Best approximative properties of exposed faces of l_1

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1 Introduction

Throughout X denotes a real Banach space , X^*, the dual of X, $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$, the unit sphere of X.

For any x in X and a subset C of X, $d(x, C)$ denotes the distance of x from C and the set $P_C(x)$ of best approximations to x from C is given by

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}.$$

If $P_C(x)$ is a non-empty set for each x in X, we say the subset C is proximinal in X. For any $\delta > 0$ we set

$$P_C(x, \delta) = \{z \in C : \|x - z\| < d(x, C) + \delta\}.$$

We say a proximinal set C of a normed linear space X is *strongly proximinal* if for each x in X and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup\{d(z, P_C(x)) : z \in P_C(x, \delta)\} \ < \ \epsilon.$$

Let X and Y be normed linear spaces. We say F is a set valued map from X into Y if F is a map from X into the class of all non-empty subsets of Y and in this case, we write $F : X \longrightarrow Y$ is a set valued map. Let x_0 be in X. The set valued map F is said to be

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1. **lower semi-continuous** at x_0 if, given $\epsilon > 0$ and z in $F(x_0)$, there exists $\delta = \delta(\epsilon, z) > 0$ such that the set $B(z,\epsilon) \cap F(x)$ is non-empty, for any x in $B(x_0, \delta)$. If δ can be chosen to be independent of z in $F(x_0)$ in the above definition, that is, given $\epsilon > 0$, there exists $\delta > 0$ such that the set $B(z,\epsilon) \cap F(x)$ is non-empty, for any x in $B(x_0,\delta)$ and any z in $F(x_0)$, then F is said to be **lower Hausdorff semi-continuous** at x_0 .

2. **upper semi-continuous** at x_0 if given any open neighbourhood U of zero in X, there exists $\delta > 0$ such that

$$F(x) \subseteq F(x_0) + U$$

for each x in $B(x_0,\delta)$. Replacing the arbitrary open set U by an open ball in the above, yields the notion of upper Hausdorff semi-continuity. More precisely, the map F is **upper Hausdorff semi-continuous** at x_0, if given $\epsilon > 0$ there exists $\delta > 0$ such that

$$F(x) \subseteq F(x_0) + \epsilon B_X$$

for each x in $B(x_0,\delta)$.

The set valued map F is lower (upper, lower Hausdorff, upper Hausdorff) semi-continuous on X if it is lower (upper, lower Hausdorff, upper Hausdorff) semi-continuous at each point of X and is called **Hausdorff metric continuous**if it is both lower and upper Hausdorff semi-continuous.

We observe that upper semi continuity implies upper Hausdorff semi continuity, while the lower semi continuity is implied by lower Hausdorff continuity.

Definition 1.1. *A selection for the set valued map F is a map $f : X \longrightarrow Y$ such that $f(x)$ is in $F(x)$, for each x in X. A selection of the set valued map F which is continuous on X, is called a continuous selection of the map F.*

It is easily verified that the metric projection onto a strongly proximinal set is upper Hausdorff metric continuous [5]. It follows from the well known Michael's selection Theorem [8, 9] that if the metric projection onto a closed, convex subset of a Banach space is lower semi continuous then it has a continuous selection.

Let X be a normed linear space. A convex, extremal subset of the closed unit ball B_X is called a **face** of X. Let $f \in X^*$ and

$$J_X(f) = \Big\{ x \in S_X : f(x) = \|f\| \Big\}.$$

The functional f is called norm attaining if the set $J_X(f)$ is non- empty. It is easily verified that if the set $J_X(f)$ is non-empty, then it is a closed, convex and extremal subset of B_X and is called an **exposed face** of B_X.

