Complex of injective words revisited

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Abstract

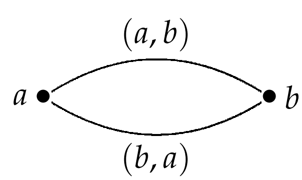
We give a simple proof that (a generalization of) the complex of injective words has vanishing homology in all except the top degree.

1 Introduction

Let A be a finite set. An *injective word* of length $r \geqslant 0$ is a sequence $(a_1, \ldots, a_r)$ of pairwise distinct elements of A. Let $K(A)$ be the semi-simplicial set whose $(r-1)$-simplices are the injective words of length r, for every $r \geqslant 1$. The face maps of $K(A)$ are defined by

$$d_{i-1}(a_1, \ldots, a_r) := (a_1, \ldots, \widehat{a_i}, \ldots, a_r) \quad \text{for } i = 1, \ldots, r,$$

where $\widehat{x}$ means that the entry x is omitted. In other words, $K(A)$ is the semi-simplicial set of ordered simplices of the abstract simplex whose set of vertices is A. We write $|K(A)|$ for the geometric realization of $K(A)$. For example, if $A = \{a, b\}$, then $|K(A)|$ is homeomorphic to a circle:



In a 1979 paper, Farmer [4, Theorem 5] shows that the reduced homology of $K(A)$ vanishes in all degrees $\neq |A| - 1$. Subsequently, a new proof was found

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by Björner and Wachs [2, Theorem 6.1] using their theory of CL-shellable posets. A simpler proof of Farmer's result was given by Kerz [6, Theorem 1] in 2004. Both the proofs of Farmer and Kerz proceed by somewhat ad hoc calculations. More recently, topological proofs of Farmer's result are given by Bestvina [1, Claim in the proof of Proposition 6] and Randal-Williams [9, Proposition 3.2]. We should mention that Björner-Wachs and Randal-Williams actually proved the stronger result that $|K(A)|$ is homotopy equivalent to a wedge of spheres of dimension $|A|-1$.

The purpose of our present note is to give a simple and natural algebraic proof of Farmer's result; indeed, our proof is a straightforward exercise on the spectral sequence of a filtered complex. Our interest in this result stems from the crucial role it plays in Quillen's method [8] for proving homological stability of the symmetric groups. Following Hatcher and Wahl [5], we shall formulate and prove a slightly more general theorem so that it can be applied in the proof of homological stability of wreath-product groups.

Notation 1. We write S_n for the symmetric group on $\{1,\ldots,n\}$. For any group G, we write G_n for the wreath-product group $G \wr S_n$, that is, $G_n := S_n \ltimes G^n$. In particular, G_0 is the trivial group.

2 The main result

Recall that A denotes a finite set. Let Γ be a nonempty set. We define a chain complex $C_*(A)$ concentrated in degrees $0,\ldots,|A|$ as follows. Let $C_r(A)$ be the free abelian group generated by the set $\Delta_r(A)$ consisting of all elements of the form $(a_1,\ldots,a_r,\gamma_1,\ldots,\gamma_r)$ where $a_1,\ldots,a_r$ are pairwise distinct elements of A, and $\gamma_1,\ldots,\gamma_r$ are any elements of Γ; in particular, $C_r(A)=\mathbb{Z}$ if $r=0$. The differential is defined by

$$d(a_1,\ldots,a_r,\gamma_1,\ldots,\gamma_r) := \sum_{i=1}^{r}(-1)^{i-1}(a_1,\ldots,\widehat{a_i},\ldots,a_r,\gamma_1,\ldots,\widehat{\gamma_i},\ldots,\gamma_r).$$

In particular, $d(a_1,\gamma_1)=1$.

For any chain complex C_* and positive integer p, we shall write C_{*-p} for the p-fold suspension of C_*.

Remark 2. If Γ is a singleton set, then $C_*(A)$ is the augmented chain complex of $K(A)$ with degrees shifted up by 1. (Topologically, it is more natural to place a word of length r in degree $r-1$. Algebraically, it seems more natural for us to place a word of length r in degree r.)

