On a frictional contact problem with adhesion in piezoelectricity

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Abstract

We consider a mathematical model describing the quasistatic frictional contact between an electro-elasto-viscoplastic body and an adhesive conductive foundation. The contact is described with a normal compliance condition with adhesion, the associated general version of Coulomb's law of dry friction in which the adhesion of contact surfaces is taken into account and a regularized electrical conductivity condition. The existence of a unique weak solution is established under smallness assumption on the surface conductance. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

1 Introduction

The piezoelectric phenomenon is characterized by the apparition of electric charges on the surfaces of some crystals after their deformation. The reverse effect consists on the generation of stress and strain in crystals under the action of electric field on the boundary. Materials which present such a behavior are called piezoelectric materials, their study require techniques and results from electromagnetic theory and continuum mechanics. Piezoelectric materials are used extensively as switches and actuary in many engineering systems, in radioelectronics, electroacoustics and measuring equipment. Currently there is a considerable interest in contact problems involving piezoelectric materials.

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General models for elastic materials with piezoelectric effect, called electro-elastic materials, can be found in [4, 15]. A static contact problem for electro-elastic materials was considered in [5, 17] and a slip-dependent frictional contact problem for electro-elastic materials was studied in [28]. Frictional and frictionless contact problems involving electro-viscoelastic materials were studied in [1, 2, 3, 13]. Frictional contact problems for elastic-viscoplastic materials with piezoelectric effect, also called electro-elasto-viscoplastic materials were considered in [19, 31]. Contact problems involving piezoelectric materials when the foundation is perfectly insulated were studied in [13, 25, 28, 29], and recently in [1, 2, 18, 26, 31] and the monograph [33] when the foundation is electrically conductive.

Processes of adhesion are important in many industrial settings where parts, usually nonmetallic, are glued together. For this reason, adhesive contact between bodies, when a glue is added to prevent the surfaces from relative motion, has recently received increased attention in the literature. Basic modeling can be found in [10, 11, 12]. Analysis of models for adhesive contact can be found in [6, 7, 8] and in the monographs [27, 30] . An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [23, 24].

The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β, it describes the fractional density of adhesion of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [10, 11], the bonding field satisfies the restriction $0 \leq \beta \leq 1$. When $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active. When $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion. When $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [12, 13, 27, 30, 32] and the references therein.

This paper represents a contribution to the study of the contact problems for piezoelectric materials. Here we investigate a mathematical model which describes the frictional contact between a deformable body assumed to be electro-elasto-viscoplastic with internal state variable and a conductive adhesive foundation. The novelty in this paper consists on the fact that the contact is modeled with a normal compliance condition, the associated general version of Coulomb's law of dry friction in which the adhesion is taken into account and a regularized electrical conductivity condition.

The paper is structured as follows. In section 2 we present notation and some preliminaries. The model is described in section 3 where the variational formulation is given. In section 4 we present our main result stated in Theorem 4.1 and its proof which is based on arguments of time-dependent variational inequalities, differential equations and fixed point.

Elastic-viscoplastic material with internal state variable and piezoelectric effect, also called electro-elasto-viscoplastic material with internal state variable is given by

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \mathcal{E}^*\mathbf{E}(\varphi(t)) \\ &+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)) + \mathcal{E}^*\mathbf{E}(\varphi(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s))\, ds, \end{aligned} \tag{1.1}$$

$$\mathbf{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathbf{B}\mathbf{E}(\varphi(t)), \tag{1.2}$$

$$\dot{\mathbf{k}}(t) = S(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{E}^*\mathbf{E}(\varphi(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)), \tag{1.3}$$

where $\mathbf{u}$ is the displacement field, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\mathbf{u})$ are the stress and the linearized strain tensor, respectively. Here $\mathcal{A}$ and $\mathcal{F}$ are operators describing the purely viscous and the elastic properties of the material, respectively. $\mathcal{G}$ is a nonlinear constitutive function describing the viscoplastic behavior of the material and depends on the internal state variable $\mathbf{k}$ and S is also a nonlinear constitutive function depending on $\mathbf{k}$. We suppose that $\mathbf{k}$ is a vector-valued function whose evolution is governed by differential equation (1.3), the set of admissible internal state variable is given by $Y = L^2(\Omega)^m$. $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor; $\mathcal{E}^*$ is its transposed and $\mathbf{B}$ denotes the electric permittivity tensor. We use dots for derivatives with respect to the time variable t. Constitutive laws of the form (1.1)-(1.3) without internal state variable have been considered in [19, 31]. To this end, we assume the decomposition of the form $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{EVP} + \boldsymbol{\sigma}^E$, where $\boldsymbol{\sigma}^E = -\mathcal{E}^*\mathbf{E}(\varphi) = \mathcal{E}^*\nabla\varphi$ is the electric part of the stress and $\boldsymbol{\sigma}^{EVP}$ is the elastic-viscoplastic part of the stress which satisfies the following behavior

