Weighted composition operators on algebras of differentiable functions

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Abstract

Let X be a perfect compact plane set, $n \in \mathbb{N}$ and $D^n(X)$ be the algebra of complex-valued functions on X with continuous n-th derivative. In this paper we study weighted composition operators on algebras $D^n(X)$. We give a necessary and sufficient condition for these operators to be compact. As a consequence, we characterize power compact composition operators on these algebras. Then we determine the spectra of Riesz weighted composition operators on these algebras.

1 Introduction

A complex-valued function f defined on a perfect plane set X is called differentiable on X if at each point $z_0 \in X$ the limit

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \in X}} \frac{f(z) - f(z_0)}{z - z_0},$$

exists. We denote the n-th derivative of f by $f^{(n)}$ when it exists. The algebra of complex-valued functions f on a perfect compact plane set X with continuous n-th derivative is denoted by $D^n(X)$. This algebra with the norm

$$\|f\|_n = \sum_{r=0}^{n} \frac{\|f^{(r)}\|_X}{r!} \qquad (f \in D^n(X)),$$

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is a normed function algebra on X which is not necessarily complete, where

$$\|f\|_X = \sup_{x \in X} |f(x)|.$$

For example, Bland and Feinstein showed that $D^1(X)$ is incomplete whenever X has infinitely many components [4, Theorem 2.3]. By standard methods one can show that if $D^1(X)$ is complete, then $D^n(X)$ is complete for each $n \in \mathbb{N}$, see [4, 15]. To provide a sufficient condition for the completeness of $D^1(X)$, let us recall the definition of pointwise regularity and uniform regularity for compact plane sets.

Definition 1.1. Let X be a rectifiably connected compact plane set and let $\delta(z, w)$ be the geodesic metric on X, the infimum of the lengths of the rectifiable path from z to w in X.

(i) X is called pointwise regular if for each $z_0 \in X$ there exists a constant c_{z_0} such that for all $z \in X$, $\delta(z, z_0) \leq c_{z_0}|z - z_0|$.

(ii) X is called uniformly regular if there exists a constant c such that for all $z, w \in X$, $\delta(z, w) \leq c|z - w|$.

Dales and Davie [8, Theorem 1.6] showed that $D^1(X)$ is complete whenever X is a finite union of uniformly regular sets. Indeed, they proved that for each z_0 in such set X, there exists a constant c_{z_0} such that for all $f \in D^1(X)$ and each $z \in X$,

$$|f(z) - f(z_0)| \leq c_{z_0}|z - z_0|(\|f\|_X + \|f'\|_X), \tag{1.1}$$

and using this inequality, they showed that $D^1(X)$ is complete. Later in [12], it was shown that the condition (1.1) is still valid when X is a finite union of pointwise regular sets, in fact, it is a necessary and sufficient condition for the completeness of $D^1(X)$ (see also [15]).

Let $C(X)$ be the algebra of all continuous complex-valued functions on a compact Hausdorff space X. A unital subalgebra A of $C(X)$ that separates the points of X is a *function algebra* on X. A function algebra A on X is said to be *natural* if every nonzero complex homomorphism (character) on A is an evaluation homomorphism at some point of X [7, Definition 4.1.3]. As it was proved in [8], the algebra $D^n(X)$ is natural when X is uniformly regular. However, as mentioned in [13], applying the same method used in it, one can show that the algebra $D^n(X)$ is natural for every perfect compact plane set X (see also [9, Theorem 4.1]).

Let A be a linear space of functions on a set X. Let u be a complex-valued function on X and φ be a self-map of X. A linear operator $T := uC_\varphi$ defined by $uC_\varphi(f) = u \cdot (f \circ \varphi)$ is a weighted composition operator on A if $u \cdot (f \circ \varphi) \in A$ whenever $f \in A$. In the case where $u = 1$, the operator uC_φ reduces to the composition operator C_φ. In [2], Behrouzi obtained some results on compactness of composition operators between algebras $D^n(X)$. In this paper, we study weighted composition operators acting on algebras $D^n(X)$ when perfect compact plane sets X satisfy the condition (1.1). Let $\mathrm{coz}(u) = \{z \in X : u(z) \neq 0\}$. In Section 2, for $u, \varphi \in D^n(X)$ we show that if either φ is constant or $\varphi(\mathrm{coz}(u)) \subseteq \mathrm{intX}$, then uC_φ is compact on $D^n(X)$. We also show that these conditions are necessary for certain compact plane sets X. Using these results, we give a necessary and sufficient condition for a composition operator (endomorphism) on $D^n(X)$ to be power compact.

Let X be a compact plane set and $A(X)$ be the uniform algebra of all continuous functions on X which are analytic on int$X$. Suppose A is a unital Banach subalgebra of $A(X)$, containing the coordinate function z. In Section 3, we study the spectrum of a weighted composition operator on such algebras A. In [3], the spectrum of a compact composition operator C_φ on A was determined as

$$\sigma_A(C_\varphi) = \{\varphi'(z_0)^k : k \text{ is a positive integer}\} \cup \{0, 1\},$$

when $\varphi(X) \subseteq \text{int}X$ and z_0 is a fixed point of φ. We show that the spectrum of a Riesz weighted composition operator uC_φ on A is

$$\sigma_A(uC_\varphi) = \{u(z_0)\varphi'(z_0)^k : k \text{ is a positive integer}\} \cup \{0, \mathrm{u}(\mathrm{z}_0)\},$$

when φ has a fixed point $z_0 \in \text{int}X$. Then we conclude this result for the Banach algebra $D^n(X)$. In the case that φ has all its fixed points on boundary, we show that $\sigma(uC_\varphi) = \{0\}$ for a compact operator uC_φ on $D^n(\overline{\mathbb{D}})$ where $\mathbb{D}$ is the open unit disc in the complex plane.

