A discrete-time approach in the qualitative theory of skew-product three-parameter semiflows

Ciprian Preda   Adriana-Paula Popiţiu

Abstract

We obtain a characterization of the uniform exponential stability for the continuous-time skew-product three-parameter semiflows in Banach spaces, using a discrete-time approach. Our technique is based on the classical "test-function" method of O. Perron and Ta Li.

1 Introduction

The notion of a cocycle over a (semi)flow appears naturally when taking into account the linearization along an invariant manifold of a dynamical system generated by an autonomous differential equation in an infinite dimensional space (see for instance [19] Chapter 4). V. A. Pliss and G. R. Sell [14] proved that well-known equations like Navier-Stokes, Taylor-Couette, Bubnov-Galerkin, can be modeled asymptotically by associating a cocycle over a semiflow. Lately, there has been an increasing interest in the study of the exponential dichotomy of the exponentially bounded, strongly continuous cocycles over continuous flows on a locally compact metric space Θ and acting on a Banach space X (see for instance [7]). The results obtained showed that the cocycle has an exponential dichotomy if and only if the associated evolution semigroup is hyperbolic and if and only if the imaginary axis is contained in the resolvent set of the generator

Received by the editors in May 2014 - In revised form in February 2017.
Communicated by P. De Maesschalck.
2010 *Mathematics Subject Classification* : 34D05, 47D06.
Key words and phrases : skew-product semiflows, evolution families, uniform exponential stability.

of the evolution semigroup. Based on the notion of exponential dichotomy, R. J. Sacker and G. R. Sell (for details we refer the reader to [18]) developed a theory for linear skew-product flows with compact base flow, the so-called *Sacker-Sell theory*. Thus, it was proved that by associating to a linear nonautonomous differential equation an exponentially bounded, strongly continuous cocycle, its asymptotic behavior (such as exponential stability) is closely related to that of the equation under consideration.

Other important results in this direction have been obtained by Y. Latushkin, S. Montgomery-Smith and T. Randolph [6], Y. Latushkin, A.M. Stepin [8], V.A. Pliss, G.R. Sell [13], C. Chicone, Y. Latushkin [1].

The more general concept of (non)linear skew-product three-parameter semiflows, also known as (evolution) cocycles has become popular recently. M. Rasmussen (see for instance [17]) obtained relationships between the concepts of exponential dichotomy, dichotomy spectra and Morse decompositions for a linear cocycle $\Phi : \mathbb{I} \times \mathbb{I} \times X \to X$, where $\mathbb{I}$ denotes a real interval of the form $(-\infty, 0], [0, \infty)$ or $\mathbb{R}$, respectively and (X, d) represents a metric space.

In our case, the natural question that appeared was whether the "input-output" techniques introduced by O. Perron [12] and developed later by Ta Li [21] for discrete-time systems could be extended for this case. Therefore, in order to obtain a condition for the uniform exponential stability of the (non)linear skew-product three-parameter semiflows we will use the admissibility of a well-known pair of spaces $(l^p(X), l^q(X))$, where $(p, q) \neq (1, \infty)$, $1 \leq p \leq q \leq \infty$ and X is a Banach space. This means that a (non)linear skew-product three-parameter semiflow acting on X is uniformly exponentially stable if and only if for any "input" f from $l^p(X)$, the corresponding "output" x_f belongs to $l^q(X)$, where $\frac{1}{p} + \frac{1}{q}$ is *not* necessarily equal to 1. It is worth mentioning that the techniques used in the present paper do not require any continuity condition on the (non)linear skew-product three-parameter semiflow. Also, we must note that we obtain characterizations for the continuous-time (non)linear skew-product three-parameter semiflows using a discrete-time approach.

Other extensions of Perron's results for the infinite-dimensional Banach spaces were obtained by J. L. Daleckij, M. G. Krein [2], J. L. Massera, J.J Schäffer [9], [10] and N. van Minh, F. Räbiger, R. Schnaubelt [11].

Some other results concerning the property of stability for the (non)linear skew-product three-parameter semiflows in the framework of infinite-dimensional Banach spaces were also obtained by C. Stoica, M. Megan [20], C. Preda, P. Preda, A. P. Petre [16] and P. V. Hai [3], [4].

2 Preliminary results

Let us consider (Θ, d) a metric space, X a Banach space, $\mathcal{B}(X)$ the space of all bounded operators acting on X and $\Delta = \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0 \geq 0\}$. We denote the norm of vectors on X and operators on $\mathcal{B}(X)$ by $|| \cdot ||$.

Definition 2.1. A three-parameter (non)linear semiflow $\sigma : \Theta \times \Delta \to \Theta$ is defined by the properties:

i) $\sigma(\theta, t, t) = \theta$ for all $t \geq 0$ and all $\theta \in \Theta$

ii) $\sigma(\sigma(\theta, s, t_0), t, s) = \sigma(\theta, t, t_0)$ for all $t \geq s \geq t_0 \geq 0$ and all $\theta \in \Theta$.