$$\boldsymbol{\sigma}^{EVP}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}(t))$$

$$+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}^{EVP}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s))\, ds, \tag{1.4}$$

$$\dot{\mathbf{k}}(t) = S(\boldsymbol{\sigma}^{EVP}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)), \tag{1.5}$$

where the viscosity operator $\mathcal{A}$ and the elasticity operator $\mathcal{F}$ are assumed to be nonlinear and, moreover, $\mathcal{G}$ and S are nonlinear functions depending on $\mathbf{k}$. Constitutive laws of the form (1.4)-(1.5) without internal state variable have been considered in [14, 16].

When $\mathcal{G} = 0$ the constitutive law (1.1)-(1.3) reduces to the electro-viscoelastic constitutive law given by (1.2) and

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathcal{E}^*\nabla\varphi(t). \tag{1.6}$$

Frictional and frictionless contact problems involving electro-viscoelastic constitutive law are studied in [1, 2, 3, 13].

When $\mathcal{G} = 0$ and $\mathcal{A} = 0$ the constitutive law (1.1)-(1.3) becomes the electro-elastic constitutive law given by (1.2) and

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathcal{E}^*\nabla\varphi(t). \tag{1.7}$$

General models for linearly elastic materials with piezoelectric effect can be found in [20, 21, 22] and more recently, in [4, 15]. Frictional and frictionless contact problems involving electro-elastic materials of the form (1.7) and (1.2) were studied in [5, 17].

2 Notation and preliminaries

In this section we present notation and some preliminary material. For further details, we refer the reader to [9]. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "." and $|\ .\ |$ will represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces

$$H = L^2(\Omega)^d = \left\{ \mathbf{u} = (u_i)\ /\ u_i \in L^2(\Omega) \right\},$$

$$\mathcal{H} = \left\{ \boldsymbol{\sigma} = (\sigma_{ij})\ /\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\},$$

$$H_1 = \{ \mathbf{u} = (u_i)\ /\ \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H} \},$$

$$\mathcal{H}_1 = \{ \boldsymbol{\sigma} \in \mathcal{H}\ /\ Div\, \boldsymbol{\sigma} \in H \}.$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators, respectively, defined by $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$, $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $Div\, \boldsymbol{\sigma} = (\sigma_{ij,j})$. The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i v_i\, dx \qquad \forall\, \mathbf{u}, \mathbf{v} \in H,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},$$

$$(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \qquad \forall\, \mathbf{u}, \mathbf{v} \in H_1,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, Div\ \boldsymbol{\tau})_H \ \ \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\ .\ |_H$, $|\ .\ |_{\mathcal{H}}$, $|\ .\ |_{H_1}$ and $|\ .\ |_{\mathcal{H}_1}$, respectively. For every element $\mathbf{v} \in H_1$ we also use the notation $\mathbf{v}$ for the trace of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on Γ given by $v_\nu = \mathbf{v}\ .\ \boldsymbol{\nu}$, $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$. We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, and we recall that when $\boldsymbol{\sigma}$ is a regular function then $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\ .\ \boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\ .\ \mathbf{v}\, da \qquad \forall\, \mathbf{v} \in H_1. \tag{2.1}$$

Let $T > 0$. For every real Banach space X we use the notation $C(0, T; X)$ and $C^1(0, T; X)$ for the space of continuous and continuously differentiable functions from $[0, T]$ to X, respectively; $C(0, T; X)$ is a real Banach space with the norm

$| \mathbf{f} |_{C(0,T;X)} = \max_{t\in[0,T]} | \mathbf{f}(t) |_X$, while $C^1(0,T;X)$ is a real Banach space with the norm $| \mathbf{f} |_{C^1(0,T;X)} = \max_{t\in[0,T]} | \mathbf{f}(t) |_X + \max_{t\in[0,T]} | \dot{\mathbf{f}}(t) |_X$. Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0,r\}$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