We denote by $NA(X)$, the set of all norm attaining functionals on X and by $NA_1(X)$ the set $NA(X) \cap S_X$. If $H = \ker f$, then it is well known that [11]and [4]

$$f \in NA(X) \Leftrightarrow J_X(f) \neq \emptyset \Leftrightarrow H \text{ is proximinal in X.}$$

The hyperplane H is called ball proximinal if B_H is a proximinal set. It is easily verified that ball proximinality implies proximinality.[1]

Best approximative properties of hyperplanes or subspaces in general, of Banach spaces are closely related to structure of the unit ball and its exposed faces and study of geometric structure of the unit ball in view of this link, is not new [2]. We also refer to [3], [4],[5], [7] and Proposition 1 of [10] in this regard. In [6] it was shown that if a hyperplane H, kernel of a functional f in the dual of X is ball proximinal then the set $J_X(f)$ satisfies a restricted proximinality condition. It was also shown in that paper that exposed faces of $C(Q)$, the Banach space of real valued continuous functions defined on a compact, Hausdorff space Q with sup norm, are proximinal sets. Here we prove that the exposed faces of the real sequence space l_1 are strongly proximinal sets and also that the metric projection onto them is Hausdorff metric continuous.

2 The Space l_1 and proximinality of exposed faces

Let $\mathbb{N}$ denote the set of natural numbers. We consider the real Banach space l_1, the space of sequences (x_n) of real scalars with $\|x\|_1 = \sum_{n\in\mathbb{N}} |x_n|$.

In this section, we first prove that the exposed faces of the unit ball of l_1 are proximinal sets . That is, we show that if $\phi \in NA_1(l_1)$ then the set $J_{l_1}(\phi) = \{x \in S_{l_1} : \phi(x) = 1\}$ is proximinal in l_1 and further characterize the set of best approximations to any element l_1, from this set.

For a real number α, let

$$sgn\alpha = \begin{cases} 1 & if\ \alpha \geq 0 \\ -1 & if\ \alpha < 0. \end{cases}$$

For a sequence $x = (x_n)$ of scalars, we set $supp(x) = \{n \in \mathbb{N} : x_n \neq 0\}$ and for any subset Λ of $\mathbb{N}$, we set

$$\|x\|_\Lambda = \sum_{n\in\Lambda} |x_n|.$$

Let $z = (z_n) \in l_\infty$ and $\|z\|_\infty = 1$. If f_z is the element of the dual of l_1, induced by z then f_z is in $NA_1(l_1)$ if and only if

$$\{n : |z_n| = \|z\|_\infty\} \neq \emptyset.$$

We denote the exposed face $J_{l_1}(f_z)$ by $J_{l_1}(z)$.Throughout he article,the following notation is used. An element z in l_∞ is fixed with f_z in $NA_1(l_1)$ and we set

$$C = J_{l_1}(z)$$

and

$$\Lambda^+ = \{n : z_n = \|z\|_\infty\}, \Lambda^- = \{n : z_n = -\|z\|_\infty\}\ \ and\ \ \Lambda = \Lambda^+ \cup \Lambda^-.$$

It is easily verified that

$$J_{l_1}(z) = \{y = (y_n) \in l_1 : \|y\|_1 = 1, supp(y) \subseteq \Lambda, sgn\, y_n = sgn\, z_n\ if\ y_n \neq 0\}. \quad (1)$$

Also, for a fixed x in l_1, we set

$$S^+ = S_x^+ = \{n : x_n \geq 0\} \ \ and \ \ S^- = S_x^- = \{n : x_n < 0\}$$

set

$$\Lambda_1 = (\Lambda^+ \cap S^+) \cup (\Lambda^- \cap S^-) \ \ and \ \ \Lambda_2 = (\Lambda^+ \cap S^-) \cup (\Lambda^- \cap S^+). \tag{2}$$

Then $\Lambda = \Lambda_1 \cup \Lambda_2$. Further for any y in C, it follows from (1) that x_n and y_n are of the same sign for $n \in \Lambda_1 \cap \ supp\ (y)$ and are of opposite sign for $n \in \Lambda_2 \cap \ supp\ (y)$. Thus for y in C we have

$$\|x-y\|_{\Lambda_1} = \sum_{n\in\Lambda_1} |x_n - y_n| = \sum_{n\in\Lambda_1} |\, |x_n| - |y_n| \,|$$

and

$$\|x-y\|_{\Lambda_2} = \sum_{n\in\Lambda_2} |x_n - y_n| = \sum_{n\in\Lambda_2} (|x_n| + |y_n|) = \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2}.$$

