Yamabe solitons on three-dimensional Kenmotsu manifolds

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Abstract

Let the Riemannian metric of a three-dimensional Kenmotsu manifold be a Yamabe soliton. In this paper, we prove that the Kenmotsu manifold is of constant sectional curvature -1 and the Yamabe soliton is expanding with the soliton constant $\lambda = -6$.

1 Introduction

It is well known that a Riemannian metric g of an n-dimensional complete Riemannian manifold (M^n, g) is said to be a *Yamabe soliton* if it satisfies

$$\mathcal{L}_V g = (\lambda - r)g \tag{1.1}$$

for a constant $\lambda \in \mathbb{R}$ and a smooth vector field V on M^n, where r is the scalar curvature of g and $\mathcal{L}$ denotes the Lie-derivative operator. A Yamabe soliton is said to be *shrinking*, *steady* or *expanding* according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ respectively and λ is said to the *soliton constant*.

Given a smooth Riemannian manifold (M^n, g_0), the evolution of the metric g_0 in time t to $g = g(t)$ through the following equation

$$\frac{\partial}{\partial t} g_t = -rg, \ g(0) = g_0 \tag{1.2}$$

Received by the editors in October 2015 - In revised form in February 2016.
Communicated by J. Fine.
2010 *Mathematics Subject Classification :* Primary 53D15; Secondary 53C25.
Key words and phrases : 3-dimensional Kenmotsu manifold, Yamabe soliton, Ricci soliton, constant sectional curvature.

is known as the *Yamabe flow* (which was introduced by Hamilton [6]). A Yamabe soliton is a special soliton of the Yamabe flow that moves by one parameter family of diffeomorphisms ϕ_t generated by a fixed vector field V on M^n (for more details see [2, 12]). Given a Yamabe soliton, if $V = Df$ holds for a smooth function f on M^n, equation (1.1) becomes

$$\text{Hess}f = \frac{1}{2}(\lambda - r)g, \tag{1.3}$$

where Hess$f$ denotes the Hessian of f and D denotes the gradient operator of g on M^n. In this case f is called the *potential function* of the Yamabe soliton and g is said to be a *gradient Yamabe soliton*. A Yamabe soliton (respectively, gradient Yamabe soliton) is said to be *trivial* when V is Killing (respectively, f is constant).

On the other hand, in 1969, S. Tanno in [13] classified the connected almost contact metric manifolds whose automorphism groups have maximal dimensions as follows.

(1) Homogeneous normal contact Riemannian manifolds with constant ϕ-holomorphic sectional curvature if $k(\xi, X) > 0$;

(2) Global Riemannian product of a line or a circle and a Kählerian manifold with constant holomorphic sectional curvature if $k(\xi, X) = 0$;

(3) A warped product space $\mathbb{R} \times_\lambda \mathbb{C}^n$ if $k(\xi, X) < 0$; where $k(\xi, X)$ denotes the sectional curvature of the plane section containing the characteristic vector field ξ and an arbitrary vector field X.

The manifolds of the first class were characterized by some tensor equations and have a Sasakian structure. The manifolds of the second class were characterized by some tensor relations admitting a cosymplectic structure. In 1972, K. Kenmotsu in [9] obtained some tensor equations to characterize the manifolds of the third class. Since then the manifolds of the third class were called Kenmotsu manifolds.

Yamabe solitons on a three-dimensional Sasakian manifold were studied by R. Sharma [12]. As far as we know, Yamabe solitons on the other almost contact metric manifolds have not yet been studied. In this paper, we start the study of Yamabe solitons on a three-dimensional Kenmotsu manifold and obtain some local classification theorems.

Theorem 1.1. *Suppose that the Riemannian metric of a three-dimensional Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ is a Yamabe soliton. Then the manifold is of constant sectional curvature -1 and the Yamabe soliton is expanding with $\lambda = -6$.*

According to Chow-Lu-Ni [2], the metric of any compact Yamabe gradient soliton is a metric of constant scalar curvature (see also Daskalopoulos-Sesum [3] and Hsu [7]). Notice that Sharma [12, Theorem 1] implies that the scalar curvature of a Yamabe soliton on a three-dimensional Sasakian manifold is a constant. From our Theorem 1.1, we see easily that the scalar curvature of a Yamabe soliton on a three-dimensional Kenmotsu manifold M^3 is also a constant. However, a three-dimensional Kenmotsu manifold can not be compact because of div$\xi = 2$ (see

Kenmotsu [9, Section 3]), where ξ is the Reeb vector field of M^3 and div denotes the divergence operator on M^3.