3 Mechanical and variational formulations

We consider a piezoelectric body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with outer Lipschitz surface Γ. The body is submitted to the action of body forces of density $\mathbf{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. We consider a partition of Γ into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ on two measurable parts Γ_a and Γ_b, on the other hand, such that *meas* $(\Gamma_1) > 0$ and *meas* $(\Gamma_a) > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on Γ_1, so the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface free electrical charge of density q_2 is prescribed on Γ_b. In the reference configuration the body may come in contact over Γ_3 with an adhesive conductive obstacle, which is also called the foundation. The contact is modeled with a normal compliance condition with adhesion, the associated general version of Coulomb's law of dry friction in which the adhesion of contact surfaces is taken into account and a regularized electrical conductivity condition. Also, there may be electrical charges on the part of the body which is in contact with the foundation and which vanish when contact is lost. We are interested in the evolution of the deformation of the body and of the electric potential on the time interval $[0,T]$. The process is assumed to be isothermal, electrically static, i.e., all radiation effects are neglected, and mechanically quasistatic, i.e., the inertial terms in the momentum balance equations are neglected. Then, the classical model for the process is as follows.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$, *an electric potential field* $\varphi : \Omega \times [0,T] \to \mathbb{R}$, *an electric displacement field* $\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$, *a bonding field* $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ *and an internal state variable field* $\mathbf{k} : \Omega \times [0,T] \to \mathbb{R}^m$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathcal{E}^*\nabla\varphi(t)$$

$$+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)) - \mathcal{E}^*\nabla\varphi(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s))\, ds \text{ in } \Omega \times (0,T), \qquad (3.1)$$

$$\mathbf{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \mathbf{B}\nabla\varphi(t) \text{ in } \Omega \times (0,T), \qquad (3.2)$$

$$\dot{\mathbf{k}}(t) = S(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t) - \mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)) \text{ in } \Omega \times (0,T), \qquad (3.3)$$

$$Div\, \boldsymbol{\sigma} + \mathbf{f}_0 = 0 \text{ in } \Omega \times (0,T), \qquad (3.4)$$

$$div\mathbf{D} = q_0 \text{ in } \Omega \times (0,T), \tag{3.5}$$

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_1 \times (0,T), \tag{3.6}$$

$$\sigma\boldsymbol{\nu} = \mathbf{f}_2 \text{ on } \Gamma_2 \times (0,T), \tag{3.7}$$

$$-\sigma_\nu = p_\nu(u_\nu - h) - \gamma_\nu\,\beta^2\, R_\nu(u_\nu) \text{ on } \Gamma_3 \times (0,T), \tag{3.8}$$

$$\begin{cases} \left|\sigma_\tau + \gamma_\tau\,\beta^2\,\mathbf{R}_\tau(\mathbf{u}_\tau)\right| \le p_\tau(u_\nu - h), \\ \left|\sigma_\tau + \gamma_\tau\,\beta^2\,\mathbf{R}_\tau(\mathbf{u}_\tau)\right| < p_\tau(u_\nu - h) \Rightarrow \dot{\mathbf{u}}_\tau = \mathbf{0} \\ \left|\sigma_\tau + \gamma_\tau\,\beta^2\,\mathbf{R}_\tau(\mathbf{u}_\tau)\right| = p_\tau(u_\nu - h) \Rightarrow \exists\lambda \ge 0 \\ \text{such that} \quad \sigma_\tau + \gamma_\tau\,\beta^2\,\mathbf{R}_\tau(\mathbf{u}_\tau) = -\lambda\dot{\mathbf{u}}_\tau \text{ on } \Gamma_3 \times (0,T), \end{cases} \tag{3.9}$$

$$\dot{\beta} = -(\beta(\gamma_\nu(R_\nu(u_\nu))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_\tau) \mid^2) - \varepsilon_a)_+ \text{ on } \Gamma_3 \times (0,T), \tag{3.10}$$

$$\varphi = 0 \text{ on } \Gamma_a \times (0,T), \tag{3.11}$$

$$\mathbf{D}\,.\,\boldsymbol{\nu} = q_2 \text{ on } \Gamma_b \times (0,T), \tag{3.12}$$

$$\mathbf{D}\,.\,\boldsymbol{\nu} = \psi(u_\nu - h)\phi_L(\varphi - \varphi_0) \text{ on } \Gamma_3 \times (0,T), \tag{3.13}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \mathbf{k}(0) = \mathbf{k}_0 \text{ on } \Omega. \tag{3.14}$$

