf}(A) = \{\lambda \in \sigma(A) : \lambda \notin \Phi_{Bf}(A)\}$$
$$(\sigma_{uBf}(A) = \{\lambda \in \sigma(A) : \lambda \notin \Phi_{uBf}(A)\})$$

and the B-Weyl (resp.,upper or approximate B-Weyl) spectrum of A is the set

$$\sigma_{Bw}(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_{Bf}(A) \text{ or } \text{ind}(A-\lambda) \neq 0\}$$
$$(\sigma_{uBw}(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_{uBf}(A) \text{ or } \text{ind}(A-\lambda) > 0\}).$$

It is clear that $\sigma_{Bw}(A) \subseteq \sigma_w(A)$ and $\sigma_{uBw}(A) \subseteq \sigma_{aw}(A)$.

Let $H_0(A)$ and $K(A)$ denote, respectively, the *quasinilpotent part*

$$H_0(A) = \{x \in \mathcal{X} : \lim_{n\to\infty} ||A^n x||^{\frac{1}{n}} = 0\}$$

and the *analytic core*

$$K(A) = \{x \in \mathcal{X} : \text{ there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which } x = x_0, Ax_{n+1} = x_n \text{ and } ||x_n|| \leq \delta^n ||x|| \text{ for all } n = 0, 1, 2, \cdots\}$$

of A. It is well known, [1], that $(A-\lambda)^{-p}(0) \subseteq H_0(A-\lambda)$, for all integers $p \geq 1$, and $(A-\lambda)K(A-\lambda) = K(A-\lambda)$ for all complex λ. A necessary and sufficient condition for $\lambda \in \text{iso}\sigma(A)$ to be a pole of A is that $H_0(A-\lambda) = (A-\lambda)^{-p}(0)$ for some integer $p \geq 1$: This is seen as follows. If $\lambda \in \text{iso}\sigma(A)$, then (by the Riesz representation theorem [1, 14])

$$\begin{aligned} & \mathcal{X} = H_0(A-\lambda) \oplus K(A-\lambda) = (A-\lambda)^{-p}(0) \oplus K(A-\lambda) \\ \Longrightarrow\ & (A-\lambda)^p(\mathcal{X}) = 0 \oplus (A-\lambda)^p K(A-\lambda) = K(A-\lambda) \\ \Longrightarrow\ & \mathcal{X} = (A-\lambda)^{-p}(0) \oplus (A-\lambda)^p(\mathcal{X}) \\ \Longrightarrow\ & \lambda \text{ is a pole of order } p \text{ of } A \\ \Longrightarrow\ & H_0(A-\lambda) = (A-\lambda)^{-p}(0). \end{aligned}$$

For every $\lambda \notin \sigma_{Bw}(A)$ such that A has SVEP at λ, $\text{asc}(A-\lambda) < \infty$ (implying thereby that there exists an integer $p \geq 1$ such that $H_0(A-\lambda) = (A-\lambda)^{-p}(0)$) and $\lambda \in \text{iso}\sigma(A)$ [9]. Hence A has SVEP at $\lambda \notin \sigma_{Bw}(A)$ implies λ is a pole of A.

If $\mathcal{X} = \mathcal{H}$ is a Hilbert space, and $A \in B(\mathcal{H})$ is such that $\lambda \in \Phi_{sf}(A)$, then *the minimal index of* $A - \lambda$ is the integer

$$\min\{\alpha(A-\lambda), \beta(A-\lambda)\} = \min\{\alpha(A-\lambda), \alpha(A-\lambda)^*\}.$$

It is well known that the function $\lambda \to \text{min.ind}(A-\lambda)$ is constant on every component of $\Phi_{sf}(A)$ (except perhaps for a denumerable subset without limit points in $\Phi_{sf}(A)$) [11, Corollary 1.14]

3. Results

Given $A \in B(\mathcal{X})$, the *reduced minimum modulus function* $\gamma(A)$ is the function

$$\gamma(A) = \inf_{x \notin A^{-1}(0)}\{\frac{||Ax||}{\text{dist}(x, A^{-1}(0))}\},$$

where $\gamma(A) = \infty$ if $A = 0$. Recall that $A(\mathcal{X})$ is closed if and only if $\gamma(A) > 0$. Let $\sigma_p(A)$ (resp., $\sigma_a(A)$) denote the point spectrum (resp., the approximate point spectrum) of the operator A, and let $\text{acc}\sigma(A)$ denote the set of accumulation points of $\sigma(A)$.