2 Compactness

It is known that if $u, \varphi \in D^n(X)$, then uC_φ is a weighted composition operator on $D^n(X)$. Conversely, if uC_φ is a weighted composition operator on $D^n(X)$, then $u \in D^n(X)$ although φ does not necessarily belong to $D^n(X)$, even it may not be continuous on X. Here we give a necessary and sufficient condition on u and φ for uC_φ to be a weighted composition operator on $D^1(X)$.

Theorem 2.1. *Let X be a perfect compact plane set. Let u be a complex-valued function on X and φ be a self-map of X not necessarily continuous. Then uC_φ is a weighted composition operator on $D^1(X)$ if and only if u and $u\varphi$ belong to $D^1(X)$.*

Proof. Let uC_φ be a weighted composition operator on $D^1(X)$. Then $u, u\varphi \in D^1(X)$, since this algebra contains constant functions and the coordinate function z.

Conversely, let u and $u\varphi$ belong to $D^1(X)$. Then $\varphi = \frac{u\varphi}{u}$ is differentiable on $\text{coz}(u)$ and $\varphi' = \frac{(u\varphi)' - u'\varphi}{u}$. If $z \in X$ with $u(z) = 0$ and $u'(z) \neq 0$, then u is nonzero on a punctured neighborhood of z and

$$\lim_{w \to z} \varphi(w) = \lim_{w \to z} \frac{\frac{u(w)\varphi(w) - u(z)\varphi(z)}{w - z}}{\frac{u(w) - u(z)}{w - z}} = \frac{(u\varphi)'(z)}{u'(z)}.$$

Hence, in this case, $\varphi_1(z) := \lim_{w \to z} \varphi(w)$ exists and belongs to X, so we can write $(u\varphi)'(z) = u'(z)\varphi_1(z)$. When $u(z) = u'(z) = 0$, we have $(u\varphi)'(z) = 0$, since φ is bounded. These relations along with the continuity of $(u\varphi)'$ imply that

$$\lim_{\substack{w \to z \\ w \in \text{coz}(u)}} (u\varphi')(w) = 0, \tag{2.1}$$

whenever $u(z) = 0$ and z is in the closure of $\operatorname{coz}(u)$. Let $f \in D^1(X)$. Then

$$\begin{aligned}(u\cdot(f\circ\varphi))'(z) &= \lim_{w\to z}\frac{u(w)f(\varphi(w))}{w-z} = \lim_{w\to z}\frac{u(w)-u(z)}{w-z}f(\varphi(w))\\ &= \begin{cases} u'(z)f(\varphi_1(z)) & u'(z)\neq 0\\ 0 & u'(z)=0,\end{cases}\end{aligned}$$

whenever $u(z) = 0$. Therefore, for each $z \in X$ we have

$$(u\cdot(f\circ\varphi))'(z) = \begin{cases} u'(z)f(\varphi(z)) + u(z)\varphi'(z)f'(\varphi(z)) & u(z)\neq 0\\ u'(z)f(\varphi_1(z)) & u(z)=0, u'(z)\neq 0\\ 0 & u(z)=0, u'(z)=0.\end{cases}$$

We show that $(u\cdot(f\circ\varphi))'$ is continuous on X. Obviously, it is continuous on $\operatorname{coz}(u)$. Now let $z \in X$ with $u(z) = 0$ and (z_n) be a sequence in X such that $z_n \neq z$ and $\lim z_n = z$. Without loss of generality we can assume that either $(z_n) \subseteq \operatorname{coz}(u)$ or $u(z_n) = 0$ for all $n \in \mathbb{N}$. In the case that $(z_n) \subseteq \operatorname{coz}(u)$, by using (2.1), $\lim_n u(z_n)\varphi'(z_n) = 0$, hence

$$\begin{aligned}\lim_n (u\cdot(f\circ\varphi))'(z_n) &= \lim_n [u'(z_n)f(\varphi(z_n)) + u(z_n)\varphi'(z_n)f'(\varphi(z_n))]\\ &= \begin{cases} u'(z)f(\varphi_1(z)) & u'(z)\neq 0\\ 0 & u'(z)=0.\end{cases}\end{aligned}$$

In the second case, $u(z_n) = 0$ for all $n \in \mathbb{N}$, by the definition of derivative, $u'(z) = 0$ and hence $\lim_n(u\cdot(f\circ\varphi))'(z_n) = 0$. This argument shows that $(u\cdot(f\circ\varphi))'$ is continuous and the proof is complete. ∎

To give a necessary and sufficient condition for compactness of uC_φ on $D^n(X)$ we need the following notations.

Let φ and f belong to $D^n(X)$ with $\varphi : X \to X$. The following equality for higher derivatives of composite functions is known as Faà di Bruno's formula [1, page 823],

$$(f\circ\varphi)^{(n)} = \sum_{j=1}^{n}(f^{(j)}\circ\varphi)\cdot\psi_{j,n},$$

where

$$\psi_{j,n} = \sum_a\left(\frac{n!}{a_1!a_2!\cdots a_n!}\prod_{i=1}^{n}\left(\frac{\varphi^{(i)}}{i!}\right)^{a_i}\right),$$

the sum $\sum_a$ is taken over all non-negative integers $a_1, a_2, \dots, a_n$ satisfying $a_1 + a_2 + \cdots + a_n = j$ and $a_1 + 2a_2 + \cdots + na_n = n$. For example, $\psi_{1,n} = \varphi^{(n)}$ and $\psi_{n,n} = (\varphi')^n$. We also need the Leibniz's formula of products of functions. For $f, g \in D^n(X)$ we have

$$(fg)^{(n)} = \sum_{j=0}^{n}\tbinom{n}{j}f^{(j)}\cdot g^{(n-j)}.$$

In the case that X satisfies the condition (1.1), for each $z_0 \in X$ we define

$$p_{z_0}(f) := \sup_{\substack{z\in X\\ z\neq z_0}} \frac{|f(z)-f(z_0)|}{|z-z_0|} \qquad (f \in D^1(X)).$$

Then for each $z_0 \in X$ there exists a constant c_{z_0} such that

$$p_{z_0}(f) \leq c_{z_0}(\|f\|_X + \|f'\|_X) \qquad (f \in D^1(X)). \tag{2.2}$$

In general, for a constant self-map φ of X, the weighted composition operator uC_φ on a normed function algebra A on X is a rank one operator, so it is compact. We next give a sufficient condition for compactness of uC_φ on $D^n(X)$ for those φ which are not constant self-maps of X.