If Γ is a group G, then $C_*(A)$ is the augmented chain complex of the semi-simplicial set $W_{|A|}(\emptyset,\{1\})$ (defined by Randal-Williams and Wahl in [10, Definition 2.1]) with degrees shifted up by 1, associated to the category FI_G (defined by Sam and Snowden in [12]).

Theorem 3. *If $r < |A|$, then $H_r(C_*(A)) = 0$.*

Proof. Set $n = |A|$. We use induction on n. The base case $n = 0$ is trivial.

Suppose $n > 0$. Choose and fix an element $a \in A$. For each $r \geqslant 0$, there is an increasing filtration on $C_r(A)$ defined by letting $F_pC_r(A)$ be the subgroup spanned by all elements $(a_1, \ldots, a_r, \gamma_1, \ldots, \gamma_r)$ such that none of $a_{p+1}, \ldots, a_r$ is equal to a. This gives an increasing filtration on the complex $C_*(A)$ and hence a first-quadrant spectral sequence:

$$E^1_{p,q} = H_{p+q}\left(F_pC_*(A)/F_{p-1}C_*(A)\right) \Rightarrow H_{p+q}(C_*(A)).$$

Observe that $F_0C_*(A) = C_*(A \setminus \{a\})$. For each $p \geqslant 1$, there is an isomorphism of chain complexes:

$$F_pC_*(A)/F_{p-1}C_*(A) \cong \bigoplus_{(a_1,\ldots,a_{p-1},a,\gamma_1,\ldots,\gamma_p)\in\Delta_p(A)} C_{*-p}(A \setminus \{a_1, \ldots, a_{p-1}, a\})$$

where an element on the left hand side represented by $(a_1, \ldots, a_r, \gamma_1, \ldots, \gamma_r) \in F_pC_r(A)$ with $a_p = a$ is identified, on the right hand side, with the element $(a_{p+1}, \ldots, a_r, \gamma_{p+1}, \ldots, \gamma_r)$ in the direct summand $C_{r-p}(A \setminus \{a_1, \ldots, a_{p-1}, a\})$ indexed by $(a_1, \ldots, a_p, \gamma_1, \ldots, \gamma_p) \in \Delta_p(A)$.

By the induction hypothesis, one has:

$$E^1_{0,q} = 0 \quad \text{whenever} \quad q < n-1;$$
$$E^1_{p,q} = 0 \quad \text{whenever} \quad p \geqslant 1 \text{ and } p+q < n.$$

Therefore, it only remains to show that $E^\infty_{0,n-1} = 0$.

We have:

$$E^1_{0,n-1} = H_{n-1}(C_*(A \setminus \{a\})), \qquad E^1_{1,n-1} = \bigoplus_{\gamma_1\in\Gamma} H_{n-1}(C_*(A \setminus \{a\})).$$

The restriction of the differential $d^1 : E^1_{1,n-1} \to E^1_{0,n-1}$ to each direct summand in the above decomposition of $E^1_{1,n-1}$ is the identity map on $H_{n-1}(C_*(A \setminus \{a\}))$; this follows from the identity

$$\begin{aligned} d(a, a_2, \ldots, a_n, \gamma_1, \ldots, \gamma_n) &= (a_2, \ldots, a_n, \gamma_2, \ldots, \gamma_n) \\ &\quad - \sum_{i=1}^{n-1}(-1)^{i-1}(a, a_2, \ldots, \widehat{a_{i+1}}, \ldots, \gamma_1, \gamma_2, \ldots, \widehat{\gamma_{i+1}}, \ldots). \end{aligned}$$

In particular, the map $d^1 : E^1_{1,n-1} \to E^1_{0,n-1}$ is surjective. It follows that $E^2_{0,n-1} = 0$, and we are done. ∎