Also for y in C we have by (2),

$$sgn\ x_n = sgn\ z_n = sgn\ y_n \ \ for \ \ n \in \Lambda_1 \cap \ supp\ (y). \tag{3}$$

In the following we prove that the set $C = J_{l_1}(z)$ is proximinal in l_1. Further, for $x \in l_1$, we characterize $P_C(x)$, the set of all best approximations to x from C. The proofs of both these results involve discussion of two cases, namely $\|x\|_{\Lambda_1} \geq 1$ and $\|x\|_{\Lambda_1} < 1$.

Fact 2.1. *The set $C = J_{l_1}(z)$ is a proximinal set.*

Proof. Fix $x \in l_1$. We first assume that Λ_1 is the emptyset . Then for any $y \in C$ we have $1 = \|y\| = \|y\|_{\Lambda_2}$ and further,

$$\begin{aligned} \|x-y\| &= \|x-y\|_{\Lambda_1} + \|x-y\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} \\ &= \|x-y\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} \\ &= \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} \\ &= \|x\|_{\Lambda_2} + \|y\| + \|x\|_{\mathbb{N}\setminus\Lambda} \\ &= \|x\| + 1. \end{aligned}$$

Hence

$$d(x,C) = \|x\| + 1 \ \ and \ \ P_C(x) = C \tag{4}$$

in this case.

So we now assume that the set Λ_1 is non- empty. We now consider two cases.
Case (i) $\|x\|_{\Lambda_1} \geq 1$.
We have

$$\begin{aligned} \|x-y\| &= \|x-y\|_{\Lambda_1} + \|x-y\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} \\ &\geq \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} \\ &\geq \|x\|_{\Lambda_1} - 1 + \|x\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda}. \end{aligned} \tag{5}$$

since $\|y\|_{\Lambda_1} \leq 1$.

Case (ii) $\|x\|_{\Lambda_1} < 1$.

$$\begin{aligned}\|x-y\| &= \|x-y\|_{\Lambda_1} + \|x-y\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} \\ &\geq \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} \\ &= 1 - \|x\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} \end{aligned} \tag{6}$$

as $1 = \|y\|_{\Lambda} = \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2}$. It is clear from (5) and (6) that

$$d(x,C) \geq |1 - \|x\|_{\Lambda_1}| + \|x\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda}. \tag{7}$$

Define

$$y_n = \begin{cases} \frac{x_n}{\|x\|_{\Lambda_1}} & if\ n \in \Lambda_1 \\ 0 & otherwise. \end{cases}$$

Then $\|y\| = 1$ and using (1) and (3) we conclude y is in C. Also, $\|x-y\| = |1 - \|x\|_{\Lambda_1}| + \|x\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda}$. This together with (7) implies that $d(x,C) = \|x-y\|$ and therefore y is a nearest element to x from the exposed face C. This proves the proximinality of the set C. Also it is clear that

$$d(x,C) = |1 - \|x\|_{\Lambda_1}| + \|x\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} = |1 - \|x\|_{\Lambda_1}| + \alpha, \tag{8}$$

where $\alpha = \|x\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda}$. ■

We recall that $P_C(x)$, the set of best approximations to x from C is the set C itself, if Λ_1 is an empty set. We now characterize the set $P_C(x)$, when the set Λ_1 is non- empty.

Fact 2.2. *For $x \in l_1$ and $z \in NA(l_1)$, assume that the set Λ_1 is non-empty. Then the set $P_C(x)$, where $C = J_{l_1}(z)$, is given by*

$$P_C(x) = \begin{cases} \{y \in C : \|y\|_{\Lambda_2} = 0 \ \ and \ \ \|x-y\|_{\Lambda_1} = \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1}\}, & if\ \|x\|_{\Lambda_1} \geq 1 \\ \{y \in C : \|x-y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}\}, & if\ \|x\|_{\Lambda_1} < 1. \end{cases}$$

Proof. The proof involves discussion of two cases, as in the case of the previous Fact.