Theorem 2.2. *Let X be a perfect compact plane set satisfying the condition* (1.1). *Let $u, \varphi \in D^n(X)$. If $\varphi(\mathrm{coz}(u)) \subseteq \mathrm{int}X$, then the weighted composition operator uC_φ is compact on $D^n(X)$.*

Proof. Let $\{f_k\}$ be a bounded sequence in $D^n(X)$ with $\|f_k\|_n = \sum_{r=0}^n \frac{\|f_k^{(r)}\|_X}{r!} \leq 1$. Using the condition (1.1), the uniformly bounded sequences $\{f_k^{(r)}\}$, $r = 0, \ldots, n-1$ are equicontinuous at each point of X. Then by Arzela-Ascoli Theorem, $\{f_k\}$ has a subsequence $\{f_{k_j}\}$, say it $\{f_k\}$ again, such that each $\{f_k^{(r)}\}$, $0 \leq r \leq n-1$ is uniformly convergent and hence is uniformly Cauchy on X. Moreover, using Leibniz's and Faà di Bruno's formulas we have

$$\begin{aligned}(uC_\varphi(f))^{(r)} &= \sum_{j=0}^{r} \tbinom{r}{j} u^{(r-j)} (f\circ\varphi)^{(j)} \\ &= u^{(r)}(f\circ\varphi) + \sum_{j=1}^{r} \tbinom{r}{j} u^{(r-j)} \sum_{i=1}^{j} (f^{(i)}\circ\varphi)\psi_{i,j},\end{aligned}$$

for any $f \in D^n(X)$ and for each $0 \leq r \leq n$. Using this relation for the differences $f_k - f_\ell$ we get

$$\begin{aligned}\|(uC_\varphi(f_k - f_\ell))^{(r)}\|_X \leq{}& \|u^{(r)}\|_X \|f_k - f_\ell\|_X \\ &+ \sum_{j=1}^{r} \tbinom{r}{j} \|u^{(r-j)}\|_X \sum_{i=1}^{j} \|f_k^{(i)} - f_\ell^{(i)}\|_X \|\psi_{i,j}\|_X,\end{aligned}$$

for each $0 \leq r \leq n-1$ and

$$\begin{aligned}\|(uC_\varphi(f_k - f_\ell))^{(n)}\|_X \leq{}& \|u^{(n)}\|_X \|f_k - f_\ell\|_X \\ &+ \sum_{j=1}^{n-1} \tbinom{n}{j} \|u^{(n-j)}\|_X \sum_{i=1}^{j} \|f_k^{(i)} - f_\ell^{(i)}\|_X \|\psi_{i,j}\|_X \\ &+ \|u\|_X \sum_{i=1}^{n-1} \|f_k^{(i)} - f_\ell^{(i)}\|_X \|\psi_{i,n}\|_X \\ &+ \|u(\varphi')^n((f_k^{(n)} - f_\ell^{(n)})\circ\varphi)\|_X.\end{aligned}$$

Therefore, to show that $\{uC_\varphi(f_k)\}$ is a Cauchy and hence a convergent sequence in $D^n(X)$, it is enough to show that $\{u(\varphi')^n(f_k^{(n)} \circ \varphi)\}$ is uniformly Cauchy on X.

As we know, each $f_k \in D^n(X)$ is analytic in int$X$, thus the sequence $\{f_k^{(n)}\}$ is uniformly convergent on every compact subset of int$X$, [6, VII, Theorem 2.1]. Let $\varepsilon > 0$ and $K = \{z \in X : |u(z)| \geq \varepsilon\}$. Then K is a compact subset of $\operatorname{coz}(u)$ and $\varphi(K)$ is a compact subset of $\varphi(\operatorname{coz}(u)) \subseteq$ intX. Hence $\{f_k^{(n)}\}$ is uniformly Cauchy on $\varphi(K)$, so $\|f_k^{(n)} - f_\ell^{(n)}\|_{\varphi(K)} < \varepsilon$, for large enough k, ℓ.

Let $z \in X$, we consider two cases. First, let $z \in K$. In this case $\varphi(z) \in \varphi(K)$ and

$$\begin{aligned} |u(z)(\varphi')^n(z)(f_k^{(n)}(\varphi(z)) - f_\ell^{(n)}(\varphi(z)))| &\leq \|u\|_X \|\varphi'\|_X^n \|f_k^{(n)} - f_\ell^{(n)}\|_{\varphi(K)} \\ &< \varepsilon \|u\|_X \|\varphi'\|_X^n, \end{aligned}$$

for large enough k, ℓ. Next, let $z \notin K$. In this case,

$$\begin{aligned} |u(z)(\varphi')^n(z)(f_k^{(n)}(\varphi(z)) - f_\ell^{(n)}(\varphi(z)))| &\leq |u(z)| \|\varphi'\|_X^n (\|f_k^{(n)}\|_X + \|f_\ell^{(n)}\|_X) \\ &< 2n!\varepsilon \|\varphi'\|_X^n. \end{aligned}$$

Therefore,

$$\|u(\varphi')^n(f_k^{(n)} \circ \varphi - f_\ell^{(n)} \circ \varphi)\|_X < \varepsilon \|\varphi'\|_X^n (2n! + \|u\|_X),$$

for large enough k, ℓ. ∎

We now show that the above conditions are also necessary for compactness of weighted composition operators uC_φ on algebras $D^n(X)$ for certain compact plane sets X. For this we introduce the type of plane sets which we shall consider.