$$\beta(0) = \beta_0 \text{ on } \Gamma_3. \tag{3.15}$$

First, equations (3.1)-(3.3) represent the electro-elasto-viscoplastic constitutive law introduced in the first section. Equations (3.4)-(3.5) represent equilibrium equations for the stress and electric-displacement fields. Equations (3.6)-(3.7) are the displacement-traction boundary conditions, respectively. Equations (3.11)-(3.12) represent the electric boundary conditions. In (3.14)-(3.15), $\mathbf{u}_0$ is the given initial displacement, $\mathbf{k}_0$ is the initial internal state variable and β_0 is the initial bonding. Condition (3.8) represents the normal compliance condition with adhesion and condition (3.9) is the associated general version of Coulomb's law of dry friction on the contact surface Γ_3. Here p_ν and p_τ are given functions, h represents the initial gap in direction of $\boldsymbol{\nu}$. γ_ν, γ_τ are material parameters, also R_ν and $\mathbf{R}_\tau$ are truncation operators defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0, \end{cases} \quad \mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if } \mid \mathbf{v} \mid \le L, \\ L\frac{\mathbf{v}}{|\mathbf{v}|} & \text{if } \mid \mathbf{v} \mid > L, \end{cases}$$

with $L > 0$ being the characteristic length of the bond, beyond which it does not offer any additional traction. It follows from (3.8) that the contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\beta^2 R_\nu(u_\nu)$, the adhesive traction is tensile, and is proportional to the square of the adhesion and to the normal displacement, but as long as it does not exceed the bond length L. Also, it follows from (3.9) that the contribution of the adhesive to the tangential shear on the contact surface is represented by the term $\gamma_\tau\beta^2\mathbf{R}_\tau(\mathbf{u}_\tau)$, the adhesive shear is proportional to the square of the adhesion and to the tangential displacement, but again, only up to the bond length L. Next, equation (3.10) represents the ordinary differential equation which describes the evolution of the bonding field in which ε_a is a given material parameter. Here and below, for the simplicity, we use the notation $R_\nu(u_\nu)^2 = (R_\nu(u_\nu))^2$. We note that the adhesive process is irreversible and, indeed, once debonding occurs bonding cannot be reestablished, since $\dot{\beta} \leq 0$. Also, it is easy to see that if $0 \leq \beta_0 \leq 1$ a.e. on Γ_3, then $0 \leq \beta \leq 1$ a.e. on Γ_3 during the process.

Next, we use (3.13) as the electrical contact condition on Γ_3 which represents a regularized condition which may be obtained as follows. First, we assume that the foundation is electrically conductive and its potential is maintained at φ_0. When there is no contact at a point on the surface (i.e., $u_\nu < h$), the gap is assumed to be an insulator, there are no free electrical charges on the surface and the normal component of the electric displacement field vanishes. Thus,

$$u_\nu < h \Rightarrow \mathbf{D} \,.\, \boldsymbol{\nu} = 0. \tag{3.16}$$

During the process of contact (i.e., when $u_\nu \geq h$) the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with k as the proportionality factor. Thus

$$u_\nu \geq h \Rightarrow \mathbf{D} \,.\, \boldsymbol{\nu} = k(\varphi - \varphi_0). \tag{3.17}$$

We combine (3.16) and (3.17) to obtain

$$\mathbf{D} \,.\, \boldsymbol{\nu} = k\chi[0,\infty)(u_\nu - h)(\varphi - \varphi_0), \tag{3.18}$$

where $\chi[0,\infty)$ is the characteristic function of the interval $[0,\infty)$, that is

$$\chi[0,\infty)(r) = \begin{cases} 0 & \text{if } r < 0 \\ 1 & \text{if } r \geq 0. \end{cases}$$

Condition (3.18) describes perfect electrical contact and is somewhat similar to the well-known Signorini contact condition. Both conditions may be over-idealizations in many applications.

To make it more realistic, we regularize condition (3.18) and write it as (3.13) in which $k\chi[0,\infty)(u_\nu - h)$ is replaced with ψ which is a regular function which will be described below, and ϕ_L is the truncation function

$$\phi_L(s) = \begin{cases} -L & \text{if} \quad s < -L \\ s & \text{if } -L \leq s \leq L \\ L & \text{if} \quad s > L, \end{cases}$$

where L is a large positive constant. We note that this truncation does not pose any practical limitations on the applicability of the model, since L may be arbitrarily large, higher than any possible peak voltage in the system, and therefore in applications $\phi_L(\varphi - \varphi_0) = \varphi - \varphi_0$.