Remark 4. It was pointed out to us by the referee that the last step in the above proof can be replaced by the following argument. Fixing an element $e \in \Gamma$, the map

$$(a_1, \ldots, a_r, \gamma_1, \ldots, \gamma_r) \mapsto (a, a_1, \ldots, a_r, e, \gamma_1, \ldots, \gamma_r)$$

gives a null-homotopy for the inclusion map $C_*(A - \{a\}) \to C_*(A)$. Thus, the edge map $E^1_{0,n-1} \longrightarrow E^\infty_{0,n-1} \subset H_{n-1}(C_*(A))$ is zero; hence, $E^\infty_{0,n-1} = 0$.

Remark 5. Let $n = |A|$. Since $H_n(C_*(A))$ is a subgroup of the free abelian group $C_n(A)$, it is a free abelian group; we make the following observations on its rank.

(i) Suppose $|\Gamma| = \infty$. If $n = |A| \geqslant 1$, then $H_n(C_*(A))$ is a free abelian group of infinite rank. This is clear if $n = 1$. For $n > 1$, it follows from noticing that by induction, the kernel of $d^1 : E^1_{1,n-1} \to E^1_{0,n-1}$ (in the spectral sequence in the proof above) is a free abelian group of infinite rank, and which is $E^\infty_{1,n-1}$.

(ii) Suppose $|\Gamma| = \ell < \infty$. If $n = |A| \geqslant 1$, then it follows from Theorem 3 and the Euler-Poincaré principle that $H_n(C_*(A))$ is a free abelian group of rank $d(\ell)_n$, where

$$d(\ell)_n := \sum_{i=0}^{n} \frac{(-1)^i n! \ell^{n-i}}{i!}.$$

For $\ell = 1$, this observation is due to Farmer [4, Remark on page 613] and Reiner-Webb [11, Proposition 2.1]. It is well known that $d(1)_n$ is equal to the number of derangements in the symmetric group S_n. Let us give a similar interpretation of $d(\ell)_n$ for any $\ell \geqslant 1$. Fixing an element $e \in \Gamma$, we claim that $d(\ell)_n$ is the number of elements $(\pi, \gamma_1, \ldots, \gamma_n) \in S_n \times \Gamma^n$ such that: if $1 \leqslant a \leqslant n$ and $\pi(a) = a$, then $\gamma_a \neq e$. To see this, let

$$T_a := \{(\pi, \gamma_1, \ldots, \gamma_n) \in S_n \times \Gamma^n \mid \pi(a) = a \text{ and } \gamma_a = e\} \quad \text{for } a = 1, \ldots, n.$$

Then, for any $1 \leqslant a_1 < \cdots < a_i \leqslant n$, one has $|T_{a_1} \cap \cdots \cap T_{a_i}| = (n-i)! \ell^{n-i}$. Hence, by the inclusion-exclusion principle, we have $d(\ell)_n = |S_n \times \Gamma^n| - |T_1 \cup \cdots \cup T_n|$, as claimed.

We note that for any group G, the wreath-product group G_n (see Notation 1) acts on $\{1, \ldots, n\} \times G$ by $(\pi; g_1, \ldots g_n) \cdot (a, \gamma) := (\pi(a), g_a \gamma)$, where $(\pi; g_1, \ldots, g_n) \in G_n$ and $(a, \gamma) \in \{1, \ldots, n\} \times G$. When G is a finite group of order ℓ, the integer $d(\ell)_n$ is equal to the number of elements of G_n which has no fixed point in $\{1, \ldots, n\} \times G$.

3 Application to homological stability

Nakaoka [7, Corollary 6.7] proved that the natural inclusion map $S_{n-1} \to S_n$ induces an isomorphism in homology $H_m(S_{n-1}) \to H_m(S_n)$ if $n > 2m$. His result was generalized by Hatcher and Wahl [5, Proposition 1.6] to wreath-product groups (although as they noted in their paper the generalization might have been known for a long time).