The present paper is organized as follows. In Section 2, we recall some well known basic formulas and properties of Kenmotsu manifolds. Section 3 is devoted to giving the detailed proof of Theorem 1.1 after we present some key lemmas. Finally, in the last section, we discuss the relation between the Yamabe solitons and Ricci solitons on three-dimensional Kenmotsu manifolds.

2 Preliminaries

In this section, we shall recall some basic notions and properties of Kenmotsu manifolds. An *almost contact structure* (see [1]) on a $(2n+1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η), where ϕ is a $(1,1)$-type tensor, ξ a global vector field and η a 1-form, such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity mapping and relation (2.1) implies that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. Throughout this paper, $\mathcal{D}$ is denoted by the contact distribution defined by $\mathcal{D} = \ker(\eta) = \mathrm{Im}(\phi)$. A Riemannian metric g on M^{2n+1} is said to be *compatible* with the almost contact structure (ϕ, ξ, η) if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X, Y on M^{2n+1}. An almost contact structure endowed with a compatible Riemannian metric is said to be an *almost contact metric structure*.

The *fundamental 2-form* Φ on an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M^{2n+1}. An almost contact metric manifold satisfying $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ is said to be an *almost Kenmotsu manifold* (see [8]).

The normality condition of an almost contact structure is expressed by the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ. For more details regarding the normality of an almost contact metric structure we refer the reader to Blair's book [1]. A normal almost Kenmotsu manifold is said to be a *Kenmotsu manifold* (see [8]).

On an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ we set $h = \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ is the Lie differentiation. Generally, the vanishing of tensor field h means that the Reeb foliation of M^{2n+1} is conformal (see [10]). In particular, a three-dimensional almost Kenmotsu manifold M^3 is a Kenmotsu manifold if and only if the $(1,1)$-type tensor field h vanishes (see Proposition 3 of [4]). This is equivalent to

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.3}$$

for any vector fields X, Y on M^3, where ∇ denotes the Levi-Civita connection of the metric g. For a $(2n+1)$-dimensional Kenmotsu manifold, the following four formulas can be found in Kenmotsu [9]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$R(X,Y)\xi = -\eta(Y)X + \eta(X)Y, \tag{2.5}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.6}$$

$$S(\xi,\xi) = g(Q\xi,\xi) = -2n, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where S, Q and $\Gamma(TM)$ denote the Ricci curvature tensor, the Ricci operator with respect to the metric g and the Lie algebra of all vector fields on M^{2n+1}, respectively.

On an almost contact metric manifold M, if the Ricci operator satisfies

$$Q = \alpha \mathrm{id} + \beta \eta \otimes \xi, \tag{2.8}$$

where both α and β are smooth functions, then the manifold is said to be an *η-Einstein manifold*. An η-Einstein manifold with β vanishing and α a constant is obviously an Einstein manifold. An η-Einstein manifold is said to be *proper η-Einstein* if $\beta \neq 0$.

3 Main Results

Before giving the detailed proof of our main result, we first present some key lemmas used later. By equation (1.1), we obtain easily that for a Yamabe soliton the vector field V is a *conformal vector field*, that is,

$$\mathcal{L}_V g = 2\rho g, \tag{3.1}$$

where ρ is called the *conformal coefficient* (in this context by relation (1.1) we have $\rho = \frac{\lambda - r}{2}$). In particular, a conformal vector field with a vanishing conformal coefficient reduces to a *Killing vector field*.

Lemma 3.1 (Yano [14]). *On an n-dimensional Riemannian manifold (M^n, g) endowed with a conformal vector field V, we have*

$$(\mathcal{L}_V S)(X,Y) = -(n-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X,Y),$$

$$\mathcal{L}_V r = -2\rho r + 2(n-1)\Delta\rho$$

for any vector fields X and Y, where D denotes the gradient operator and $\Delta := -\mathrm{div}D$ denotes the Laplacian operator of g.