Theorem 3.1. *If, for an operator $A \in B(\mathcal{X})$, there exists a compact operator $K \in B(\mathcal{X})$ such that $A + K \in (\mathcal{HP})$, then $A + K$ has SVEP at points $\lambda \in \Phi_{sf}(A)$. Consequently, $\Phi^+_{sf}(A) = \{\lambda \in \Phi_{sf}(A) : ind(A - \lambda) > 0\} = \emptyset$.*

Proof. Suppose to the contrary that $A + K$ does not have SVEP at a point $\lambda \in \Phi_{sf}(A) = \Phi_{sf}(A + K)$. Recall from [1, Theorem 3.23] that if an operator $T \in B(\mathcal{X})$ has SVEP at a point $\mu \in \Phi_{sf}(T)$, then $\mu \in \text{iso}\sigma_a(T)$. Hence, since $A + K$ does not have SVEP at $\lambda \in \Phi_{sf}(A + K)$, we must have $\lambda \in \text{acc}\sigma_p(A + K)$. Consequently, there exists a sequence $\{\lambda_n\} \subset \sigma_p(A + K)$ of non-zero eigenvalues of $A + K$ converging to λ. Choose α, $\beta \in \{\lambda_n\}$, and let M denote the subspace generated by the eigenvectors $(A + K - \alpha)^{-1}(0) \cup (A + K - \beta)^{-1}(0)$. Then $A_1 = (A + K)|_M$ is a polaroid operator with $\sigma(A_1) = \{\alpha, \beta\}$, which implies that $(A_1 - \alpha)^{-1}(0)$ and $(A_1 - \beta)^{-1}(0)$ are mutually orthogonal spaces (in the sense of G. Birkhoff: A subspace M of $\mathcal{X}$ is orthogonal to a subspace N of $\mathcal{X}$ if $||m|| \leq ||m + n||$ for every $m \in M$ and $n \in N$ [10, P. 93]). Now choose a $\lambda_m \in \{\lambda_n\}$. Then the mutual orthogonality of the eigenspaces corresponding to distinct (non -trivial) eigenvalues implies

$$\text{dist}(x, (A + K - \lambda)^{-1}(0)) \geq 1$$

for every unit vector $x \in (A + K - \lambda_m)^{-1}(0)$. Define $\delta(\lambda_m, \lambda)$ by

$$\delta(\lambda_m, \lambda) = \sup\{\text{dist}(x, (A + K - \lambda)^{-1}(0)) : x \in (A + K - \lambda_m)^{-1}(0), ||x|| = 1\}.$$

Then $\delta(\lambda_m, \lambda) \geq 1$ for all m, and

$$\frac{|\lambda_m - \lambda|}{\delta(\lambda_m, \lambda)} \longrightarrow 0 \text{ as } m \to \infty,$$

i.e., the reduced minimum modulus function satisfies

$$\gamma(A + K - \lambda) = \frac{|\lambda_m - \lambda|}{\delta(\lambda_m, \lambda)} \longrightarrow 0 \text{ as } m \to \infty.$$

Since this implies $(A + K - \lambda)(\mathcal{X})$ is not closed, we have a contradiction (of our assumption $\lambda \in \Phi_{sf}(A + K)$). Hence $A + K$ has SVEP at every $\lambda \in \Phi_{sf}(A + K) = \Phi_{sf}(A)$. The fact that $\Phi^+_{sf}(A) = \emptyset$ is now a straightforward consequence of "A has SVEP at $\lambda \in \Phi_{sf}(A)$ implies $\text{ind}(A - \lambda) \leq 0$". ∎