Corollary 6. *Let G be a group. The natural inclusion map $G \wr S_{n-1} \to G \wr S_n$ induces an isomorphism in homology $H_m(G \wr S_{n-1}) \to H_m(G \wr S_n)$ if $n > 2m$.*

Corollary 6 follows from Theorem 3 by a standard argument of Quillen; see [5, Section 5]. We give the details of this argument below since our injectivity range $n > 2m$ is better than the one stated in [5, Proposition 1.6] by 1.

From now on, we set $A = \{1, \ldots, n\}$ and $\Gamma = G$, where n is an integer $\geqslant 1$ and G is a group.

There is a natural action of G_n (see Notation 1) on $C_r(A)$ defined by

$$(\pi; g_1, \ldots, g_n) \cdot (a_1, \ldots, a_r, \gamma_1, \ldots, \gamma_r) := (\pi(a_1), \ldots, \pi(a_r), g_{a_1}\gamma_1, \ldots, g_{a_r}\gamma_r),$$

where $(\pi; g_1, \ldots, g_n) \in G_n$ and $(a_1, \ldots, a_r, \gamma_1, \ldots, \gamma_r) \in C_r(A)$. Define the map

$$d_{i-1} : C_r(A) \longrightarrow C_{r-1}(A) \quad \text{for } i = 1, \ldots, r,$$

by

$$d_{i-1}(a_1, \ldots, a_r, \gamma_1, \ldots, \gamma_r) := (a_1, \ldots, \widehat{a_i}, \ldots, a_r, \gamma_1, \ldots, \widehat{\gamma_i}, \ldots, \gamma_r).$$

Since the map d_{i-1} is G_n-equivariant, there is an induced map

$$(d_{i-1})_* : H_*(G_n; C_r(A)) \to H_*(G_n; C_{r-1}(A)).$$

The group G_n acts transitively on the basis $\Delta_r(A)$ of $C_r(A)$. For any $x \in \Delta_r(A)$, we write $\mathrm{Stab}(x)$ for its stabilizer in G_n. By Shapiro's lemma, the natural inclusion map $\mathrm{Stab}(x) \to G_n$ and the map $\mathbb{Z} \to C_r(A)$, $\lambda \mapsto \lambda x$ induce an isomorphism

$$\alpha(x)_* : H_*(\mathrm{Stab}(x)) \longrightarrow H_*(G_n; C_r(A)).$$

Denote by $e \in G$ the identity element. Let

$$x_r := (n - r + 1, \ldots, n, e, \ldots, e) \in C_r(A).$$

Then $\mathrm{Stab}(x_r) = G_{n-r} \leqslant G_n$. In particular, $x_0 = 1 \in \mathbb{Z}$ and $\mathrm{Stab}(x_0) = G_n$. We write $\iota : \mathrm{Stab}(x_r) \to \mathrm{Stab}(x_{r-1})$ for the natural inclusion map.

Lemma 7. *For every $i = 1, \ldots, r$, the following diagram commutes:*

$$\begin{array}{ccc} H_*(\mathrm{Stab}(x_r)) & \xrightarrow{\iota_*} & H_*(\mathrm{Stab}(x_{r-1})) \\ {\scriptstyle \alpha(x_r)_*}\downarrow & & \downarrow{\scriptstyle \alpha(x_{r-1})_*} \\ H_*(G_n; C_r(A)) & \xrightarrow{(d_{i-1})_*} & H_*(G_n; C_{r-1}(A)) \end{array} .$$

Proof. The diagram clearly commutes for $i = 1$ because $d_0(x_r) = x_{r-1}$.