If in addition $(\theta, t, t_0) \mapsto \sigma(\theta, t, t_0)$ is continuous, then σ is called a *continuous* three-parameter (non)linear semiflow on Θ.

Definition 2.2. The pair $\pi = (\Phi, \sigma)$ is said to be a *(non)linear skew-product three-parameter semiflow* on X if $\Phi : \Theta \times \Delta \to \mathcal{B}(X)$ satisfies the following properties:

i) $\Phi(\theta, t, t) = I$ for all $t \geq 0$ and all $\theta \in \Theta$, where I represents the identity operator on X;

ii) $\Phi(\sigma(\theta, s, t_0), t, s)\Phi(\theta, s, t_0) = \Phi(\theta, t, t_0)$, for all $t \geq s \geq t_0 \geq 0$ and all $\theta \in \Theta$.

iii) the maps $(\theta, t, t_0) \mapsto \Phi(\theta, t, t_0)x$ are continuous on $\Theta \times \Delta$ for each $x \in X$;

iv) there exist $M, \omega \in \mathbb{R}$, $M \geq 1$ such that $||\Phi(\theta, t, t_0)|| \leq Me^{\omega(t-t_0)}$, for all $t \geq t_0 \geq 0$ and $\theta \in \Theta$.

The following example illustrates the fact that by linearization, the nonlinear and non-autonomous equations become variational (three-parameter) equations around an invariant manifold.

Example 2.3. Consider the Cauchy problem

$$\begin{cases} \dot{y} = F(t, y) \\ y(t_0) = y_0 \end{cases} \tag{1}$$

where F is taken so that the problem (1) has a unique solution and $F(t, \cdot)$ is Fréchet differentiable. Let us denote by $\varphi(y_0, t, t_0)$ the solution of the problem (1) and assume that there exists $M \subset X \times \Delta$ such that $\varphi(y_0, t, t_0) \in M$ for all $(y_0, t, t_0) \in M$. Then $\varphi(\varphi(y_0, s, t_0), t, s) = \varphi(y_0, t, t_0)$. Now take $y = x + \varphi$, and we obtain that

$$\dot{x} + \dot{\varphi} = F(t, \varphi + x) = F(t, \varphi) + d_{\varphi(t)}F(t, \cdot)(x) + ||x||\omega(t, x + \varphi).$$

Hence $\dot{x} = A(\varphi(t))x + G(t, x)$, where $G(t, 0) = 0$ and $d_0G(t, \cdot) = 0$. Taking $\varphi = \sigma$, where σ represents a (non) linear three-parameter semiflow, it follows that $\dot{x} = A(\sigma(\theta, t, t_0))x + G(t, x)$. Therefore, the variational (three-parameter) equations come from nonlinear non-autonomous equations around an invariant manifold M.

Example 2.4. Let σ be a continuous three-parameter (non)linear semiflow on Θ, $A : \Theta \to \mathcal{B}(X)$ a continuous map and f a locally integrable function on X. It is easy to see that the solution of the homogeneous Cauchy problem

$$\begin{cases} \dot{x}(t) = A(\sigma(\theta, t, t_0))x(t), \quad t \geq t_0 \geq 0 \\ x(t_0) = x_0 \end{cases}$$

satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} A(\sigma(\theta,\tau,t_0))x(\tau)d\tau \tag{2}$$

and that the solution of the inhomogeneous Cauchy problem:

$$\begin{cases} \dot{x}(t) = A(\sigma(\theta,t,t_0))x(t) + f(t), & t \geq t_0 \geq 0 \\ x(t_0) = x_0 \end{cases}$$

satisfies the integral equation

$$x(t) = \Phi(\theta,t,t_0)x_0 + \int_{t_0}^{t} A(\sigma(\theta,\tau,t_0))x(\tau)d\tau + \int_{t_0}^{t} f(\tau)d\tau \tag{3}$$

due to the fact that, in both cases, the solution of the variational Cauchy problem is an absolutely continuous function.

Example 2.5. With a similar argument as in the proof of the existence and uniqueness theorems for the non-autonomous systems (see for example [2, 10]), one can show that the solution of the variational homogeneous equation (1) is $x(t) = \Phi(\theta,t,t_0)x_0$, where $\pi = (\Phi,\sigma)$ is a (non)linear skew-product three-parameter semiflow, and that (2) has the solution

$$x(t) = \Phi(\theta,t,t_0)x_0 + \int_{t_0}^{t} \Phi(\sigma(\theta,\tau,t_0),t,\tau)f(\tau)d\tau.$$

Definition 2.6. A family of linear and bounded operators $\{U(t,t_0)\}_{t\geq t_0\geq 0}$ is said to be a *two-parameter evolution family* if it satisfies the following conditions:

i) $U(t,t) = I$, for all $t \geq 0$;

ii) $U(t,s)U(s,t_0) = U(t,t_0)$, for all $t \geq s \geq t_0 \geq 0$;

iii) $U(\cdot,t_0)x$ is continuous on $[t_0,\infty)$, for all $t_0 \geq 0,\ x \in X$; $U(t,\cdot)x$ is continuous on $[0,t]$, for all $t \geq 0,\ x \in X$;

iv) there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $||U(t,t_0)|| \leq Me^{\omega(t-t_0)}$ for all $t \geq t_0 \geq 0$.