Case 1: $\|x\|_{\Lambda_1} \geq 1$.

Pick $y \in P_C(x)$. Then $\|y\|_{\Lambda_1} \leq 1$ and also using (8), we write

$$\begin{aligned} d(x,C) = |1 - \|x\|_{\Lambda_1}| + \alpha &= \|x-y\| \\ &= \|x-y\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus\Lambda} \\ &= \|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha \\ &\geq \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha. \\ &\geq \|x\|_{\Lambda_1} - 1 + \|y\|_{\Lambda_2} + \alpha. \\ &\geq |1 - \|x\|_{\Lambda_1}| + \alpha. \end{aligned}$$

So equality holds throughout and therefore

$$\|y\|_{\Lambda_2} = 0 \ \ and \ \ \|x-y\|_{\Lambda_1} = \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1}.$$

Conversely if y is in C and the above two conditions are satisfied then $\|y\| = \|y\|_{\Lambda_1} = 1$ and

$$\|x-y\| = \|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha = \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} + \alpha = \|x\|_{\Lambda_1} - 1 + \alpha = d(x,C)$$

from (8) and y is in $P_C(x)$.

It is now seen if $\|x\|_{\Lambda_1} \geq 1$

$$y \in P_C(x) \Leftrightarrow y \in C, \|y\|_{\Lambda_2} = 0 \ \textit{and} \ \|x-y\|_{\Lambda_1} = \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1}. \tag{9}$$

Case (ii) $\|x\|_{\Lambda_1} < 1$.

In this case, for $y \in P_C(x)$, we have again using (8),

$$d(x,C) = 1 - \|x\|_{\Lambda_1} + \alpha = \|x-y\| = \|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha.$$

Hence

$$\|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} = 1 - \|x\|_{\Lambda_1}.$$

Now

$$\begin{aligned} \|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} = 1 - \|x\|_{\Lambda_1} &\Leftrightarrow \|x-y\|_{\Lambda_1} + \|x\|_{\Lambda_1} = 1 - \|y\|_{\Lambda_2} \\ &\Leftrightarrow \|x-y\|_{\Lambda_1} + \|x\|_{\Lambda_1} = \|y\|_{\Lambda_1} \\ &\Leftrightarrow \|x-y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}. \end{aligned}$$

Hence y in $P_C(x)$ implies that $\|x-y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}$.

Conversely, if y is in C we have $1 = \|y\| = \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2}$. Further if y is such that $\|x-y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}$, then

$$\begin{aligned} \|x-y\| &= \|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha \\ &= \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha \\ &= \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2} - \|x\|_{\Lambda_1} + \alpha \\ &= 1 - \|x\|_{\Lambda_1} + \alpha \\ &= d(x,C). \end{aligned}$$

Hence y is in $P_C(x)$. Thus if $\|x\|_{\Lambda_1} < 1$

$$y \in P_C(x) \Leftrightarrow y \in C \ \textit{and} \ \|x-y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}. \tag{10}$$

■

3 Strong proximinality and Semi continuity of metric projection

We now prove the strong proximinality of the set $J_{l_1}(z)$.

Theorem 3.1. *Let $C = J_{l_1}(z)$, for z in $NA_1(l_1)$ and x in l_1. If y is in C and $\|x-y\| < d(x,C) + \delta$ for some $\delta > 0$ then there exists w in $P_C(x)$ such that $\|y-w\| < 2\delta$. In particular, C is strongly proximinal in l_1.*

Proof. If the set Λ_1, given by (2), is empty, then $P_C(x) = C$ by (4). Hence the theorem is trivially true in this case. So we assume that the set Λ_1 is non-empty.