Lemma 3.2. *On any three-dimensional Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ we have*

$$\xi(r) = -2(r+6). \tag{3.2}$$

Proof. It is well known that on any three-dimensional Riemannian manifold (M^3, g) the following formula holds

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ & - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{3.3}$$

for any vector fields X, Y, Z on M^3. Replacing $Y = Z$ by ξ in the above equation and using (2.5), (2.7), we have

$$Q = \left(\frac{r}{2} + 1\right) \mathrm{id} - \left(\frac{r}{2} + 3\right) \eta \otimes \xi. \tag{3.4}$$

This means that M^3 is an η-Einstein manifold. On the other hand, we also have the following well known formula on Riemannian manifolds

$$\mathrm{trace}\{Y \to (\nabla_Y Q)X\} = \frac{1}{2}\nabla_X r$$

for any vector field X. Making using of (3.4) in the above formula and taking into account (2.4) and (2.7) we obtain

$$\xi(r)\eta(X) = -2(r+6)\eta(X)$$

for any vector field $X \in \Gamma(TM)$. Substituting X with ξ in the above equation we obtain (3.2). This completes the proof. ∎

Lemma 3.3. *Suppose that the Riemannian metric of a three-dimensional Kenmotsu manifold* $(M^3, \phi, \xi, \eta, g)$ *is a Yamabe soliton. Then the Yamabe soliton is expanding with* $\lambda = -6$ *and the scalar curvature of* M^3 *is harmonic, that is,*

$$\Delta r = 0. \tag{3.5}$$

Proof. Notice that the Reeb vector field ξ is a unit vector field, that is, $g(\xi, \xi) = 1$. Taking the Lie-derivative of this relation along the vector field V and using the second equation of (2.1) and (1.1), we obtain

$$\eta(\mathcal{L}_V \xi) = -(\mathcal{L}_V \eta)(\xi) = \frac{r - \lambda}{2}. \tag{3.6}$$

As the Riemannian metric g of M^3 is a Yamabe soliton, applying $\rho = \frac{\lambda - r}{2}$ and $n = 3$ in Lemma 3.1 we have

$$(\mathcal{L}_V S)(X,Y) = \frac{1}{2} g(\nabla_X Dr, Y) - \frac{1}{2}(\Delta r) g(X,Y), \tag{3.7}$$

$$\mathcal{L}_V r = r(r - \lambda) - 2\Delta r \tag{3.8}$$

for any vector fields $X, Y \in \Gamma(TM)$. On the other hand, equation (3.4) can be re-written as

$$S(X,Y) = \left(\frac{r}{2} + 1\right) g(X,Y) - \left(\frac{r}{2} + 3\right) \eta(X)\eta(Y)$$

for any vector fields $X, Y \in \Gamma(TM)$. Taking the Lie-derivative of this relation along the vector field V and making use of (3.6), (3.8) and (1.1), we obtain

$$\begin{aligned}(\mathcal{L}_V S)(X,Y) =& (-\Delta r + \lambda - r) g(X,Y) + \left(\Delta r + \frac{r}{2}(\lambda - r)\right) \eta(X)\eta(Y) \\ & - \left(\frac{r}{2} + 3\right) \{(\mathcal{L}_V \eta)(X)\eta(Y) + (\mathcal{L}_V \eta)(Y)\eta(X)\}\end{aligned} \tag{3.9}$$

for any vector fields $X, Y \in \Gamma(TM)$. Consequently, combining (3.7) with (3.9) we have

$$\begin{aligned} g(\nabla_X Dr, Y) =& (-\Delta r + 2(\lambda - r))g(X,Y) + (2\Delta r + r(\lambda - r))\eta(X)\eta(Y) \\ &- (r+6)\{(\mathcal{L}_V\eta)(X)\eta(Y) + (\mathcal{L}_V\eta)(Y)\eta(X)\} \end{aligned} \tag{3.10}$$

for any vector fields $X, Y \in \Gamma(TM)$. Putting $X = Y = \xi$ into (3.10) and making use of (2.3) and (3.6), we obtain

$$\xi(\xi(r)) = \Delta r + 4(r - \lambda).$$

Clearly, making use of (3.2) in the above relation we obtain

$$\Delta r = 4(\lambda + 6). \tag{3.11}$$

Substituting Y with ξ in equation (3.10) and applying Lemma 3.2, relations (3.6) and (3.11), we obtain

$$(r+6)(\mathcal{L}_V\eta)X = \left(4(\lambda+6) - \left(\frac{r}{2} - 1\right)(r - \lambda) + 2(r+6)\right)\eta(X) + 3X(r)$$

for any vector field $X \in \Gamma(TM)$. Putting the above equation into (3.10) we get

$$\begin{aligned} \nabla_X Dr =& -2(\lambda + r + 12)X \\ &+ 2(\lambda - 3r - 12)\eta(X)\xi - 3X(r)\xi - 3\eta(X)Dr \end{aligned} \tag{3.12}$$

for any vector field $X \in \Gamma(TM)$. Substituting X with ξ in equation (3.12) and making use of (3.2), we have $\nabla_\xi Dr = -2(r+6)\xi - 3Dr$.