Example 2.7. If $\{U(t,t_0)\}_{t\geq t_0\geq 0}$ is a two-parameter evolution family on X and σ is a (non)linear three-parameter semiflow on Θ, then $\pi = (\Phi,\sigma)$ is a (non)linear skew-product three-parameter semiflow, where $\Phi(\theta,t,t_0) = U(t,t_0)$. Thus we can consider that evolution families are particular cases of (non)linear skew-product three-parameter semiflows.
Conversely, considering $\Theta = \mathbb{R}_+$, $\sigma : \mathbb{R}_+ \times \Delta \to \mathbb{R}_+$, $\sigma(\theta,t,t_0) = \theta$, and $\pi = (\Phi,\sigma)$ a (non)linear skew-product three-parameter semiflow on X, we have that $\{U(t,t_0)\}_{t\geq t_0\geq 0}$, $U(t,t_0) = \Phi(0,t,t_0)$ is an evolution family on X.

Example 2.8. Let $\Theta = \mathbb{R}_+$ and let $\sigma : \Theta \times \Delta \to \Theta$, $\sigma(\theta,t,t_0) = \theta + t - t_0$. If $\{U(t,t_0)\}_{t\geq t_0\geq 0}$ is a two-parameter evolution family on X, then $\pi = (\Phi,\sigma)$ is a (non)linear skew-product three-parameter semiflow, where

$$\Phi(\theta,t,t_0) = U(\theta + t - t_0,\theta).$$

Definition 2.9. Let $\pi = (\Phi, \sigma)$ be a (non)linear skew-product three-parameter semiflow.
$\pi = (\Phi, \sigma)$ is *uniformly exponentially stable* if there exist $N \geq 1$ and $\nu > 0$ such that $||\Phi(\theta, t, t_0)x|| \leq Ne^{-\nu(t-t_0)}||x||$, for all $x \in X$, $t \geq t_0 \geq 0$ and $\theta \in \Theta$.
$\pi = (\Phi, \sigma)$ is *uniformly stable* if there exists $N > 0$ such that $||\Phi(\theta, t, t_0)x|| \leq N||x||$, for all $x \in X$, $t \geq t_0 \geq 0$ and $\theta \in \Theta$.

3 Main result

Throughout this paper (Θ, d) is a metric space and X is a Banach space. Also, we denote by σ a three-parameter (non)linear semiflow on Θ and Φ satisfying the conditions from Definition 1.2.

$$\begin{aligned} l^\infty(X) &= \{f : \mathbb{N} \to X : \sup_{n\in\mathbb{N}} ||f(n)|| < \infty\}) \text{ and} \\ l^p(X) &= \{f : \mathbb{N} \to X \mid \sum_{n=0}^{\infty} ||f(n)||^p < \infty\} \ \ (p \in [1, \infty)) \end{aligned}$$

are Banach spaces endowed with norms

$$||f||_\infty = \sup_{n\in\mathbb{N}} ||f(n)|| \quad \text{and} \quad ||f||_p = \Big(\sum_{n=0}^{\infty} ||f(n)||^p\Big)^{\frac{1}{p}}.$$

For $f \in l^p(X)$ we consider the map

$$\tilde{x}_f : \Theta \times \mathbb{N} \to l^q(X), \ \ \tilde{x}_f(\theta, n_0) = x_f(\theta, \cdot, n_0),$$

where

$$x_f(\theta, n, n_0) = \sum_{k=n_0}^{n} \Phi(\sigma(\theta, k, n_0), n, k) f(k),$$

for all $\theta \in \Theta$, and $n, n_0 \in \mathbb{N}, n \geq n_0 \geq 0$.

Definition 3.1. For $p, q \in [1, \infty]$, the pair $(l^p(X), l^q(X))$ is *admissible* to $\pi = (\Phi, \sigma)$ if for all $f \in l^p(X)$, the map $\tilde{x}_f \in C_b(\Theta \times \mathbb{N}, l^q(X))$, the set of all bounded functions on $\Theta \times \mathbb{N}$.