We have, by (8),

$$\begin{aligned}\|x-y\| &= \|x-y\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{\mathbb{N}\setminus A} \\ &= \{\|x-y\|_{\Lambda_1} - |1-\|x\|_{\Lambda_1}|\} + \|y\|_{\Lambda_2} + d(x,C) \\ &< d(x,C) + \delta\end{aligned}$$

which implies

$$\{\|x-y\|_{\Lambda_1} - |1-\|x\|_{\Lambda_1}|\} + \|y\|_{\Lambda_2} < \delta. \tag{11}$$

We now discuss three cases: $\|x\|_{\Lambda_1} = 1$, $\|x\|_{\Lambda_1} > 1$ *and* $\|x\|_{\Lambda_1} < 1$. In the following, for $y \in C$, we set

$$L_y = \{n \in \Lambda_1 : |x_n| > |y_n|\}, \ \ E_y = \{n \in \Lambda_1 : |x_n| = |y_n|\}$$

and

$$G_y = \{n \in \Lambda_1 : |y_n| > |x_n|\}.$$

Then $L_y \cup E_y \cup G_y = \Lambda_1$.

Case (i): $\|x\|_{\Lambda_1} = 1$.

In this case, by (11), $\|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} < \delta$. Setting

$$v_n = \begin{cases} x_n & if\ n \in \Lambda_1 \\ 0 & otherwise. \end{cases}$$

Then $\|v\| = \|x\|_{\Lambda_1} = 1$. Also, it follows from (1) that v is in C. We have $\|v\|_{\Lambda_2} = 0$ and $0 = \|x-v\|_{\Lambda_1} = \|x\|_{\Lambda_1} - 1 = \|x\|_{\Lambda_1} - \|v\|_{\Lambda_1}$. So using (9), we conclude v is in $P_C(x)$. Further

$$\|y-v\| = \|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} < \delta.$$

Taking $w = v$, completes the proof for this case.

Case (ii) $\|x\|_{\Lambda_1} > 1$.

Now

$$\begin{aligned}\|x-y\|_{\Lambda_1} &= \sum_{n\in\Lambda_1} |x_n - y_n| \\ &= \sum_{n\in\Lambda_1} |\,|x_n| - |y_n|\,| \\ &= \sum_{n\in(L_y\cup E_y)} \{|x_n| - |y_n|\} + \sum_{n\in G_y} \{|y_n| - |x_n|\} \\ &= \sum_{L_y\cup E_y\cup G_y} |x_n| - \sum_{L_y\cup E_y\cup G_y} |y_n| - 2\sum_{G_y}|x_n| + 2\sum_{G_y}|y_n| \\ &= \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} - 2\sum_{G_y}|x_n| + 2\sum_{G_y}|y_n|\end{aligned}$$

This together with (11) implies that

$$\|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} + 2[\sum_{G_y}\{|y_n| - |x_n|\}] - \{\|x\|_{\Lambda_1} - 1\} + \|y\|_{\Lambda_2} < \delta$$

which, in turn, implies

$$2[\sum_{G_y}[|y_n| - |x_n|] + \|y\|_{\Lambda_2}] < \delta, \tag{12}$$

as $1 - \|y\|_{\Lambda_1} = \|y\|_{\Lambda_2}$.

Define $v \in l_1$ by

$$v_n = \begin{cases} y_n & on \;\; L_y \\ x_n & on \;\; E_y \cup G_y \\ 0 & on \;\; \mathbb{N} \setminus (L_y \cup E_y \cup G_y) = \mathbb{N} \setminus \Lambda_1. \end{cases}$$

Then $\|y - v\| < \delta$ by (12) and so $\|v\| > 1 - \delta$. Further it is clear from the definition of v that $\|v\| \leq \|y\| = 1$. Thus $1 - \delta \leq \|v\| \leq 1$.