It is well known (indeed, easily proved) that

$$A \in B(\mathcal{X}) \cap (\mathcal{P}) \Longleftrightarrow A^* \in B(\mathcal{X}^*) \cap (\mathcal{P}).$$

This equivalence does not extend to $(\mathcal{HP})$ operators. To see this, consider an operator $A \in B(\mathcal{X})$ such that both A and A^* are in $(\mathcal{HP})$. Then both A and A^* have SVEP at points in $\Phi_{sf}(A)$ $(= \Phi_{sf}(A^*))$ by the preceding theorem. Hence, for every such operator A,

$$\lambda \in \Phi_{sf}(A) \Longrightarrow \lambda \in \Phi_w(A) = \{\lambda : \lambda \in \Phi(A), \operatorname{ind}(A-\lambda) = 0\}.$$

But then $\lambda \in \Phi_{sf}(A) \cap \sigma(A)$ is (an isolated point of $\sigma(A)$ which happens to be) a finite rank pole of A. That this is (in general) false follows from a consideration of the forward unilateral shift $U \in B(\mathcal{H})$ (which is trivially $(\mathcal{HP})$ and satisfies $\lambda \in \Phi_{sf}(U)$ for all $|\lambda| < 1$).

We consider next a sufficient condition for $A + K \in (\mathcal{P})$ to imply $A + K \in (\mathcal{HP})$. If $A \in B(\mathcal{X})$ and M is an invariant (assumed, as before, to be closed) subspace of A, then A has an upper triangular matrix representation

$$A = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix} \in B(M \oplus M^{\perp})$$

with main diagonal (A_1, A_2). Generally, $\sigma(A) \subseteq \sigma(A_1) \cup \sigma(A_2)$ and $\sigma_w(A) \subseteq \sigma_w(A_1) \cup \sigma_w(A_2)$: Indeed,

$$\begin{aligned} &\sigma(A_1) \cup \sigma(A_2) = \sigma(A) \cup \{\sigma(A_1) \cap \sigma(A_2)\} \text{ and} \\ &\sigma_w(A_1) \cup \sigma_w(A_2) = \sigma_w(A) \cup \{\sigma_w(A_1) \cap \sigma_w(A_2)\}. \end{aligned}$$

Recall from [18, Exercise 7, P. 293] that

$$\operatorname{asc}(A_1 - \lambda) \leq \operatorname{asc}(A - \lambda) \leq \operatorname{asc}(A_1 - \lambda) + \operatorname{asc}(A_2 - \lambda)$$

for every complex λ; hence, if $H_0(A - \lambda) = (A - \lambda)^{-p}(0)$ for a $\lambda \in \{\sigma(A) \cap \sigma(A_1)\}$ (and some integer $p \geq 1$), then

$$\begin{aligned} H_0(A_1 - \lambda) &= H_0(A-\lambda)|_M \subseteq (A-\lambda)^{-p}(0) \cap M \\ &= (A_1 - \lambda)^{-p}(0) \subseteq H_0(A_1 - \lambda); \end{aligned}$$

consequently

$$H_0(A_1 - \lambda) = (A_1 - \lambda)^{-p}(0)$$

(with $\operatorname{asc}(A_1 - \lambda) \leq p$).

Let $\Pi_0(A)$ denote the set of Riesz points (i.e., finite rank poles), and let $\Pi(A)$ denote the set of poles, of $A \in B(\mathcal{X})$. If A has SVEP on the complement of $\sigma_w(A)$ in $\sigma(A)$, then

$$\sigma(A) \setminus \sigma_w(A) = \Pi_0(A) \Longleftrightarrow \sigma(A) \setminus \sigma_{Bw}(A) = \Pi(A)$$

[5, Theorem 2.1].

Theorem 3.2. *If, for an operator $A \in B(\mathcal{X})$, there exists a compact operator $K \in B(\mathcal{X})$ such that $A+K \in (\mathcal{P})$ and if $\sigma(A+K) \setminus \sigma_w(A+K) = \Pi_0(A+K)$, then a sufficient condition for $A+K \in (\mathcal{HP})$ is that:*
(i) $\sigma_w(A_1) \cup \sigma_w(A_2) \subseteq \sigma_w(A+K)$ for every invariant subspace M of $A+K$ such that $(A+K)|_M = A_1$ (and (A_1, A_2) is the main diagonal in the upper triangular representation of $A+K \in B(M \oplus M^{\perp})$);
(ii) $\text{iso}\sigma_w(A_1) \subseteq \text{iso}\sigma_w(A)$.