Definition 2.3. A plane set X has an *internal circular tangent* at $\zeta \in \partial X$ if there exists an open disc U such that $\zeta \in \partial U$ and $\overline{U} \setminus \{\zeta\} \subseteq$ int$X$. A plane set X is *strongly accessible from the interior* if it has an internal circular tangent at each point of its boundary.

A compact plane set X is said to have a *peak boundary with respect to* $B \subseteq C(X)$ if for each $\zeta \in \partial X$ there exists a non-constant function $h \in B$ such that $\|h\|_X = h(\zeta) = 1$.

Such sets include the closed unit disc $\overline{\mathbb{D}}$ and $\overline{\Delta}(z_0, r) \setminus \cup_{k=1}^n \Delta(z_k, r_k)$ where closed discs $\overline{\Delta}(z_k, r_k)$ are mutually disjoint in $\Delta(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$. Moreover, if X is a compact plane set such that $\mathbb{C} \setminus X$ is strongly accessible from the interior, then X has peak boundary with respect to $R_0(X)$, the algebra of rational functions with poles off X, and hence with respect to $D^n(X)$, since $R_0(X) \subseteq D^n(X)$. For this, suppose $\zeta \in \partial X$, then there exists a disc $U = \Delta(z_0, r)$ such that $\zeta \in \partial U$ and $\overline{U} \setminus \{\zeta\} \subseteq \mathbb{C} \setminus X$. The function $h(z) = \dfrac{r}{z - z_0}$ satisfies the definition of peak boundary, (see [3, 16]).

We shall also require the following lemma due to Julia [5, Chapter I of Part Six].

Lemma 2.4. *Let $\overline{\mathbb{D}}$ be the closed unit disc in $\mathbb{C}$ and let h be a continuously differentiable function on $\overline{\mathbb{D}}$. If $h(\zeta) = \|h\|_{\overline{\mathbb{D}}}$ for some $\zeta \in \overline{\mathbb{D}}$, then either h is constant or $h'(\zeta) \neq 0$.*

Theorem 2.5. *Let X be a perfect compact plane set with connected interior satisfy the condition* (1.1)*, be strongly accessible from the interior and have a peak boundary with respect to $D^n(X)$. Let a complex function u and a self-map φ of X be in $D^n(X)$. If the weighted composition operator uC_φ on $D^n(X)$ is compact, then either φ is constant or $\varphi(\mathrm{coz}(u)) \subseteq \mathrm{int}X$.*

Proof. Let uC_φ be compact on $D^n(X)$ and suppose $u(\zeta) \neq 0$ and $\varphi(\zeta) \in \partial X$ for some $\zeta \in X$. Then by open mapping theorem for analytic functions, $\zeta \in \partial X$. Since X has a peak boundary with respect to $D^n(X)$, there exists a non-constant function $h \in D^n(X)$ such that $h(\varphi(\zeta)) = \|h\|_X = 1$. Also, the plane set X is strongly accessible from the interior, hence there exists an open disc U such that $\zeta \in \partial U$ and $\overline{U} \setminus \{\zeta\} \subseteq \mathrm{int}X$. Thus, $(h \circ \varphi)(\zeta) = \|h \circ \varphi\|_{\overline{U}} = \|h\|_X = 1$. Define

$$f_k(z) = \frac{h^k(z)}{k(k-1)\cdots(k-n+1)} \qquad (z \in X, k \geq n).$$

It is not hard to show that $\{f_k\}$ is a bounded sequence in $D^n(X)$ and $f_k^{(r)} \to 0$ uniformly on X for each $r = 0, 1, 2, \ldots, n-1$. Also by (2.2), $p_\zeta(f_k^{(r)}) \to 0$ and $p_{\varphi(\zeta)}(f_k^{(r)}) \to 0$ for each $r = 0, 1, 2, \ldots, n-2$. Using Faà di Bruno's formulas, one can conclude that

$$\|(f_k \circ \varphi)^{(r)}\|_X \to 0 \quad \text{as} \quad k \to \infty \quad (r = 0, 1, 2, \ldots, n-1), \tag{2.3}$$

hence by (2.2),

$$p_\zeta((f_k \circ \varphi)^{(r)}) \to 0 \quad \text{as} \quad k \to \infty \quad (r = 0, 1, 2, \ldots, n-2). \tag{2.4}$$

By compactness of uC_φ, there exists a subsequence of $\{f_k\}$ which is denoted by $\{f_k\}$ again, such that $\{uC_\varphi(f_k)\}$ converges in $D^n(X)$. Since $\|f_k\|_X \to 0$, $uC_\varphi(f_k) \to 0$ in $D^n(X)$. Hence, $\|(uC_\varphi(f_k))^{(r)}\|_X \to 0$, as $k \to \infty$ for each r, $0 \leq r \leq n$. These limits along with the relation (2.2) imply that

$$p_\zeta((uC_\varphi(f_k))^{(n-1)}) = p_\zeta((u \cdot (f_k \circ \varphi))^{(n-1)}) \to 0, \quad \text{as} \quad k \to \infty. \tag{2.5}$$