Next we shall consider a local orthonormal frame $\{e_i : i = 1,2,3\}$ on M^3. Making use of (3.12), (2.4), (3.11) and applying Lemma 3.2 we may obtain

$$\begin{aligned} &S(\xi, Dr) \\ =& -\sum_{i=1}^{3} g(\nabla_\xi \nabla_{e_i} Dr, e_i) + \sum_{i=1}^{3} g(\nabla_{e_i}\nabla_\xi Dr, e_i) + \sum_{i=1}^{3} g(\nabla_{[\xi, e_i]} Dr, e_i) \\ =& 4(r + 4\lambda + 30). \end{aligned} \tag{3.13}$$

On the other hand, it follows from equations (3.4) and (3.2) that

$$S(\xi, Dr) = -2\xi(r) = 4(r+6). \tag{3.14}$$

Consequently, subtracting (3.13) from (3.14) we obtain

$$\lambda = -6. \tag{3.15}$$

This means that the Yamabe soliton is expanding. Finally, taking into account (3.15) in relation (3.11) we obtain (3.5). This completes the proof. ■

Proof of Theorem 1.1. Since the metric g is parallel with respect to the Levi-Civita connection ∇, then by taking the covariant differentiation of (1.1) along arbitrary vector field X we may obtain

$$\nabla_X \mathcal{L}_V g = -X(r)g. \tag{3.16}$$

According to Yano [14], we have the following well known formula

$$(\mathcal{L}_V\nabla_X g - \nabla_X\mathcal{L}_V g - \nabla_{[V,X]}g)(Y,Z) = \\ -g((\mathcal{L}_V\nabla)(X,Y),Z) - g((\mathcal{L}_V\nabla)(X,Z),Y).$$

Making use of the parallelism of the metric g again on the above formula we have

$$(\nabla_X\mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V\nabla)(X,Y),Z) + g((\mathcal{L}_V\nabla)(X,Z),Y). \tag{3.17}$$

Since $(\mathcal{L}_V\nabla)$ is a symmetric operator, i.e., $(\mathcal{L}_V\nabla)(X,Y) = (\mathcal{L}_V\nabla)(Y,X)$, making use of (3.16) in (3.17) we obtain

$$(\mathcal{L}_V\nabla)(X,Y) = \frac{1}{2}g(X,Y)Dr - \frac{1}{2}X(r)Y - \frac{1}{2}Y(r)X, \tag{3.18}$$

where D denotes the divergence operator on M^3. Taking the covariant differentiation of (3.18) along arbitrary vector field $Z \in \Gamma(TM)$ we have

$$(\nabla_Z\mathcal{L}_V\nabla)(X,Y) = \frac{1}{2}g(X,Y)\nabla_Z Dr - \frac{1}{2}g(X,\nabla_Z Dr)Y - \frac{1}{2}g(Y,\nabla_Z Dr)X.$$

Applying the above relation on the following well known formula

$$(\mathcal{L}_V R)(X,Y)Z = (\nabla_X\mathcal{L}_V\nabla)(Y,Z) - (\nabla_Y\mathcal{L}_V\nabla)(X,Z),$$

it follows that

$$\begin{aligned}&(\mathcal{L}_V R)(X,Y)Z \\ =&\frac{1}{2}g(Y,Z)\nabla_X Dr - \frac{1}{2}g(X,Z)\nabla_Y Dr - \frac{1}{2}g(Z,\nabla_X Dr)Y \\ &+ \frac{1}{2}g(Z,\nabla_Y Dr)X + \frac{1}{2}[g(X,\nabla_Y Dr) - g(Y,\nabla_X Dr)]Z.\end{aligned} \tag{3.19}$$