Proof. We claim that $\sigma(A+K) = \sigma(A_1) \cup \sigma(A_2)$. To prove the claim, we start by combining hypothesis (i) with the observation that $\sigma_w(A+K) \subseteq \sigma_w(A_1) \cup \sigma_w(A_2)$ for every upper triangular operator with main diagonal (A_1, A_2) to obtain $\sigma_w(A+K) = \sigma_w(A_1) \cup \sigma_w(A_2)$. Consider a complex $\lambda \notin \sigma(A+K)$. Since

$$A+K-\lambda = \begin{pmatrix} 1 & 0 \\ 0 & A_2-\lambda \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1-\lambda & 0 \\ 0 & 1 \end{pmatrix},$$

$A_1-\lambda$ is left invertible, $A_2-\lambda$ is right invertible, $\alpha(A_1-\lambda) = 0 = \beta(A_2-\lambda)$ and $\text{ind}(A+K-\lambda) = (\text{ind}(A_1-\lambda) + \text{ind}(A_2-\lambda) = 0 \Longrightarrow) \beta(A_1-\lambda) = \alpha(A_2-\lambda)$. If $\beta(A_1-\lambda) \neq 0$, then $\lambda \in \sigma_w(A_1) \cup \sigma_w(A_2) = \sigma_w(A+K)$, a contradiction (since $\lambda \notin \sigma(A+K)$). Consequently, $\beta(A_1-\lambda) = \alpha(A_2-\lambda) = 0$, and hence $\lambda \notin \sigma(A_1) \cup \sigma(A_2)$. This proves our claim. Consider now a $\lambda \in \text{iso}\sigma(A_1)$. Then either $\lambda \notin \sigma_{Bw}(A_1)$ or $\lambda \in \sigma_{Bw}(A_1)$. If $\lambda \notin \sigma_{Bw}(A_1)$ and $\lambda \in \text{iso}\sigma(A_1)$, then λ is a pole of A_1. If, instead, $\lambda \in \sigma_{Bw}(A_1) \subseteq \sigma_w(A_1)$, then $\lambda \in \text{iso}\sigma_w(A_1) \subseteq \text{iso}\sigma_w(A) = \text{iso}\sigma_w(A+K)$. Since $\sigma_w(A+K) = \sigma(A+K) \setminus \Pi_0(A+K)$, $\lambda \in \text{iso}\sigma_w(A+K)$ implies $\lambda \in \text{iso}\sigma(A+K)$ and $\lambda \notin \Pi_0(A+K)$. Again, since $A+K \in (\mathcal{P})$, $\lambda \in \Pi(A+K)$ (is a pole of $A+K$ of infinite multiplicity), and there exists an integer $p > 0$ such that $H_0(A+K-\lambda) = (A+K-\lambda)^{-p}(0)$. But then, as seen above, $H_0(A_1-\lambda) = (A_1-\lambda)^{-p}(0)$, and hence λ is a pole of A_1. This contradiction implies $\lambda \notin \sigma_{Bw}(A_1)$, and $A \in (\mathcal{HP})$. ∎