Using Leibniz's formula, we have

$$\begin{aligned} p_\zeta(u \cdot (f_k \circ \varphi)^{(n-1)}) \leq & p_\zeta((u \cdot (f_k \circ \varphi))^{(n-1)}) + \sum_{j=1}^{n-1} \tbinom{n-1}{j} p_\zeta(u^{(j)}) \|(f_k \circ \varphi)^{(n-1-j)}\|_X \\ & + \sum_{j=1}^{n-1} \tbinom{n-1}{j} \|u^{(j)}\|_X p_\zeta((f_k \circ \varphi)^{(n-1-j)}). \end{aligned}$$

This inequality, along with limits (2.3), (2.4) and (2.5) gives

$$p_\zeta(u \cdot (f_k \circ \varphi)^{(n-1)}) \to 0 \quad \text{as} \quad k \to \infty. \tag{2.6}$$

Using Faà di Bruno's formula,

$$\begin{aligned}
p_\zeta(u(\varphi')^{n-1}\cdot(f_k^{(n-1)}\circ\varphi)) \le& p_\zeta(u\cdot(f_k\circ\varphi)^{(n-1)}) + \sum_{j=1}^{n-2} p_\zeta((f_k^{(j)}\circ\varphi)\cdot u\psi_{j,n-1})\\
\le& p_\zeta(u\cdot(f_k\circ\varphi)^{(n-1)}) + \sum_{j=1}^{n-2} \|f_k^{(j)}\circ\varphi\|_X p_\zeta(u\psi_{j,n-1})\\
&+\sum_{j=1}^{n-2} p_\zeta(f_k^{(j)}\circ\varphi)\|u\psi_{j,n-1}\|_X\\
\le& p_\zeta(u\cdot(f_k\circ\varphi)^{(n-1)}) + \sum_{j=1}^{n-2} \|f_k^{(j)}\|_X p_\zeta(u\psi_{j,n-1})\\
&+\sum_{j=1}^{n-2} p_{\varphi(\zeta)}(f_k^{(j)})p_\zeta(\varphi)\|u\psi_{j,n-1}\|_X.
\end{aligned}$$

This inequality, along with the limit (2.6) and the properties of $\{f_k\}$ which mentioned after its definition implies that

$$p_\zeta(u(\varphi')^{n-1}\cdot(f_k^{(n-1)}\circ\varphi)) \to 0 \quad \text{as} \quad k\to\infty. \tag{2.7}$$

By the definition of $f_k^{(n-1)}$,

$$\begin{aligned}
\frac{1}{k-n+1}p_\zeta(u\cdot((h\circ\varphi)')^{n-1}\cdot(h^{k-n+1}\circ\varphi)) \le& p_\zeta(u\cdot(\varphi')^{n-1}\cdot(f_k^{(n-1)}\circ\varphi))\\
&+\frac{P(k)}{k(k-1)\cdots(k-n+1)}p_\zeta(\psi),
\end{aligned} \tag{2.8}$$

where the function ψ is a combination of u, φ, h and the derivatives of h, and $P(k)$ is a polynomial in terms of k with degree less than n. Hence $\frac{P(k)}{k(k-1)\cdots(k-n+1)} \to 0$ as $k\to\infty$. Using this limit together with the limit (2.7) and the inequality (2.8), we obtain

$$\frac{1}{k-n+1}p_\zeta(u\cdot((h\circ\varphi)')^{n-1}\cdot(h^{k-n+1}\circ\varphi)) \to 0 \quad \text{as} \quad k\to\infty. \tag{2.9}$$

On the other hand, we have

$$\begin{aligned}
&\sup_{\substack{z\in\overline{U}\\ z\neq\zeta}} |u(z)||(h\circ\varphi)'(z)|^{n-1}\frac{|h^{k-n+1}(\varphi(z))-h^{k-n+1}(\varphi(\zeta))|}{(k-n+1)|z-\zeta|}\\
&\le \frac{1}{k-n+1}\{p_\zeta(u\cdot((h\circ\varphi)')^{n-1}\cdot(h^{k-n+1}\circ\varphi)) + p_\zeta(u\cdot((h\circ\varphi)')^{n-1})\|h\|_X^{k-n+1}\}.
\end{aligned}$$

Using (2.9) and the fact that $\|h\|_X = 1$, one can conclude from the above inequality that

$$\sup_{\substack{z\in\overline{U}\\ z\neq\zeta}} |u(z)||(h\circ\varphi)'(z)|^{n-1}\frac{|h^{k-n+1}(\varphi(z))-h^{k-n+1}(\varphi(\zeta))|}{(k-n+1)|z-\zeta|} \to 0, \qquad \text{as} \quad k\to\infty.$$

Let $\varepsilon > 0$. Then

$$|u(z)||(h \circ \varphi)'(z)|^{n-1} \frac{|h^{k-n+1}(\varphi(z)) - h^{k-n+1}(\varphi(\zeta))|}{(k-n+1)|z-\zeta|} < \varepsilon,$$

for some positive integer $k > n$ and for all $z \in \overline{U}$ with $z \neq \zeta$. Taking limit as $z \to \zeta$, we get $|u(\zeta)||(h \circ \varphi)'(\zeta)|^n \leq \varepsilon$, for each $\varepsilon > 0$, since $h(\varphi(\zeta)) = 1$. Consequently, $|u(\zeta)||(h \circ \varphi)'(\zeta)|^n = 0$, and since $u(\zeta) \neq 0$, $(h \circ \varphi)'(\zeta) = 0$. By Julia's Lemma 2.4, $h \circ \varphi$ is constant on $\overline{U}$. Using the identity Theorem [6, IV, Theorem 3.7], the analytic function $h \circ \varphi$ is constant on connected set int$X$. The hypothesis, X is strongly accessible from the interior, implies that X has dense interior, so $h \circ \varphi$ is constant on X. But h is not constant, thus φ must be constant. ∎

In the case $u = 1$, we have the following corollary for composition operators on $D^n(X)$.