Theorem 3.2 is essential in the proof of our main result, Theorem 3.5

Theorem 3.2. *Let $\pi = (\Phi, \sigma)$ be a (non)linear skew-product three-parameter semiflow. If the pair $(l^p(X), l^q(X))$ is admissible to $\pi = (\Phi, \sigma)$, then there exists $L > 0$ such that*

$$||\tilde{x}_f(\theta, n_0)||_q \leq L||f||_p, \text{ for all } f \in l^p(X) \text{ and } (\theta, n_0) \in \Theta \times \mathbb{N}.$$

Proof. Define

$$\mathcal{U} : l^p(X) \to C_b(\Theta \times \mathbb{N}, l^q(X)), \ \mathcal{U}f = \tilde{x}_f,$$

where $\tilde{x}_f$ is defined as above. We note that $\mathcal{U}$ is a linear operator. In order to show that $\mathcal{U}$ is also closed, we consider $(f_m)_{m\in\mathbb{N}} \in l^p(X), f \in l^p(X)$ such that:

$$||f_m - f||_p \to 0, \text{ for } m \to \infty$$

which leads to

$$f_m(k) \to f(k) \text{ for } m \to \infty \text{ and for all } k \in \mathbb{N}$$

and take $\mathcal{U} f_m \to g$ for $m \to \infty$ in $C_b(\Theta \times \mathbb{N}, l^q(X))$. Thus

$$||x_{f_m}(\theta, n, n_0) - x_f(\theta, n, n_0)|| \leq \sum_{k=n_0}^{n} ||\Phi(\sigma(\theta, k, n_0), n, k)|| \, ||f_m(k) - f(k)|| \to 0$$

for $m \to \infty$ which implies that $\mathcal{U} f = g$, therefore $\mathcal{U}$ is also bounded, which means that there exists $L > 0$ such that

$$||\tilde{x}_f(\theta, n_0)||_q \leq L||f||_p, \text{ for all } f \in l^p(X) \text{ and } (\theta, n_0) \in \Theta \times \mathbb{N}.$$ ■

Remark 3.3. Let $\pi = (\Phi, \sigma)$ be a (non)linear skew-product three-parameter semiflow. If the pair $(l^p(X), l^q(X))$ is admissible to $\pi = (\Phi, \sigma)$, then there exists a constant $L > 0$ such that

$$||x_f(\theta, k, n_0)|| \leq L||f||_p, \text{ for all } f \in l^p(X), k \in \mathbb{N} \text{ and } (\theta, n_0) \in \Theta \times \mathbb{N}.$$

Another crucial tool in the proof of Theorem 3.5 is Lemma 3.4, which is inspired by [9, Lemma 5.3, p. 539].

Lemma 3.4. [15] If $g : \{(t, s) \in \mathbb{R}^2 : \ t \geq s \geq 0\} \to \mathbb{R}_+$ satisfies the conditions

i) $g(t, s) \leq g(t, \tau) g(\tau, s)$ for all $t \geq \tau \geq s \geq 0$;

ii) $\sup_{0 \leq t_0 \leq t \leq t_0 + 1} g(t, t_0) < \infty$;

iii) there exists $h : \mathbb{N} \to \mathbb{R}_+$ with $\lim_{n \to \infty} h(n) = 0$ such that:

$$g(m + n_0, n_0) \leq h(m), \ m, n_0 \in \mathbb{N}$$

then there exist $N, \nu > 0$ such that: $g(t, t_0) \leq N e^{-\nu(t - t_0)}$, for all $t \geq t_0 \geq 0$.

Theorem 3.5. *Let $\pi = (\Phi, \sigma)$ be a (non)linear skew-product three-parameter semiflow. $\pi = (\Phi, \sigma)$ is uniformly exponentially stable if and only if there exist $p, q \in [1, \infty]$, with $(p, q) \neq (1, \infty)$ such that the pair $(l^p(X), l^q(X))$ is admissible to $\pi = (\Phi, \sigma)$.*

Proof. In order to prove the *necessity*, we used as a source of inspiration the work P. Hartman [5]. Let $f \in l^p(X)$ and $\tilde{x}_f : \Theta \times \mathbb{N} \to l^q(X)$ be the application defined above. If $p = 1$, then

$$\begin{aligned} ||x_f(\theta, n, n_0)|| &\leq \sum_{k=n_0}^{n} ||\Phi(\sigma(\theta, k, n_0), n, k)|| \, ||f(k)|| \\ &\leq N \sum_{k=n_0}^{n} e^{-\nu(n-k)} ||f(k)|| = N \sum_{i=0}^{n-n_0} e^{-\nu i} ||f(n-i)||, \end{aligned}$$

for all $\theta \in \Theta$, and $n, n_0 \in \mathbb{N}, n \geq n_0 \geq 0$. Applying Fubini's Theorem, we obtain that:

$$\begin{aligned}
\sum_{n=n_0}^{\infty} ||x_f(\theta, n, n_0)|| &\leq N \sum_{n=n_0}^{\infty} \Big(\sum_{i=0}^{n-n_0} e^{-\nu i} ||f(n-i)|| \Big) \\
&\leq N \sum_{i=0}^{\infty} \Big(\sum_{n=n_0}^{\infty} e^{-\nu i} ||f(n-i)|| \Big) = N \sum_{i=0}^{\infty} e^{-\nu i} \Big(\sum_{n=n_0}^{\infty} ||f(n-i)|| \Big) \\
&\leq \frac{N}{1-e^{-\nu}} ||f||_1 < \infty \qquad (4)
\end{aligned}$$