Now , using (3) we get

$$sgn\ v_n = \ sgn\ z_n = \ sgn\ x_n \ \ for\ all \ \ n \in \Lambda_1 \cap \ supp\ (v). \tag{13}$$

This with (1) would imply that v is in C if $\|v\| = 1$. Thus if $\|v\| = 1$, since $\|v\|_{\Lambda_2} = 0$ and

$$\begin{aligned} \|x - v\|_{\Lambda_1} &= \sum_{L_y} |x_n - y_n| \\ &= \sum_{L_y} [\ |x_n| - |y_n|\ ] \\ &= \sum_{L_y} |x_n| + \sum_{G_y \cup E_y} |x_n| - \sum_{G_y \cup E_y} |x_n| - \sum_{L_y} |y_n| \\ &= \|x\|_{\Lambda_1} - \|v\|_{\Lambda_1}, \end{aligned} \tag{14}$$

we can conclude using (9) that $v \in P_C(x)$ and take $w = v$ to complete the proof for this case. So we assume that $\|v\| < 1$.

Let $\|x\|_{\Lambda_1} = 1 + \epsilon$. Then $\epsilon > 0$ as $\|x\|_{\Lambda_1} > 1$. Let $\|v\| = 1 - \delta_1$. Then $0 < \delta_1 \leq \delta$. We have $|v_n| \leq |x_n|$ for all $n \in \Lambda_1$ and so

$$\begin{aligned} \|x - v\|_{\Lambda_1} &= \sum_{\Lambda_1} [\ |x_n| - |v_n|\ ] \\ &= \|x\|_{\Lambda_1} - \|v\|_{\Lambda_1} = \epsilon + \delta_1. \end{aligned}$$

Let $\lambda = \frac{\epsilon}{\epsilon + \delta_1}$ and define

$$w_n = \begin{cases} \lambda v_n + (1 - \lambda) x_n & if\ n \in \Lambda_1 \\ 0 & otherwise. \end{cases}$$

We have, using (13),

$$\begin{aligned}\|w\| = \|w\|_{\Lambda_1} &= \sum_{\Lambda_1} |w_n| \\ &= \sum_{\Lambda_1} |\lambda\, v_n + (1-\lambda)x_n| \\ &= \sum_{\Lambda_1} [\, \lambda\, |v_n| + (1-\lambda)|x_n|\,] \\ &= \lambda\, \|v\|_{\Lambda_1} + (1-\lambda)\|x\|_{\Lambda_1} \\ &= \lambda(1-\delta_1) + (1-\lambda)(1+\epsilon) \\ &= \frac{\epsilon}{\epsilon+\delta_1}(1-\delta_1) + \frac{\delta_1}{\epsilon+\delta_1}(1+\epsilon) \\ &= 1\end{aligned}$$

Further

$$\begin{aligned}\|w-v\| &\leq (1-\lambda)\|x-v\|_{\Lambda_1} \\ &= \frac{\delta_1}{\epsilon+\delta_1}\,(\epsilon+\delta_1) \\ &= \delta_1 \leq \delta\end{aligned}$$

Now $\|w\| = 1$ and also it follows from (13) that $sgn\ w_n = sgn\ z_n$ for all $n \in \Lambda_1$. Therefore w is in C by (1). Clearly, $\|w\|_{\Lambda_2} = 0$. Further it is easily verified that

$$\|x-w\|_{\Lambda_1} = \lambda\|x-v\|_{\Lambda_1} \ \ \textit{and}\ \ \|x\|_{\Lambda_1} - \|w\|_{\Lambda_1} = \lambda\, [\, \|x\|_{\Lambda_1} - \|v\|_{\Lambda_1}\,].$$

Now, using (9) and (14), we see that w is in $P_C(X)$. Also,

$$\begin{aligned}\|y-w\| &= \|y-v\| + \|v-w\| \\ &< \delta + \delta = 2\delta.\end{aligned}$$

This completes the proof for this case.
Case (iii) $\|x\|_{\Lambda_1} < 1$.