Corollary 2.6. *Let X be a perfect compact plane set satisfying the condition* (1.1)*. Let a self-map φ of X be in $D^n(X)$.*

(i) If either φ is constant or $\varphi(X) \subseteq$ int$X$*, Then C_φ is compact on $D^n(X)$.*

(ii) Let X be strongly accessible from the interior, have a peak boundary with respect to $D^n(X)$ and let int$X$ *be connected. If C_φ is compact on $D^n(X)$, then either φ is constant or $\varphi(X) \subseteq$* int$X$.

Using this corollary we can get some results about quasicompactness and power compactness of C_φ on $D^n(X)$. First we state their definitions. If E is an infinite dimensional Banach space, we denote by $\mathcal{B}(E)$ and $\mathcal{K}(E)$ the Banach algebra of all bounded linear operators and compact linear operators on E, respectively. The essential spectral radius $r_e(T)$ of $T \in \mathcal{B}(E)$ is the spectral radius of $T + \mathcal{K}(E)$ in the Calkin algebra $\mathcal{B}(E)/\mathcal{K}(E)$, that is

$$r_e(T) = \lim_{n \to \infty} \|T^n + \mathcal{K}(E)\|^{\frac{1}{n}}.$$

The operator $T \in \mathcal{B}(E)$ is called *quasicompact* if $r_e(T) < 1$ and it is called *Riesz* if $r_e(T) = 0$. Also, we say T is *power compact* if T^N is compact for some positive integer N. Clearly every power compact operator is Riesz.

It was shown in [11, Theorem 1.2 (iii)] that if φ induces a quasicompact endomorphism of a unital commutative semi-simple Banach algebra B with connected maximal ideal (character) space X, then $\bigcap \varphi_n(X) = \{x_0\}$ for some $x_0 \in X$, where φ_n denotes the n-th iterate of φ. By using this relation and the obtained condition for compactness of composition operators on algebras $D^n(X)$, we get the following results.

Theorem 2.7. *Let X be a perfect compact plane set satisfying the condition* (1.1)*. Let a self-map φ of X be in $D^n(X)$.*

(i) If $\bigcap \varphi_n(X) = \{z_0\}$ for some $z_0 \in$ int$X$*, then C_φ is power compact on $D^n(X)$.*

(ii) Let X be strongly accessible from the interior, have a peak boundary with respect to $D^n(X)$ and let int$X$ *be connected. If φ is non-constant and C_φ is power compact on $D^n(X)$, then $\bigcap \varphi_n(X) = \{z_0\}$ for some $z_0 \in$* int$X$.

Proof. (i) Since $z_0 \in \text{int}X$ and $\bigcap \varphi_n(X) = \{z_0\}$, there is a positive integer N such that $\varphi_N(X) \subseteq \text{int}X$. Hence, by Corollary 2.6, $(C_\varphi)^N = C_{\varphi_N}$ is compact and hence C_φ is power compact.

(ii) suppose C_φ is power compact, then C_φ is quasicompact and using [11, Theorem 1.2 (iii)], $\bigcap \varphi_n(X) = \{z_0\}$ for some $z_0 \in X$. Also, by power compactness of C_φ, there is a positive integer N such that $(C_\varphi)^N = C_{\varphi_N}$ is compact. Next by connectedness of X, φ_N is non-constant. Thus by Corollary 2.6, $\varphi_N(X) \subseteq \text{int}X$. Consequently, $z_0 \in \text{int}X$. ∎

Using the same argument as in the proof of [11, Lemma 2.1], one can show that for a connected perfect compact plane set X and a self-map φ with fixed point x_0, if C_φ is a quasicompact composition operator on $D^n(X)$, then $|\varphi'(x_0)| < 1$.

It was also shown in [11, Theorem 3.2] that if $T = C_\varphi$ acts on $C^1[0,1]$, the Banach algebra of continuously differentiable functions on $[0,1]$, and $\bigcap \varphi_n([0,1]) = \{x_0\}$ for some $x_0 \in [0,1]$, then $r_e(T) = |\varphi'(x_0)|$. Giving the following example we show that this is not true for $D^1(X)$, in general.

Example 2.8. Let $\varphi(z) = \frac{1-z}{2}$ for every $z \in \overline{\mathbb{D}}$. Then $z_0 = \frac{1}{3}$ is the fixed point of φ in $\mathbb{D}$ and $|\varphi'(z_0)| = \frac{1}{2}$. On the other hand, $\varphi(-1) = 1$, so $\varphi(\overline{\mathbb{D}}) \nsubseteq \mathbb{D}$ and the composition operator C_φ on $D^1(X)$ is not compact. However, $|\varphi_2(z)| \le \frac{1}{2} < 1$ for all $z \in \overline{\mathbb{D}}$. Hence, C_φ is power compact on $D^1(X)$ and then $r_e(C_\varphi) = 0$.

Also if C_φ is a quasicompact composition operator on $D^n(X)$, then by [11, Theorem 1.2] the induced function φ has a fixed point in X. As the following example which is similar to [17, Example 3.1], shows the fixed point of φ does not necessarily belong to int$X$ and consequently there is a quasicompact operator on $D^n(X)$ which is not necessarily power compact.