Remark 3.3. The hypothesis $\sigma_w(A_1) \cup \sigma_w(A_2) \subseteq \sigma_w(A+K)$ is not necessary in Theorem 3.2. For example, if $\sigma_w(A_1) \subseteq \sigma_w(A)$, then $\lambda \notin \sigma(A+K)$ implies $\alpha(A_1-\lambda) = 0 = \beta(A_2-\lambda)$ and $\beta(A_1-\lambda) = \alpha(A_2-\lambda)$; hence, since $\lambda \notin \sigma_w(A_1)$, $\beta(A_1-\lambda) = \alpha(A_2-\lambda) = 0$. Consequently, $\sigma(A+K) = \sigma(A_1) \cup \sigma(A_2)$. The hypothesis $\sigma(A+K) = \sigma(A_1) \cup \sigma(A_2)$ on its own does not guarantee $A+K \in (\mathcal{HP})$ in Theorem 3.2. Let $R \in B(\mathcal{X})$ be a Riesz operator and let $Q \in B(\mathcal{X})$ be a compact quasinilpotent operator. Define $A \in B(\mathcal{X} \oplus \mathcal{X})$ by $A = R \oplus 0$, and let $K = 0 \oplus Q$. Then $A+K$ is a Riesz operator. Since the restriction of a Riesz operator to an invariant subspace is again a Riesz operator [14, 1], $\sigma_w(A_1) \subseteq \sigma_w(A+K)$ for every part (i.e., restriction an invariant subspace) $A_1 = (A+K)|_M$ of $A+K$. Hence $\sigma(A+K) = \sigma(A_1) \cup \sigma(A_2)$ for every upper triangular representation, with main diagonal (A_1, A_2), of $A+K$. Evidently, $A+K \in (\mathcal{P})$. Observe however that $\text{iso}\sigma_w(A_1) \subseteq \text{iso}\sigma_w(A+K)$ fails for the (upper triangular matrix) representation $A+K = Q \oplus A$ of $A+K$. Clearly, $A+K \notin (\mathcal{HP})$. In the presence of the hypothesis $\sigma(A+K) = \sigma(A_1) \cup \sigma(A_2)$, a sufficient condition for $A_1 \in (\mathcal{HP})$ is (of course) that $\text{iso}\sigma(A_1) \cap \text{acc}\sigma(A) = \emptyset$.

Hilbert Space Operators. Given an operator $A \in B(\mathcal{H})$, there always exists a compact operator $K \in B(\mathcal{H})$ satisfying (the hypotheses of Theorem 3.2 that) $\sigma(A+K) \setminus \sigma_w(A+K) = \Pi_0(A+K)$ and $A+K \in (\mathcal{P})$: This follows from the following familiar (see [11, 19, 17]) argument. Every $A \in B(\mathcal{H})$ has an upper triangular matrix representation

$$A = \begin{pmatrix} A_0 & * \\ 0 & A_1 \end{pmatrix} \in B(\mathcal{H}_0 \oplus \mathcal{H}_1),\ \mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0,\ \sigma(A_0) = \Pi_0(A),$$
$$\sigma(A_1) = \sigma(A) \setminus \Pi_0(A).$$

Consider $A_1 \in B(\mathcal{H}_1)$. If we define d by $d = \max\{\text{dist}(\lambda, \partial\Phi_{sf}(A_1) : \lambda \in \Pi_0(A)\} < \epsilon/2$ (for some arbitrarily small $\epsilon > 0$), then there exists a compact operator $K_1 \in B(\mathcal{H}_1)$, $||K_1|| < \epsilon/2 + d < \epsilon$, such that $\text{min.ind}(A_1 + K_1 - \lambda) = 0$ for all $\lambda \in \Phi_{sf}(A_1)$ and $\sigma(A_1 + K_1) = \sigma_w(A_1)$ [11, Theorem 3.48]. Let $A_{11} = A_1 + K_1$. Then $\lambda \in \text{iso}\sigma_w(A_1)$ and $\lambda \notin \sigma_{ulf}(A_{11})$ implies $\lambda \in \Pi_0(A)$; hence $\text{iso}\sigma(A_{11}) \cap \sigma_{ulf}(A_{11}) \neq \emptyset$. Let $(\emptyset \neq)\ \Gamma \subset \{\text{iso}\sigma(A_{11}) \cap \sigma_{ulf}(A_{11})\}$. Then, for every $\epsilon > 0$, there exists a compact operator $K_{11} \in B(\mathcal{H}_1)$, $||K_{11}|| < \epsilon$, such that

$$A_{11} + K_{11} = \begin{pmatrix} N & C \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{H}_{11} \oplus \mathcal{H}_{12}),\ \ \mathcal{H}_{11} = \mathcal{H}_1 \ominus \mathcal{H}_{12},\ \ \dim(\mathcal{H}_{11}) = \infty,$$