Substituting Z with ξ in (3.19) and making use of (3.12), (3.15), we get

$$\begin{aligned}(\mathcal{L}_V R)(X,Y)\xi =& \left(2(r+6)\eta(X) + \frac{3}{2}X(r)\right)Y - \frac{3}{2}X(r)\eta(Y)\xi \\ &- \left(2(r+6)\eta(Y) + \frac{3}{2}Y(r)\right)X + \frac{3}{2}Y(r)\eta(X)\xi\end{aligned} \tag{3.20}$$

for any vector fields X, Y. On the other hand, by (1.1) and (2.5) and a straightforward calculation we obtain that

$$\begin{aligned}(\mathcal{L}_V R)(X,Y)\xi =& \left(g(\mathcal{L}_V\xi,X) - (r+6)\eta(X)\right)Y \\ &- \left(g(\mathcal{L}_V\xi,Y) - (r+6)\eta(Y)\right)X - R(X,Y)\mathcal{L}_V\xi\end{aligned} \tag{3.21}$$

for any vector fields X, Y. Subtracting (3.20) from (3.21) we obtain

$$\begin{aligned}&R(X,Y)\mathcal{L}_V\xi \\ =&\left(g(\mathcal{L}_V\xi,X) - 3(r+6)\eta(X) - \frac{3}{2}X(r)\right)Y + \frac{3}{2}X(r)\eta(Y)\xi \\ &- \left(g(\mathcal{L}_V\xi,Y) - 3(r+6)\eta(Y) - \frac{3}{2}Y(r)\right)X - \frac{3}{2}Y(r)\eta(X)\xi.\end{aligned}$$

Contracting X in this equation we may obtain that

$$S(Y, \mathcal{L}_V\xi) = -2g(Y, \mathcal{L}_V\xi) + 3(r+6)\eta(Y) + \frac{3}{2}Y(r) \tag{3.22}$$

for any vector field Y. Making use of (3.4) and (3.6) in (3.22), we have

$$(r+6)\mathcal{L}_V\xi = \frac{1}{2}(r+6)(r+18)\xi + 3Dr. \tag{3.23}$$

Next we suppose that on an open subset U of M^3 there holds $r \neq -6$, then it follows from equation (3.23) that

$$\mathcal{L}_V\xi = \frac{r+18}{2}\xi + \frac{3}{r+6}Dr. \tag{3.24}$$

Substituting Y with ξ in the following well known formula (see [14])

$$(\mathcal{L}_V\nabla)(X, Y) = \mathcal{L}_V\nabla_X Y - \nabla_X\mathcal{L}_V Y - \nabla_{[V,X]}Y$$

and making use of (1.1), (2.4) and (3.24) we obtain

$$\begin{aligned}(\mathcal{L}_V\nabla)(X, \xi) =& \frac{6}{r+6}\eta(X)Dr + \frac{3}{(r+6)^2}X(r)Dr \\ &+ \frac{6-r}{2(r+6)}X(r)\xi + \left(\frac{r}{2}+15\right)\eta(X)\xi - \left(\frac{r}{2}+3\right)X\end{aligned} \tag{3.25}$$

for any vector field X, where we have used the following relation

$$\nabla_X Dr = -2(r+6)X - 6(r+6)\eta(X)\xi - 3X(r)\xi - 3\eta(X)Dr$$

deduced from (3.12) and (3.15). On the other hand, replacing Y by ξ in (3.18) and applying (3.2) we obtain that

$$(\mathcal{L}_V\nabla)(X, \xi) = \frac{1}{2}\eta(X)Dr - \frac{1}{2}X(r)\xi + (r+6)X \tag{3.26}$$

for any vector field X. Comparing (3.25) with (3.26) implies that

$$\begin{aligned}&\frac{3}{(r+6)^2}X(r)Dr + \frac{6-r}{2(r+6)}\eta(X)Dr - 3\left(\frac{r}{2}+3\right)X \\ &\qquad + \frac{6}{r+6}X(r)\xi + \left(\frac{r}{2}+15\right)\eta(X)\xi = 0\end{aligned} \tag{3.27}$$

for any vector field X. Finally, replacing X by ξ in this relation and applying (3.2) we obtain

$$Dr = -2(r+6)\xi. \tag{3.28}$$

Using (3.28) again and replacing X by ϕX in (3.27) we obtain $r = -6$, this contradicts the earlier assumption $r \neq -6$.