Example 2.9. Let $c > 1$ and $\varphi(z) = \frac{z+(c-1)}{c}$ for every $z \in \overline{\mathbb{D}}$. Then $T := C_\varphi$ is a composition operator on $D^n(\overline{\mathbb{D}})$ and $\varphi_m(z) = \frac{z+(c^m-1)}{c^m}$ for each positive integer m and every $z \in \overline{\mathbb{D}}$. To show that T is a quasicompact operator on $D^n(\overline{\mathbb{D}})$, let $S(f) = f(1) \cdot 1$ for every $f \in D^n(\overline{\mathbb{D}})$, then S is a (rank one) compact operator on $D^n(\overline{\mathbb{D}})$ and for each $f \in D^n(\overline{\mathbb{D}})$ we have

$$|f(\varphi_m(z)) - f(1)| \le \|f'\|_{\overline{\mathbb{D}}} |\varphi_m(z) - 1| \le \frac{2}{c^m} \|f'\|_{\overline{\mathbb{D}}},$$

for every $z \in \overline{\mathbb{D}}$. Thus

$$\|T^m f - Sf\|_{\overline{\mathbb{D}}} \le \frac{2}{c^m} \|f'\|_{\overline{\mathbb{D}}}. \tag{2.10}$$

Also,

$$(T^m f - Sf)^{(k)} = \frac{1}{c^{mk}} f^{(k)} \circ \varphi_m \qquad k = 1, \ldots, n.$$

Hence

$$\|(T^m f - Sf)^{(k)}\|_{\overline{\mathbb{D}}} \le \frac{1}{c^{mk}} \|f^{(k)}\|_{\overline{\mathbb{D}}} \le \frac{1}{c^m} \|f^{(k)}\|_{\overline{\mathbb{D}}} \qquad k = 1, \ldots, n.$$

This and (2.10) imply that

$$\begin{aligned}\|T^m f - Sf\|_n &= \sum_{k=0}^{n} \frac{\|(T^m f - Sf)^{(k)}\|_{\overline{\mathbb{D}}}}{k!} \\ &\leq \frac{2}{c^m}\|f'\|_{\overline{\mathbb{D}}} + \sum_{k=1}^{n} \frac{\|f^{(k)}\|_{\overline{\mathbb{D}}}}{c^m k!} \\ &\leq \frac{3}{c^m}\|f\|_n.\end{aligned}$$

Therefore, $\|T^m - S\| \leq \frac{3}{c^m}$ and hence $\|T^m + \mathcal{K}\| \leq \frac{3}{c^m}$ where $\mathcal{K} = \mathcal{K}(D^n(\overline{\mathbb{D}}))$. This implies that

$$r_e(T) = \lim_{m\to\infty} \|T^m + \mathcal{K}\|^{\frac{1}{m}} \leq \frac{1}{c} < 1.$$

Consequently, T is a quasicompact operator on $D^n(X)$. On the other hand $\bigcap \varphi_m(\overline{\mathbb{D}}) = \{1\}$, hence by Theorem 2.7 (ii), T is not power compact.

A question which may be asked here is whether every Riesz operator on $D^n(X)$ is necessarily power compact. Feinstein and Kamowitz showed that this is no longer true by giving a Riesz operator on $C^1[0,1]$ which is not power compact [11, Corollary 3.3].

3 Spectrum

Suppose A is a Banach space of functions on a plane set X which contains constant functions and coordinate function z. If $\varphi : X \to X$ is a constant function, $\varphi(z) = z_0$ for all $z \in X$, and uC_φ is a weighted composition operator on A, then uC_φ is a rank one operator on A and $\sigma(uC_\varphi) = \{0, u(z_0)\}$. Thus in what follows we assume that φ is a non-constant self-map of X and u is a non-zero complex-valued function on X.

Kamowitz proved two interesting and useful lemmas [14, Lemmas 2.3 and 2.4] and by using them determined the spectrum of a compact weighted composition operator uC_φ on disc algebra $A(\overline{\mathbb{D}})$, when φ has a fixed point in $\mathbb{D}$. These lemmas still valid for general case as follows.

Lemma 3.1. *Let X be a compact plane set with nonempty interior. Suppose A is a unital subalgebra of $A(X)$ containing the coordinate function z. If u is a complex-valued function on X and φ is a self-map of X which is analytic on* int$X$ *and $\varphi(z_0) = z_0$ for some $z_0 \in$* int$X$. *Then for the weighted composition operator uC_φ on A, we have*

$$\{u(z_0)\varphi'(z_0)^k : k \text{ is a positive integer}\} \cup \{\mathrm{u(z_0)}\} \subseteq \sigma(uC_\varphi).$$

Proof. Since $u(z_0)f - uC_\varphi f \neq 1$ for all $f \in A$, $u(z_0) - uC_\varphi$ is not surjective and so invertible. Thus $u(z_0) \in \sigma(uC_\varphi)$.

If $u(z_0) = 0$, then the same as the above argument uC_φ is not surjective. When $\varphi'(z_0) = 0$, the operator uC_φ is not surjective too. Since otherwise, we must have $uC_\varphi f = 1$ for some $f \in A$. In particular, $u(z_0)f(\varphi(z_0)) = 1$. This implies that $f(z_0) \neq 0$. Moreover, for such function f we have $u'(f \circ \varphi) + u\varphi'(f' \circ \varphi) = 0$ on int$X$ and hence $u'(z_0)f(z_0) + u(z_0)\varphi'(z_0)f'(z_0) = 0$, which implies $u'(z_0) = 0$, since $\varphi'(z_0) = 0$ and $f(z_0) \neq 0$. Therefore, when $\varphi'(z_0) = 0$, the surjectivity of uC_φ implies that $u'(z_0) = 0$ which lead to $(uC_\varphi g)'(z_0) = 0$ for all $g \in A$, in particular, for a function $g \in A$ with $uC_\varphi g = z$ which is impossible. Therefore, if $u(z_0)\varphi'(z_0) = 0$, the operator uC_φ is not (surjective) invertible and consequently $u(z_0)\varphi'(z_0)^k = 0 \in \sigma(uC_\varphi)$ for every positive integer k.