It follows from (4) that $x_f(\theta, \cdot, n_0) \in l^1(X)$. It is obvious that $x_f(\theta, \cdot, n_0) \in l^\infty(X)$, and therefore $x_f(\theta, \cdot, n_0) \in l^1(X) \cap l^\infty(X)$, hence $x_f(\theta, \cdot, n_0) \in l^q(X)$, for all $q \in [1, \infty]$. Now let $1 < p < q < \infty$, and take α, β such that $\alpha + \beta = 1$. Applying Hölder's inequality, we find that

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} \Big(\sum_{i=0}^{n-n_0} e^{-\nu i} ||f(n-i)|| \Big)^q \\
&= \sum_{n=n_0}^{\infty} \Big(\sum_{i=0}^{n-n_0} e^{-\nu \alpha i} ||f(n-i)||^{\frac{p}{q}} e^{-\nu \beta i} ||f(n-i)||^{1-\frac{p}{q}} \Big)^q \\
&\leq \sum_{n=n_0}^{\infty} \Big[\Big(\sum_{i=0}^{n-n_0} e^{-\nu \alpha q i} ||f(n-i)||^p \Big)^{\frac{1}{q}} \\
&\quad \cdot \Big(\sum_{i=0}^{n-n_0} e^{-\nu \beta \frac{q}{q-1} i} ||f(n-i)||^{\frac{q-p}{q} \cdot \frac{q}{q-1}} \Big)^{\frac{q-1}{q}} \Big]^q \\
&= \sum_{n=n_0}^{\infty} \Big[\Big(\sum_{i=0}^{n-n_0} e^{-\nu \alpha q i} ||f(n-i)||^p \Big) \cdot \Big(\sum_{i=0}^{n-n_0} e^{-\nu \beta \frac{q}{q-1} i} ||f(n-i)||^{\frac{q-p}{q-1}} \Big)^{q-1} \Big]. \qquad (5)
\end{aligned}$$

Let $x = \frac{p(q-1)}{q-p}$, and $x' = \frac{p(q-1)}{q(p-1)}$. Then $\frac{1}{x} + \frac{1}{x'} = 1$. Applying Hölder's Inequality using x and x' to the second sum in (5), we obtain that

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} \Big(\sum_{i=0}^{n-n_0} e^{-\nu i} ||f(n-i)|| \Big)^q \\
&\leq \sum_{n=n_0}^{\infty} \Big[\Big(\sum_{i=0}^{n-n_0} e^{-\nu \alpha q i} ||f(n-i)||^p \Big) \cdot \Big(\sum_{i=0}^{n-n_0} e^{-\nu \beta \frac{q}{q-1} i x'} \Big)^{\frac{q-1}{x'}} \\
&\quad \cdot \Big(\sum_{i=0}^{n-n_0} ||f(n-i)||^{\frac{q-p}{q-1} x} \Big)^{\frac{q-1}{x}} \Big] \\
&= \sum_{n=n_0}^{\infty} \Big[\Big(\sum_{i=0}^{n-n_0} e^{-\nu \alpha q i} ||f(n-i)||^p \Big) \cdot \Big(\sum_{i=0}^{n-n_0} e^{-\nu \beta \frac{p}{p-1} i} \Big)^{\frac{q(p-1)}{p}} \\
&\quad \cdot \Big(\sum_{i=0}^{n-n_0} ||f(n-i)||^p \Big)^{\frac{q-p}{p}} \Big].
\end{aligned}$$

Now

$$\Big(\sum_{i=0}^{n-n_0} e^{-\nu\beta\frac{p}{p-1}i}\Big)^{\frac{q(p-1)}{p}} \le \Big(\sum_{i=0}^{\infty} e^{-\nu\beta\frac{p}{p-1}i}\Big)^{\frac{q(p-1)}{p}} = \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{\frac{q(p-1)}{p}},$$

and

$$\Big(\sum_{i=0}^{n-n_0} ||f(n-i)||^p\Big)^{\frac{q-p}{p}} = \Big(\sum_{s=n}^{n_0} ||f(s)||^p\Big)^{\frac{q-p}{p}} \le \Big(\sum_{s=0}^{\infty} ||f(s)||^p\Big)^{\frac{q-p}{p}} = ||f||_p^{q-p},$$

hence

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\Big(\sum_{i=0}^{n-n_0} e^{-\nu i}||f(n-i)||\Big)^q \\
&\le \sum_{n=n_0}^{\infty}\Big[\Big(\sum_{i=0}^{n-n_0} e^{-\nu\alpha q i}||f(n-i)||^p\Big)\cdot\Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{\frac{q(p-1)}{p}}\cdot ||f||_p^{q-p}\Big] \\
&= \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{\frac{q(p-1)}{p}}\cdot ||f||_p^{q-p}\sum_{n=n_0}^{\infty}\Big(\sum_{i=0}^{n-n_0} e^{-\nu\alpha q i}||f(n-i)||^p\Big).
\end{aligned}$$