Suppose $i > 1$. Let $y := d_{i-1}(x_r)$, so

$$y = (n - r + 1, \ldots, \widehat{n - r + i}, \ldots, n, e, \ldots, e) \in C_{r-1}(A).$$

Write $\jmath : \mathrm{Stab}(x_r) \to \mathrm{Stab}(y)$ for the natural inclusion map. Then there is a commuting diagram:

$$\begin{array}{ccc} H_*(\mathrm{Stab}(x_r)) & \xrightarrow{\jmath_*} & H_*(\mathrm{Stab}(y)) \\ {\scriptstyle \alpha(x_r)_*}\downarrow & & \downarrow{\scriptstyle \alpha(y)_*} \\ H_*(G_n; C_r(A)) & \xrightarrow{(d_{i-1})_*} & H_*(G_n; C_{r-1}(A)) \end{array} . \tag{1}$$

Let μ be the cyclic permutation $(n - r + 1, \ldots, n - r + i) \in S_n$ and let

$$t := (\mu; e, \ldots, e) \in G_n.$$

Then t has the properties that $t \cdot y = x_{r-1}$ and

$$tut^{-1} = u \quad \text{for each } u \in \mathrm{Stab}(x_r). \tag{2}$$

We have a commuting diagram

$$\begin{array}{ccc} H_*(\mathrm{Stab}(y)) & \xrightarrow{\kappa_*} & H_*(\mathrm{Stab}(x_{r-1})) \\ \downarrow{\scriptstyle \alpha(y)_*} & & \downarrow{\scriptstyle \alpha(x_{r-1})_*} \\ H_*(G_n; C_{r-1}(A)) & \xrightarrow{\delta_*} & H_*(G_n; C_{r-1}(A)) \end{array} \tag{3}$$

where the top arrow κ_* is induced by the homomorphism $\kappa : \mathrm{Stab}(y) \to \mathrm{Stab}(x_{r-1}),\ u \mapsto tut^{-1}$ and the bottom arrow δ_* is induced by the inner automorphism $G_n \to G_n,\ u \mapsto tut^{-1}$ and the map $C_{r-1}(A) \to C_{r-1}(A),\ x \mapsto t \cdot x$.

By (2), we have $\kappa \circ \jmath = \iota$, so $\kappa_* \circ \jmath_* = \iota_*$. By [3, Proposition III.8.1], the homomorphism δ_* is the identity map on $H_*(G_n; C_{r-1}(A))$. Therefore, it follows from our two commuting diagrams (1) and (3) that

$$\begin{aligned} \alpha(x_{r-1})_* \circ \iota_* = \alpha(x_{r-1})_* \circ \kappa_* \circ \jmath_* = \delta_* \circ \alpha(y)_* \circ \jmath* \\ = \delta_* \circ (d_{i-1})_* \circ \alpha(x_r)_* = (d_{i-1})_* \circ \alpha(x_r)_*. \quad \blacksquare \end{aligned}$$

We are now ready to prove Corollary 6.

Proof of Corollary 6. We use induction on m. The base case $m = 0$ is trivial.

Suppose $m \geqslant 1$. Let $n > 2m$.

Choose any free resolution $\cdots \to F_1 \to F_0 \to \mathbb{Z}$ of $\mathbb{Z}$ over $\mathbb{Z}G_n$. By taking the tensor product over G_n of the two chain complexes $C_*(A)$ and F_*, we obtain a first-quadrant double complex D with $D_{r,s} := F_s \otimes_{G_n} C_r(A)$. Let $\mathrm{Tot}_*(D)$ be the total complex of D. From Theorem 3 and the spectral sequence associated to the horizontal filtration of $\mathrm{Tot}_*(D)$, we deduce that $H_i(\mathrm{Tot}_*(D)) = 0$ for each $i \leqslant n-1$.