Suppose now $u(z_0)\varphi'(z_0) \neq 0$. Let k be a positive integer such that $\varphi'(z_0)^j \neq 1$ for each j $(1 \leq j \leq k)$ and

$$u(z_0)\varphi'(z_0)^k f(z) - u(z)f(\varphi(z)) = (z - z_0)^k \qquad (z \in X), \tag{3.1}$$

for some $f \in A$. Choose $r > 0$ such that $\Delta_r = \{z \in \mathbb{C} : |z - z_0| < r\} \subseteq \text{int}X$. Thus the elements of A are analytic on Δ_r and by (3.1), f is not the zero function on Δ_r. Now by replacing $\mathbb{D}$ and Δ_r and applying the same argument as in the proof of [14, Lemma 2.3], the relation (3.1) leads to a contradiction. Consequently, $(z - z_0)^k$ is not in the range of $u(z_0)\varphi'(z_0)^k - uC_\varphi$. Therefore this operator is not invertible and hence $u(z_0)\varphi'(z_0)^k \in \sigma(uC_\varphi)$. ∎

Lemma 3.2. *Let X be a compact plane set with connected and dense interior. Let A be a subspace of $A(X)$ containing constant functions and the coordinate function z. Suppose u is a non-zero complex-valued function on X, $\varphi \in A(X)$ is a non-constant self-map of X with $\varphi(z_0) = z_0$ for some $z_0 \in \text{int}X$ and uC_φ is a weighted composition operator on A. If $\lambda \neq 0$ is an eigenvalue of uC_φ, then*

$$\lambda \in \{u(z_0)\varphi'(z_0)^k : k \text{ is a positive integer}\} \cup \{\mathrm{u(z_0)}\}.$$

Proof. By the property of z_0, $\Delta_r = \{z \in \mathbb{C} : |z - z_0| < r\} \subseteq \text{int}X$ for some r. Now by replacing $\mathbb{D}$ and Δ_r, and the same argument as in the proof of [14, Lemma 2.4], the result concludes. ∎

It is known that if T is a Riesz operator, then every non-zero number in $\sigma(T)$ is an eigenvalue of T [10, Theorem 3.14]. Thus we have the following theorem.

Theorem 3.3. *Let X be a compact plane set with connected and dense interior. Let A be a unital Banach subalgebra of $A(X)$ containing the coordinate function z. Suppose u is a non-zero complex-valued function on X, $\varphi \in A(X)$ is a non-constant self-map of X with $\varphi(z_0) = z_0$ for some $z_0 \in \text{int}X$ and uC_φ is a Riesz weighted composition operator on A, then*

$$\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^k : k \text{ is a positive integer}\} \cup \{0, \mathrm{u(z_0)}\}.$$

Corollary 3.4. *Let X be a perfect compact plane set with connected and dense interior and satisfy the condition* (1.1). *Suppose u is a non-zero complex-valued function on X and $\varphi \in A(X)$ is a non-constant self-map of X with $\varphi(z_0) = z_0$ for some $z_0 \in \text{int}X$. If uC_φ is a Riesz weighted composition operator on $D^n(X)$, then*

$$\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^k : k \text{ is a positive integer}\} \cup \{0, \mathrm{u(z_0)}\}.$$

Using Theorem 2.7, as an immediate consequence of the above corollary we get the following result for the spectrum of a quasicompact composition operator.

Corollary 3.5. *Let X be a perfect compact plane set with connected and dense interior and satisfy the condition* (1.1)*. Let C_φ be a quasicompact composition operator on $D^n(X)$. If $\varphi(z_0) = z_0$ for some $z_0 \in \text{int}X$, then C_φ is power compact and*

$$\sigma(C_\varphi) = \{\varphi'(z_0)^k : k \text{ is a positive integer}\} \cup \{0\}.$$

In the case that $X = \overline{\mathbb{D}}$ and all fixed points of φ are on the unit circle, we have the following theorem, due to Kamowitz, for disc algebra $A(\overline{\mathbb{D}})$.

Theorem 3.6. *[14, Theorem 2.2] Suppose $u, \varphi \in A(\overline{\mathbb{D}})$, $\|\varphi\|_{\overline{\mathbb{D}}} = 1$, φ is not a constant function and φ has all its fixed points on the unit circle. If uC_φ is a compact operator on $A(\overline{\mathbb{D}})$, then $\sigma_{A(\overline{\mathbb{D}})}(uC_\varphi) = \{0\}$.*

Using this theorem, we give a similar result for algebras $D^n(\overline{\mathbb{D}})$ as follows.

Corollary 3.7. *Suppose $u, \varphi \in D^n(\overline{\mathbb{D}})$ and φ is a non-constant self-map of $\overline{\mathbb{D}}$ whose all fixed points lie on the unit circle. If uC_φ is a compact operator on $D^n(\overline{\mathbb{D}})$, then $\sigma(uC_\varphi) = \{0\}$.*

Proof. It is clear that $D^n(\overline{\mathbb{D}}) \subseteq A(\overline{\mathbb{D}})$ and uC_φ is also a weighted composition operator on $A(\overline{\mathbb{D}})$. By Theorem 2.5, the compactness of uC_φ on $D^n(\overline{\mathbb{D}})$ implies that $\varphi(\text{coz}(u)) \subseteq \mathbb{D}$. Thus, by [14, Theorem 1.2], uC_φ is also a compact operator on $A(\overline{\mathbb{D}})$. Moreover, every eigenvalue of compact operator uC_φ on $D^n(\overline{\mathbb{D}})$ is also an eigenvalue of compact operator uC_φ on $A(\overline{\mathbb{D}})$. Hence, by Theorem 3.6, $\sigma_{D^n(\overline{\mathbb{D}})}(uC_\varphi) = \{0\}$. ■

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