It follows from Fubini's Theorem that

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\Big(\sum_{i=0}^{n-n_0} e^{-\nu i}||f(n-i)||\Big)^q \\
&\le \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{\frac{q(p-1)}{p}}\cdot ||f||_p^{q-p}\sum_{n=n_0}^{\infty}\Big(\sum_{i=0}^{n-n_0} e^{-\nu\alpha q i}||f(n-i)||^p\Big) \\
&\le \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{\frac{q(p-1)}{p}}\cdot ||f||_p^{q-p}\sum_{i=0}^{\infty}\Big(\sum_{n=n_0}^{\infty} e^{-\nu\alpha q i}||f(n-i)||^p\Big) \\
&= \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{\frac{q(p-1)}{p}}\cdot ||f||_p^{q-p}\sum_{i=0}^{\infty} e^{-\nu\alpha q i}\Big(\sum_{n=n_0}^{\infty} ||f(n-i)||^p\Big) \\
&\le \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{\frac{q(p-1)}{p}}\cdot ||f||_p^{q-p}\cdot\frac{1}{1-e^{-\nu\alpha q}}\sum_{u=0}^{\infty} ||f(u)||^p,
\end{aligned}$$

and it follows that $x_f(\theta,\cdot,n_0) \in l^q(X)$.
Now let $p = q$, and take α, β such that $\alpha + \beta = 1$. Then

$$\sum_{n=n_0}^{\infty}\Big(\sum_{i=0}^{n-n_0} e^{-\nu i}||f(n-i)||\Big)^p = \sum_{n=n_0}^{\infty}\Big(\sum_{i=0}^{n-n_0} e^{-\nu\alpha i}||f(n-i)||e^{-\nu\beta i}\Big)^p.$$

Applying successively Hölder's inequality and Fubini's Theorem, we obtain that

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\Big(\sum_{i=0}^{n-n_0} e^{-\nu i}||f(n-i)||\Big)^p \\
\leq\;& \sum_{n=n_0}^{\infty}\Big[\Big(\sum_{i=0}^{n-n_0} e^{-\nu p\alpha i}||f(n-i)||^p\Big)^{\frac{1}{p}}\cdot\Big(\sum_{i=0}^{n-n_0} e^{-\nu\beta\frac{p}{p-1}i}\Big)^{\frac{p-1}{p}}\Big]^p \\
\leq\;& \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{p-1}\cdot\sum_{n=n_0}^{\infty}\Big(\sum_{i=0}^{n-n_0} e^{-\nu p\alpha i}||f(n-i)||^p\Big) \\
=\;& \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{p-1}\cdot\sum_{i=0}^{\infty}\Big(\sum_{n=n_0}^{\infty} e^{-\nu p\alpha i}||f(n-i)||^p\Big) \\
=\;& \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{p-1}\cdot\sum_{i=0}^{\infty} e^{-\nu p\alpha i}\Big(\sum_{n=n_0}^{\infty}||f(n-i)||^p\Big) \\
=\;& \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{p-1}\cdot\frac{1}{1-e^{-\nu\alpha p}}\cdot\Big(\sum_{n=n_0}^{\infty}||f(n-i)||^p\Big) \\
\leq\;& \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{p-1}\cdot\frac{1}{1-e^{-\nu\alpha p}}\cdot||f||_p^p.
\end{aligned}$$

It follows that

$$\sum_{n=n_0}^{\infty}\Big(\sum_{i=0}^{n-n_0} e^{-\nu i}||f(n-i)||\Big)^p \leq \Big(\frac{1}{1-e^{-\nu\beta\frac{p}{p-1}}}\Big)^{p-1}\cdot\frac{1}{1-e^{-\nu\alpha p}}\cdot||f||_p^p<\infty,$$

and $x_f(\theta,\cdot,n_0)\in l^p(X)$.

In order to prove the *sufficiency*, consider $n_0\geq 0$, $\theta\in\Theta$, $x\in X$ and take

$$f(n)=\chi_{\{n_0\}}(n)\Phi(\theta,n,n_0)x.$$

Here χ_A is the characteristic function of the set A. Then $f\in l^p(X)$, $||f||_p=||x||$ and

$$x_f(\theta,n,n_0)=\sum_{k=n_0}^{n}\Phi(\sigma(\theta,k,n_0),n,k)f(k)=\Phi(\theta,n,n_0)x,$$

f or all $n\geq n_0$ and $\theta\in\Theta$. From Theorem 3.2 we obtain that

$$||\Phi(\theta,n,n_0)x||\leq L||x||, \tag{6}$$

for all $n\geq n_0\geq 0$, $\theta\in\Theta$ and $x\in X$. Now let $m>1$, $\theta\in\Theta$ and $x\in X$ and take

$$g_m(n)=\chi_{\{n_0,\cdots,n_0+m\}}(n)\Phi(\theta,n,n_0)x.$$

It follows from (6) that $||g_m(n)||\leq L\,\chi_{\{n_0,\cdots,n_0+m\}}(n)||x||$, for all $n\geq n_0$. We conclude that

$$g_m\in l^p(X) \text{ and } ||g_m||_p\leq L\,a_m||x||,$$

where

$$a_m = \begin{cases} (m+1)^{\frac{1}{p}} & \text{if } p \in [1,\infty) \\ 1 & \text{if } p = \infty \end{cases}$$