N is a diagonal normal operator of uniform infinite multiplicity, $\sigma(N) = \sigma_{ulf}(N) = \Gamma, \sigma(A_2) = \sigma(A_{11}), \sigma_{ulf}(A_2) = \sigma_{ulf}(A_{11}), \text{ind}(A_2 - \lambda) = \text{ind}(A_{11} - \lambda)$ and $\text{min.ind}(A_2 - \lambda) = 0$ for all $\lambda \in \Phi_{sf}(A_{11})$ [15, Lemma 2.10]. Assume, without loss of generality, that $N = \oplus_{i=1}^{\infty} \lambda_i 1_{\mathcal{H}_{11i}} \in B(\oplus_{i=1}^{\infty} \mathcal{H}_{11i}) = B(\mathcal{H}_{11})$, where $\dim(\mathcal{H}_{11\mathrm{i}}) = \infty$ for all $i \geq 1$. The points λ_i being isolated in $\sigma(N)$, there exists $\epsilon > 0$, an ϵ-neighbourhood $\mathcal{N}_\epsilon(\lambda_i)$ of λ_i and a sequence $\{\lambda_{ij}\} \subset \mathcal{N}_\epsilon(\lambda_i)$ such that $|\lambda_{ij} - \lambda_i| < \epsilon/2^i$ for all $i \geq 1$. Choose an orthonormal basis $\{e_{ij}\}_{j=1}^{\infty}$ of $\mathcal{H}_{11i}$, and let K_{i0} be the compact operator

$$K_{i0} = \sum_{j=1}^{\infty} (\lambda_{ij} - \lambda_i)(e_{ij} \otimes e_{ij}) \in B(H_{11i}),\ ||K_{i0}|| = \max_j |\lambda_{ij} - \lambda_i|,$$

define the compact operator K_{22} by

$$K_{22} = \oplus_{i=1}^{\infty} K_{i0} \in B(\mathcal{H}_{11}),$$

and let

$$N + K_{22} = \oplus_{i=1}^{\infty} \{\lambda_i 1_{\mathcal{H}_{11i}} + K_{i0}\} = \oplus_{i=1}^{\infty} N_i.$$

Then each N_i is a diagonal operator with $\text{diag}\{\lambda_{i1}, \lambda_{i2}, \cdots\}$, $\sigma(N + K_{22}) = \cup_{i=1}^{\infty} \sigma(N_i)$ and $\sigma_{ulf}(N + K_{22})$ is the closure of the set $\{\lambda_i : i = 1, 2, ...\}$. Define the compact operator $K \in B(H_0 \oplus \mathcal{H}_1)$ by

$$K = 0 \oplus (K_1 + K_{11} + (K_{22} \oplus 0)) \in B(\mathcal{H}_0 \oplus (\mathcal{H}_{11} \oplus \mathcal{H}_{12})),$$

and consider a point $\lambda \in \text{iso}\sigma(A+K)$: Either $\lambda \in \sigma(A_0)$, in which case $\lambda \in \Pi_0(A)$, or, $\lambda \in \text{iso}\sigma_w(A+K) = \text{iso}\sigma_w(A)$. If $\lambda \in \text{iso}\sigma_w(A)$, then $\lambda \in \text{iso}\sigma_{ulf}(A) = \text{iso}(\Gamma)$. Consequently, $\lambda = \lambda_i$ for some integer $i \geq 1$, which then forces $\lambda_i = \lim_{j\to\infty} \lambda_{ij}$. Since this contradicts $\lambda \in \text{iso}\sigma(A+K)$, we are led to conclude $A + K \in (\mathcal{P})$.