Now

$$\begin{aligned}\|x-y\|_{\Lambda_1} &= \sum_{n\in\Lambda_1} |x_n - y_n| \\ &= \sum_{G_y\cup E_y} \{|y_n| - |x_n|\} + \sum_{L_y}\{|x_n| - |y_n|\} \\ &= \sum_{G_y\cup E_y\cup L_y} |y_n| - \sum_{L_y\cup E_y\cup G_y} |x_n| + 2\sum_{E_x}\{|x_n| - |y_n|\} \\ &= \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1} + 2\sum_{L_y}\{|x_n| - |y_n|\}.\end{aligned}$$

This together with (11) we have

$$\|y\|_{\Lambda_1} - \|x\|_{\Lambda_1} + 2\sum_{L_y}\{|x_n| - |y_n|\} + \|y\|_{\Lambda_2} - |1 - \|x\|_{\Lambda_1}| < \delta.$$

Since $1 = \|y\| = \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2}$, this in turn implies

$$1 - \|x\|_{\Lambda_1} + 2\sum_{L_y}\{|x_n| - |y_n|\} - (1 - \|x\|_{\Lambda_1}) = \sum_{L_y}\{|x_n| - |y_n|\} < \frac{\delta}{2}. \tag{15}$$

Define $v \in l_1$ by

$$v_n = \begin{cases} x_n & on \;\; L_y \\ y_n & on \;\; \mathbb{N} \setminus L_y. \end{cases}$$

Then, by (15), $\|y - v\| < \delta$. Also it is clear from the definition of v that

$$sgn\ v_n = sgn\ z_n \;\; for\ n \in \Lambda_1 \cap\ supp\ (v) \tag{16}$$

and $\|v\| \geq \|y\| = 1$.

If $\|v\| = 1$, then $v \in C$ by (1). Further, since $|x_n| = |y_n| = |v_n|$ for $n \in E_y$, using the definition of v we have

$$\begin{aligned} \|x - v\|_{\Lambda_1} &= \sum_{\Lambda_1 \setminus L_y} |x_n - y_n| \\ &= \sum_{G_y} [\, |y_n| - |x_n| \,] \\ &= \sum_{\Lambda_1} [\, |v_n| - |x_n| \,] \\ &= \|v\|_{\Lambda_1} - \|x\|_{\Lambda_1}. \end{aligned} \tag{17}$$

Hence (10) holds and we conclude $v \in P_C(x)$. We take $w = v$ and complete the proof in this case. Therefore we assume that $\|v\| > 1$.

Now$\|y\| = 1$ and $\|y - v\| < \delta$ Therefore

$$1 \leq \|v\| \leq 1 + \delta.$$

Let $\|v\| = 1 + \delta_1$. Then $0 < \delta_1 \leq \delta$. Also, $\|x\|_{\Lambda_1} < 1$ and we choose $\epsilon > 0$ such that

$$1 - \epsilon = \|x\|_{\Lambda_1} = \sum_{L_y} |x_n| + \sum_{\Lambda_1 \setminus L_y} |x_n|.$$

Let $\lambda = \frac{\epsilon}{\epsilon + \delta_1}$ and define

$$w_n = \begin{cases} \lambda v_n + (1 - \lambda) x_n & if\ n \in \Lambda_1 \\ 0 & otherwise. \end{cases}$$

Then $\|w\|_{L_y} = \|x\|_{L_y}$. Moreover

$$\begin{aligned}
\sum_{\Lambda_1 \setminus L_y} |w_n| &= \sum_{\Lambda_1 \setminus L_y} |\lambda\, v_n + (1-\lambda) x_n| \\
&= \sum_{\Lambda_1 \setminus L_y} [\, \lambda\, |y_n| + (1-\lambda)|x_n|\,] \\
&= \lambda\, \|y\|_{\Lambda_1 \setminus L_y} + (1-\lambda)\|x\|_{\Lambda_1 \setminus L_y} \\
&= \lambda(1 - \|y\|_{L_y}) + (1-\lambda)(1 - \sum_{L_y} |x_n| - \epsilon) \\
&= \lambda(1 - \sum_{L_y} |x_n| + \delta_1) + (1-\lambda)(1 - \sum_{L_y} |x_n| - \epsilon) \\
&= 1 - \sum_{L_y} |x_n|
\end{aligned}$$

and this implies

$$\|w\| = \|w\|_{\Lambda_1} = \|w\|_{L_y} + \|w\|_{\Lambda_1 \setminus L_y} = 1.$$