Therefore

$$\begin{aligned} x_{g_m}(\theta,n,n_0) &= \sum_{k=n_0}^{n} \Phi(\sigma(\theta,k,n_0),n,k) g_m(k) \\ &= \begin{cases} (m+1)\Phi(\theta,n,n_0)x & \text{if } n \geq n_0+m \\ (n-n_0+1)\Phi(\theta,n,n_0)x & \text{if } n_0 \leq n < n_0+m \end{cases} \end{aligned}$$

Let $n = n_0 + m$ and $p > 1$. It follows from Theorem 3.2 that

$$||\Phi(\theta,n_0+m,n_0)x|| \leq L^2 \frac{a_m}{m+1} \, ||x||.$$

Since $\lim_{m\to\infty} \frac{a_m}{m+1} = 0$, it follows from Lemma 3.4 that there exist $N, \nu > 0$ such that $||\Phi(\theta,t,t_0)x|| \leq Ne^{-\nu(t-t_0)}||x||$ for all $t \geq t_0 \geq 0$, $\theta \in \Theta$ and $x \in X$.
If $p = 1$, then it follows from Theorem 3.2 and Hölder's Inequality that

$$\begin{aligned} \frac{m(m+1)}{2}||\Phi(\theta,n_0+m,n_0)x|| &= \sum_{k=n_0}^{n_0+m}(k-n_0+1)||\Phi(\theta,n_0+m,n_0)x|| \\ \leq \; & L\sum_{k=n_0}^{n_0+m}(k-n_0+1)||\Phi(\theta,k,n_0)x|| = L\sum_{k=n_0}^{n_0+m}||x_{g_m}(\theta,k,n_0)|| \\ \leq \; & L(m+1)^{1-\frac{1}{q}}||\tilde{x}_{g_m}(\theta,n_0)||_q \leq L^3(m+1)^{2-\frac{1}{q}}||x||, \end{aligned}$$

which implies that

$$||\Phi(\theta,n_0+m,n_0)x|| \leq 2L^3\frac{b_m}{m}||x||,$$

for all $m, n_0 \in \mathbb{N}$, where

$$b_m = \begin{cases} (m+1)^{1-\frac{1}{q}} & \text{if } q \in [1,\infty) \\ m+1 & \text{if } q = \infty \end{cases}$$

In order to apply Lemma 3.4 one more time we note that

$$\Phi(\theta,[t]+2,t_0) = \Phi(\sigma(\theta,\tau,t_0),[t]+2,\tau)\,\Phi(\theta,\tau,t_0),$$

for all $t \in [t_0, t_0+1]$, and therefore

$$\sup_{0\leq t_0\leq t\leq t_0+1} ||\Phi(\theta,[t]+2,t_0)x|| \leq M^2e^{3\omega}||x||,$$

for all $x \in X$ and $\theta \in \Theta$. Thus, there exist $N, \nu > 0$ such that $||\Phi(\theta,t,t_0)x|| \leq Ne^{-\nu(t-t_0)}||x||$, for all $t \geq t_0 \geq 0$, $\theta \in \Theta$ and $x \in X$. ■

Remark 3.6. Let $\pi = (\Phi, \sigma)$ be a (non)linear skew-product three-parameter semiflow. In general, if $p, q \in [1, \infty]$ with $p > q$ and $\pi = (\Phi, \sigma)$ is uniformly exponentially stable, then the pair $(l^p(X), l^q(X))$ is not admissible to $\pi = (\Phi, \sigma)$, as it can be seen in Example 3.7, illustrating the fact that the hypothesis that $p \leq q$ is essential throughout our paper.

Example 3.7. Take $X = \mathbb{R}$, let $\pi = (\Phi, \sigma)$ be a (non)linear skew-product three-parameter semiflow, where $\Phi(\theta, t, t_0) = e^{-(t-t_0)}$, and let σ be a three-parameter semiflow. Then the pair $(l^2(X), l^1(X))$ is not admissible to $\pi = (\Phi, \sigma)$.
In order to prove this, take $f \in l^2(X)$, $f(n) = \frac{1}{n+1}$. Then

$$x_f(\theta, n, n_0) = \sum_{k=n_0}^{n} e^{-(n-k)} \frac{1}{k+1} = e^{-n} \sum_{k=n_0}^{n} e^k \frac{1}{n+1} \geq e^{-n} \frac{e^n}{n+1} = \frac{1}{n+1},$$

and

$$\sum_{n=0}^{\infty} x_f(\theta, n, n_0) \geq \sum_{n=0}^{\infty} \frac{1}{n+1},$$

which implies that $(\sum_{n \geq 0} x_f(\theta, n, n_0))$ is divergent and therefore $x_f \notin l^1(X)$.