The operator $\begin{pmatrix} A_0 & * \\ 0 & N+K_2 \end{pmatrix} \in B(\mathcal{H}_0 \oplus \mathcal{H}_{11})$ (of the above construction) has SVEP. However, since $\text{min.ind}(A_2 - \lambda) = 0$ for all $\lambda \in \Phi_{sf}(A_2)$, either $\alpha(A_2 - \lambda) = 0$ or $\alpha(A_2 - \lambda)^* = 0$. If $\alpha(A_2 - \lambda) = 0$, then ($\text{ind}(A_2 - \lambda) < 0$, and hence) A_2^* does not have SVEP at $\overline{\lambda}$; if, instead, $\alpha(A_2 - \lambda)^* = 0$, then ($\text{ind}(A_2 - \lambda) > 0$, and hence) A_2 does not have SVEP at λ. Conclusion: *The operator $A + K$ above does not always satisfy the necessary condition of Theorem 3.1, and hence $A + K$ may or may not satisfy $A + K \in (\mathcal{HP})$.* It is extremely complicated, if not impossible, to determine the structure of the invariant subspaces of the operator $A + K$, and as such the determination of the passage from $A + K \in (\mathcal{P})$ to $A + K \in (\mathcal{HP})$ does not seem to be within reach. An amenable case is the one in which $A + K$ satisfies $\Phi^+_{sf}(A + K) = \Phi^+_{sf}(A) = \emptyset$. Recall, [11, Theorem 6.4], $A \in B(\mathcal{H})$ is *quasitriangular* if and only if $\Phi^+_{sf}(A^*) = \emptyset$; if A is quasitriangular, then there is a compact operator K such that $A + K$ is triangular. Thus, if $\Phi^+_{sf}A = \emptyset$, then there exists a compact operator $K \in B(\mathcal{H})$ and an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ such that

$$(A+K)^* = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ 0 & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \end{pmatrix},$$

for some scalars $a_{ii} \neq a_{jj}$ for all $i \neq j$. For each invariant subspace M of $A + K$

$$A+K = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} M \\ M^\perp \end{pmatrix} \Longleftrightarrow (A+K)^* = \begin{pmatrix} A_2^* & * \\ 0 & A_1^* \end{pmatrix} \begin{pmatrix} M^\perp \\ M \end{pmatrix},$$

where A_1^* has an upper triangular matrix with main diagonal $\mathrm{diag}(A_1^*) = \{a_{n_k n_k}\}_{k=1}^{\infty}$, $\sigma(A+K)^* = \sigma(A_1^*) \cup \sigma(A_2^*)$ and $\text{iso}\sigma(A_1^*) \subset \text{iso}\sigma(A+K)^*$. Applying Theorem 3.2 (to obtain [17, Theorem 5.1]) and combining with Theorem 3.1 we have:

Theorem 3.4. *Given an operator $A \in B(\mathcal{H})$, a necessary and sufficient condition that there exist a compact operator $K \in B(\mathcal{H})$ such that $A + K \in (\mathcal{HP})$ is that either (i) $\Phi^+_{sf}(A) = \emptyset$ or (equivalently) (ii) $A + K$ has SVEP at points in $\Phi_{sf}(A)$.*

We close this note with the result that given an operator $A \in B(\mathcal{H})$, a necessary and sufficient condition for there to exist a compact operator $K \in B(\mathcal{H})$ such that $A + K \in (\mathcal{HP})$ is that the component $\Omega_a(A)$ is connected. Here, given an operator A, the component $\Omega_a(A)$ of $\Phi_{sf}(A)$ is defined by

$$\Omega_a(A) = \{\lambda \in \Phi_{sf}(A) : \text{ind}(A - \lambda) \leq 0\}.$$

Theorem 3.5. *(i) If, for an operator $A \in B(\mathcal{X})$, the component $\Omega_a(A)$ of $\Phi_{sf}(A)$ is connected, then $A+K$ has SVEP on $\Phi_{sf}(A)$ for every compact operator $K \in B(\mathcal{X})$.*
(ii) If $\mathcal{X} = \mathcal{H}$ is a Hilbert space and $A \in B(\mathcal{H})$, then a necessary and sufficient condition for there to exist a compact operator $K \in B(\mathcal{H})$ such that $A+K \in (\mathcal{HP})$ is that the component $\Omega_a(A)$ of $\Phi_{sf}(A)$ is connected.