Further

$$\begin{aligned}
\|w - v\| &\le (1-\lambda)\|x - v\|_{\Lambda_1} \\
&= \frac{\delta_1}{\epsilon + \delta_1}\, \epsilon + \delta_1 \\
&= \delta_1 \le \delta
\end{aligned}$$

Now $\|w\| = 1$ and using (16) we have

$$sgn\ w_n = sgn\ z_n \quad for\ \ n \in \Lambda_1 \cap supp\ (w).$$

This with (1) implies $w \in C$.

It is easily verified using (17) that

$$\|w - x\|_{\Lambda_1} = \|w\|_{\Lambda_1} - \|x\|_{\Lambda_1}$$

and using (10), we conclude that $w \in P_C(x)$. Finally

$$\begin{aligned}
\|y - w\| &= \|y - v\| + \|v - w\| \\
&< \delta + \delta = 2\delta
\end{aligned}$$

and this completes the proof of the theorem . ∎

We now proceed to show that the metric projection P_C is Hausdorff metric continuous. Since C is strongly proximinal the metric projection P_C is upper Hausdorff semi-continuous and we need to prove only the lower Hausdorff semi-continuity of P_C.We need the following Fact for this purpose.

Fact 3.2. *Let C be a closed convex subset of a Banach space X, which satisfies a uniform version of strong proximinality, namely: there exists a function ϕ from $[0, +\infty)$ to itself such that $\phi(0) = 0$ and ϕ is right-continuous at 0, such that for any $x \in X$ and any $\delta > 0$, if $y \in C$ satisfies $\|x - y\| < d(x, C) + \delta$, there exist $w \in P_C(x)$ such that $\|y - w\| \le \phi(\delta)$. Then the metric projection P_C is Hausdorff metric continuous.*

Proof. Let $x \in l_1$ and $\epsilon > 0$ be given. We can choose $\delta > 0$, such that $\phi(\delta) < \epsilon$, since $\phi(0) = 0$ and ϕ is right-continuous at 0. We now claim that for any $y \in P_C(x)$ and $z \in l_1$ with $\|x - z\| < \frac{\delta}{2}$, there exists $w \in P_C(z)$ satisfying $\|y - w\| < \epsilon$. This would clearly imply the lower Hausdorff semi-continuity of the set valued map P_C at x.

Since $\|x - z\| < \frac{\delta}{2}$, we have

$$|d(x,C) - d(z,C)| \leq \|x - z\| < \frac{\delta}{2}.$$

Thus

$$\begin{aligned} \|z - y\| &\leq \|z - x\| + \|x - y\| \\ &< \frac{\delta}{2} + d(x,C) \\ &\leq d(z,C) + \delta \end{aligned}$$

Now, using our assumption, we conclude there exists $w \in P_C(z)$ such that $\|y - w\| < \phi(\delta) < \epsilon$. ∎

The above Fact in conjunction with Theorem 3.1 implies Hausdorff metric continuity of the map P_C, as given below.

Fact 3.3. *If z in $NA_1(l_1)$ and $C = J_{l_1}(z)$, then the metric projection P_C is Hausdorff metric continuous.*

Proof. It follows from the statement of Theorem 3.1 that $\phi(\delta) = 2\delta$ in this case. The map ϕ clearly satisfies the conditions of Fact 3.2 and hence it follows that the metric projection P_C is Hausdorff metric continuous. ∎

Thus P_C is Hausdorff metric continuous and by the well known Michael selection theorem [8, 9], P_C has a continuous selection.

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