Theorem 3.8. *Let $\pi = (\Phi, \sigma)$ be a (non)linear skew-product three-parameter semiflow. $\pi = (\Phi, \sigma)$ is uniformly stable if and only if the pair $(l^1(X), l^\infty(X))$ is admissible to $\pi = (\Phi, \sigma)$.*

Proof. The *necessity* is immediate and for the *sufficiency* part, let us consider $n_0 \geq 0, \theta \in \Theta, x \in X$ and take

$$f(n) = \chi_{\{n_0\}}(n) \Phi(\theta, n, n_0) x.$$

Then $f \in l^1(X)$, $||f||_1 = ||x||$ and

$$x_f(\theta, n, n_0) = \sum_{k=n_0}^{n} \Phi(\sigma(\theta, k, n_0), n, k) f(k) = \Phi(\theta, n, n_0) x.$$

From Theorem 3.2 we obtain that

$$||\Phi(\theta, n, n_0)x|| \leq L||x||, \text{ for all } n \geq n_0 \geq 0,\ \theta \in \Theta, x \in X,$$

which implies that there exist $N = L > 0$ such that

$$||\Phi(\theta, n, n_0)x|| \leq N||x||, \text{ for all } n \geq n_0 \geq 0,\ \theta \in \Theta, x \in X,$$

and thus $\pi = (\Phi, \sigma)$ is uniformly stable. ∎

References

[1] C. Chicone, Y. Latushkin. *Evolution Semigroups in dynamical systems and differential equations*, Amer. Math. Soc., Math. Surv. and Monographs **70** (1999).

[2] J. L. Daleckij, M. G. Krein. *Stability of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, RI, 1974.

[3] P. V. Hai. *Continuous and discrete characterizations for the uniform exponential stability of linear skew-evolution semiflows* Nonlinear Anal. 72 (2010), 4390–4396.

[4] P. V. Hai. *An extension of P. Preda, A. Pogan, C. Preda, Timisoara's theorems for the uniformly exponential stability of linear skew-product semiflows* Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 53 (101) (2010), 69–83.

[5] P. Hartman. *Ordinary differential equations* Wiley, New York, 1964.

[6] Y. Latushkin, S. Montgomery-Smith, T. Randolph. *Evolutionary semigroups and dichotomy of linear skew-product flows on locally compact spaces with Banach fibers*, J. Differential Equations 125 (1996), 73–116.

[7] Y. Latushkin, R. Schnaubelt. *Evolution semigroups, translation algebra and exponential dichotomy of cocycles*, J. Differential Equations 159 (1999), 321–369.

[8] Y. Latushkin, A.M. Stepin. *Linear skew-product flows and semigroups of weighted composition operators,* Lecture Notes Math. 1486, Springer-Verlag, Berlin, 1991.

[9] J.L. Massera, J.J. Schäffer. *Linear Differential Equations and Functional Analysis I.* Ann. of Math. 67 (1958), 517–573.

[10] J. L. Massera, J.J Schäffer. *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.

[11] N. van Minh, F. Räbiger, R. Schnaubelt. *Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half line*, Integral Equations Operator Theory 32 (1998), 332–353.

[12] O. Perron. *Die Stabilitatsfrage bei Differential gleichungen*, Math. Z. 32 (1930), 703–728.

[13] V. A. Pliss, G. R. Sell. *Robustness of exponential dichotomies in infinite dimensional dynamical systems*, J. Dynam. Differential Equations 11 (1999), 471–513.

[14] V. A. Pliss, G. R. Sell. *Perturbations of normally hyperbolic manifolds with applications to the Navier-Stokes equation*, J. Differential Equations 169 (2001), 396–492.

[15] P. Preda, A. Pogan, C. Preda. *Discrete admissibility and exponential dichotomy for evolution families*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 12 (2005), 621–631.

[16] C. Preda, P. Preda, A. P. Petre. *On the uniform exponential stability of linear skew-product three-parameter semiflows*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 54 (102) (2011), 269–279.

[17] M. Rasmussen *Dichotomy spectra and Morse decompositions of linear nonautonomous differential equations*, J. Differential Equations 246 (2009), 2242–2263.

[18] R. J. Sacker, G.R. Sell. *Dichotomies for linear evolutionary equations in Banach spaces*, J. Differential Equations 113 (1994), 17–67. (MR1296160)(96k:34136)

[19] G. R. Sell, Y. You. *Dynamics of Evolutionary Equations*, Appl. Math. Sci. 143, Springer-Verlag, Berlin, 2002.

[20] C. Stoica, M. Megan. *On uniform exponential stability for skew-evolution semiflows on Banach spaces*, Nonlinear Analysis 72 (2010), 1305–1313.

[21] L. Ta. *Die Stabilitatsfrage bei Differenzengleichungen*, Acta Math. 63 (1934), 99–141.

West University of Timişoara,
Bld. V. Pârvan, no.4,
Timişoara 300223, Romania
emails : ciprian.preda@e-uvt.ro , adrianapaula.popitiu@gmail.com