Proof. (i) We prove by contradiction. If $\Omega_a(A)$ is connected, then (it has no bounded component, and hence) it has just one component, namely itself, and hence the resolvent set $\rho(A)$ intersects $\Omega_a(A)$. Consequently, both A and A^* have SVEP at points in $\Omega_a(A)$ [1, Theorem 3.36]. Suppose now that there exists a compact operator $K \in B(\mathcal{X})$ such that $A+K$ does not have SVEP at a point $\lambda \in \Phi_{sf}(A+K) = \Phi_{sf}(A)$. Since $(A+K)^*$ has SVEP and $A+K$ fails to have SVEP at a point $\lambda \in \Phi_{sf}(A)$ implies $\operatorname{ind}(A-\lambda) > 0$, we must have that neither of $A+K$ and $(A+K)^*$ have SVEP at λ. Hence $\operatorname{asc}(A+K-\lambda) = \operatorname{dsc}(A+K-\lambda) = \infty$. On the other hand, since $\rho(A+K) \subset \Omega_a(A)$, the continuity of the index at points $\lambda \in \Omega_a(A)$ implies that $\operatorname{ind}(A+K-\lambda) = 0$. Thus $\alpha(A+K-\lambda) = 0$ (except perhaps for a countable set of λ), and it follows that $A+K-\lambda$ is bounded below (and hence $\operatorname{asc}(A+K-\lambda) < \infty$).This is a contradiction.

(ii) Start by observing that if $A+K$ has SVEP at $\lambda \in \Phi_{sf}(A)$, then (necessarily) $\operatorname{ind}(A+K-\lambda) \leq 0$, equivalently, $\Phi^+_{sf}(A) = \emptyset$, for every compact operator $K \in B(\mathcal{H})$. This, by Theorem 3.4 above or [17, Theorem 5.1], implies the existence of a compact operator $K \in B(\mathcal{H})$ such that $A+K \in (\mathcal{HP})$. Conversely, if there exists a compact operator K such that $A+K \in (\mathcal{HP})$, then $A+K$ has SVEP on $\Phi_{sf}(A)$. Assume, to the contrary, that $\Omega_a(A)$ (is not connected, and hence) has a bounded component $\Omega_0(A)$. Then $\Gamma = \partial\Omega_0(A) \subset \sigma_{ulf}(A)$, and there exists a compact operator $K_1 \in B(\mathcal{H})$ such that $A+K_1$ has the upper triangular matrix representation

$$A+K_1 = \begin{pmatrix} N & * \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2), \ \dim(\mathcal{H}_1) = \infty,$$

with respect to some decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$, where N is a normal diagonal operator of uniform infinite multiplicity, $\sigma(N) = \sigma_{uf}(N) = \Gamma$, $\sigma_{uf}(A_2) = \sigma_{uf}(A)$ and $\operatorname{ind}(A_2-\lambda) = \operatorname{ind}(A-\lambda)$ for all $\lambda \in \Phi_{sf}(A)$ (see [15, Lemma 2.10]). The spectrum $\sigma(N) = \Gamma$ of N being the boundary of a bounded connected open subset of $\mathbb{C}$, [12, Theorem 3.1] implies the existence of a compact operator $K_2 \in B(\mathcal{H}_1)$ such that $\sigma(N+K_2)$ equals the closure $\overline{\Omega_0(A)}$. Define the compact operator $K \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ by $K = K_1 + (K_2 \oplus 0)$. Then

$$A+K = \begin{pmatrix} N+K_2 & * \\ 0 & A_2 \end{pmatrix} \in B(\mathcal{H}),$$

where for every $\mu \in \Omega_0(A)$ we have $\mu \in \Phi_{sf}(N+K_2) = \Phi_{sf}(N)$ with $\operatorname{ind}(N+K_2-\mu) = 0$. It being clear that SVEP for $A+K$ at a point implies SVEP for $N+K_2$ at the point, it follows that every $\mu \in \Omega_0(A)$ is an isolated point – a contradiction. Conclusion: $\Omega_a(A)$, has no bounded component. ∎

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