Hence, we conclude that $r = -6$ and by using this relation in (3.4) we obtain that the Ricci operator of M^3 is given by $Q = -2\mathrm{id}$. Making use of this relation in equation (3.3) we obtain that $R(X, Y)Z = -g(Y, Z)X + g(X, Z)Y$ for any vector fields X, Y, Z. This means that M^3 is of constant sectional curvature -1. This completes the proof. ∎

4 Remarks and examples

A *Ricci soliton* (see [6]) is a generalization of the Einstein metric (that is, the Ricci tensor is a constant multiple of the Riemannian metric g) and is defined on a Riemannian manifold (M, g) by

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} + \mu g = 0 \tag{4.1}$$

for certain constant $\mu \in \mathbb{R}$ and a potential vector field V. Clearly, a Ricci soliton with V zero or a Killing vector field reduces to an Einstein metric. The Ricci soliton is said to be *shrinking*, *steady* or *expanding* according to μ is negative, zero or positive, respectively. A compact Ricci soliton is a fixed point of the Ricci flow projected from the space of metrics onto its quotient modulo diffeomorphisms. If the potential vector field V is the gradient of a potential function $-f$, then g is called a *gradient Ricci soliton* and equation (1.2) becomes

$$\mathrm{Hess} f = \mathrm{Ric} + \mu g. \tag{4.2}$$

According to G. Perelman [11], we know that a Ricci soliton on any compact manifold is a gradient Ricci soliton.

Let $(M^3, \phi, \xi, \eta, g)$ be a three-dimensional Kenmotsu manifold, if g is a Yamabe soliton then from Theorem 1.1 we know that the Ricci operator of M^3 is $Q = -2\mathrm{id}$ and hence equation (1.1) can be re-written as $\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} + 2g = 0$. This means that g is an expanding Ricci soliton with $\mu = 2$. Conversely, suppose that g is a Ricci soliton, then according to Ghosh [5, Theorem 1] we obtain that M^3 is of constant sectional curvature -1 and $\mu = 2$ and hence (4.1) becomes $\mathcal{L}_V g = (\lambda - r)g$, where we have $\lambda = r = -6$. Then we obtain immediately the following theorem.

Theorem 4.1. *The Riemannian metric of a three-dimensional Kenmotsu manifold is a Yabame soliton if and only it is a Ricci soliton.*

Remark 4.1. If the metric g of a three-dimensional Kenmotsu manifold (M^3, g) is a Yamabe soliton for a vector field V and a constant λ, then V can not be pointwise collinear with ξ. In fact, now we assume that V is pointwise collinear with ξ, that is, $V = f\xi$ for some smooth function f on M^3. It follows from (1.1) and Theorem 1.1 that $g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0$ for any $X, Y \in \Gamma(TM)$. Putting $V = f\xi$ in this relation and making use of (2.4) we obtain that

$$X(f)\eta(Y) + Y(f)\eta(X) + 2fg(X, Y) - 2f\eta(X)\eta(Y) = 0 \tag{4.3}$$

for any vector fields X and Y. Letting X and Y in relation (4.3) belong to the contact distribution, we get $f = 0$.

It is worth pointing out that there do exists a non-trivial Ricci soliton on a strictly almost Kenmotsu manifold (see [15]).

Example 4.1. We consider the product space $\mathbb{R} \times M^2$ endowed with a Riemannian metric g defined by

$$g = dt^2 + e^{2t}h,$$

where (M^2, h) is a Riemannian surface of constant negative curvature (Kählerian surface). Such a three-dimensional manifold is said to be a warped product and denoted by $(\mathbb{R} \times_{e^t} M^2, g)$. Then, according to Kenmotsu [9, Proposition 3] and Ghosh [5, Section 4], we know that the warped product $(\mathbb{R} \times_{e^t} M^2, g)$ is a three-dimensional Kenmotsu manifold and the metric g is a Ricci soliton. Therefore, from Theorem 4.1 we know that the metric g of three-dimensional Kenmotsu manifold $(\mathbb{R} \times_{e^t} M^2, g)$ is a Yamabe soliton.

Acknowledgement. This work was supported by the National Natural Science Foundation of China (No. 11526080), Key Scientific Research Program in Universities of Henan Province (No. 16A110004), the Research Foundation for the Doctoral Program of Henan Normal University (No. qd14145) and the Youth Science Foundation of Henan Normal University (No. 2014QK01). I would like to thank the anonymous referee for his or her valuable suggestions.

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