We now consider the spectral sequence associated to the vertical filtration of $\mathrm{Tot}_*(D)$. Since $H_i(\mathrm{Tot}_*(D)) = 0$ for each $i \leqslant n-1$, this spectral sequence has:

$$E^\infty_{r,s} = 0 \quad \text{if} \quad r + s \leqslant n-1. \tag{4}$$

The E^1-terms of the spectral sequence are:

$$E^1_{r,s} = H_s(F_* \otimes_{G_n} C_r(A)) = H_s(G_n; C_r(A)).$$

The differential $d^1 : E^1_{r,s} \to E^1_{r-1,s}$ is the map

$$d_* = \sum_{i=1}^{r} (-1)^{i-1} (d_{i-1})_* : H_s(G_n; C_r(A)) \longrightarrow H_s(G_n; C_{r-1}(A)).$$

Recall that $G_{n-r} = \mathrm{Stab}(x_r)$. We shall identify $E^1_{r,s}$ with $H_s(G_{n-r})$ via the isomorphism $\alpha(x_r)_* : H_s(G_{n-r}) \to H_s(G_n; C_r(A))$. Under this identification, we see from Lemma 7 that the differential $d^1 : E^1_{r,s} \to E^1_{r-1,s}$ is:

- the map $\iota_* : H_s(G_{n-r}) \to H_s(G_{n-r+1})$ if r is odd;
- the zero map if r is even.

Therefore, row s on the E^1-page of the spectral sequence is:

$$H_s(G_n) \xleftarrow{\iota_*} H_s(G_{n-1}) \xleftarrow{0} H_s(G_{n-2}) \xleftarrow{\iota_*} H_s(G_{n-3}) \xleftarrow{0} H_s(G_{n-4}) \xleftarrow{\iota_*} \cdots,$$

where the leftmost term is in column 0.

Our goal is to show that $\iota_* : H_m(G_{n-1}) \to H_m(G_n)$ is an isomorphism, or equivalently, that the differential $d^1 : E^1_{1,m} \to E^1_{0,m}$ is an isomorphism. Note that $E^2_{1,m}$ is the kernel of $d^1 : E^1_{1,m} \to E^1_{0,m}$, and $E^2_{0,m}$ is the cokernel of $d^1 : E^1_{1,m} \to E^1_{0,m}$. Therefore, we have to show that $E^2_{1,m} = 0$ and $E^2_{0,m} = 0$. We shall use the following:

Claim. *If $r + 2s \leqslant 2m + 1$ and $s < m$, then $E^2_{r,s} = 0$.*

Proof of Claim. We have:

$$2s \leqslant 2m + 1 - r < n - r + 1.$$

When r is even, we have the stronger inequality:

$$2s \leqslant 2m - r < n - r.$$

Hence, it follows by the induction hypothesis that in row s of the E^1-page of the spectral sequence, we have:

$$\cdots \longleftarrow H_s(G_{n-r+1}) \xleftarrow{d^1_{r,s}} H_s(G_{n-r}) \xleftarrow{d^1_{r+1,s}} H_s(G_{n-r-1}) \longleftarrow \cdots$$

where

- $d^1_{r,s}$ is an isomorphism and $d^1_{r+1,s}$ is the zero map if r is odd;
- $d^1_{r.s}$ is the zero map and $d^1_{r+1,s}$ is an isomorphism if r is even.

Hence, $E^2_{r,s} = 0$.

We have proven the Claim. ∎

The above Claim implies that $E^2_{1,m} = E^\infty_{1,m}$ and $E^2_{0,m} = E^\infty_{0,m}$. Indeed, for $k \geqslant 2$, a differential on the E^k-page has target $E^k_{1,m}$ or $E^k_{0,m}$ only if it starts at $E^k_{k+1,m-k+1}$ or $E^k_{k,m-k+1}$ respectively, but the Claim implies that $E^k_{k+1,m-k+1}$ and $E^k_{k,m-k+1}$ are both zero.

Finally, since $n \geqslant 2m + 1 \geqslant 3$ and so

$$m + 1 \leqslant \frac{n+1}{2} \leqslant n - 1,$$

it follows from (4) that $E^\infty_{1,m} = 0$ and $E^\infty_{0,m} = 0$, so $E^2_{1,m} = 0$ and $E^2_{0,m} = 0$. ∎

Remark 8. As the last step in the proof above shows, we only need to use the vanishing of $H_r(C_*(A))$ for $|A| \geqslant 3$ and $r \leqslant \frac{|A|+1}{2}$.

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