The reason for the regularization (3.13) of (3.18) is mathematical. First, we need to avoid the discontinuity in the free electric charge when contact is established and, therefore, we regularize the function $k\chi[0,\infty)$ in (3.18) with a Lipschitz continuous function ψ. A possible choice is

$$\psi(r) = \begin{cases} 0 & \text{if } r < 0 \\ k\delta r & \text{if } 0 \leq r \leq \frac{1}{\delta} \\ k & \text{if } r > \delta, \end{cases} \tag{3.19}$$

where $\delta > 0$ is a small parameter. This choice means that during the process of contact the electrical conductivity increases as the contact among the surface asperities improves, and stabilizes when the penetration $u_\nu - h$ reaches the value δ. Secondly, we need the term $\phi_L(\varphi - \varphi_0)$ to control the boundedness of $\varphi - \varphi_0$. Note that when $\psi \equiv 0$ in (3.13) then

$$\mathbf{D} \,.\, \boldsymbol{\nu} = 0 \text{ on } \Gamma_3 \times (0, T)\,, \tag{3.20}$$

which decouples the electrical and mechanical problems on the contact surface. Condition (3.20) models the case when the obstacle is a perfect insulator and was used in [5, 17, 25, 28, 29]. Condition (3.13), instead of (3.20), introduces strong coupling between the mechanical and the electric boundary conditions and leads to a new and non nonstandard mathematical model.

To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$. To obtain a variational formulation of the problem (3.1)-(3.15) we need additional notation. We introduce the following set for the bonding field.

$$Z = \left\{ \beta : [0, T] \to L^2(\Gamma_3) \;/\; 0 \le \beta(t) \le 1 \;\forall t \in [0, T]\,, \text{ a.e. on } \Gamma_3 \right\}.$$

For the displacement field we need the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 \;/\; \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$ which depends only on Ω and Γ_1 such that

$$|\, \boldsymbol{\varepsilon}(\mathbf{v}) \,|_{\mathcal{H}} \ge C_k \,|\, \mathbf{v} \,|_{H_1} \;\forall\, \mathbf{v} \in V.$$

On the space V we consider the inner product and the associated norm given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}\,, \quad |\, \mathbf{v} \,|_V = |\, \boldsymbol{\varepsilon}(\mathbf{v}) \,|_{\mathcal{H}} \quad \forall\, \mathbf{u}, \mathbf{v} \in V. \tag{3.21}$$

It follows from Korn's inequality that $|\,.\,|_{H_1}$ and $|\,.\,|_V$ are equivalent norms on V. Therefore $(V, |\,.\,|_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem and (3.21), there exists a constant $c_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$|\, \mathbf{v} \,|_{L^2(\Gamma_3)^d} \le c_0 \,|\, \mathbf{v} \,|_V \qquad \forall\, \mathbf{v} \in V. \tag{3.22}$$

We also introduce the spaces

$$W = \left\{ \phi \in H^1(\Omega) \;/\; \phi = 0 \text{ on } \Gamma_a \right\},$$

$$\mathcal{W} = \left\{ \mathbf{D} = (D_i) \;/\; D_i \in L^2(\Omega),\; div\mathbf{D} \in L^2(\Omega) \right\},$$

where $div\mathbf{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi, \phi)_W = \int_\Omega \nabla\varphi \,.\, \nabla\phi \; dx,$$

$$(\mathbf{D},\mathbf{E})_{\mathcal{W}} = \int_{\Omega} \mathbf{D} \,.\, \mathbf{E}\, dx + \int_{\Omega} div\mathbf{D} \,.\, div\mathbf{E}\, dx.$$

The associated norms will be denoted by $|\,.\,|_W$ and $|\,.\,|_{\mathcal{W}}$, respectively. Moreover, when $\mathbf{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds:

$$(\mathbf{D},\nabla\phi)_H + (div\mathbf{D},\phi)_{L^2(\Omega)} = \int_{\Gamma} \mathbf{D} \,.\, \boldsymbol{\nu}\, \phi\, da \ \ \forall\, \phi \in H^1(\Omega).$$

Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$|\, \nabla\phi\, |_H \geq C_F\, |\, \phi\, |_{H^1(\Omega)} \ \ \forall\phi \in W, \tag{3.23}$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a. It fallows from (3.23) that $|\,.\,|_{H^1(\Omega)}$ and $|\,.\,|_W$ are equivalent norms on W and therefore $(W, |\,.\,|_W)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem, there exists a constant $a_0 > 0$, depending only on Ω, Γ_a and Γ_3 such that

$$|\, \phi\, |_{L^2(\Gamma_3)} \leq a_0\, |\, \phi\, |_W \qquad \forall\, \phi \in W. \tag{3.24}$$

In the study of the mechanical problem (3.1)-(3.15), we make the following assumptions. The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ |\, \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)\, | \leq L_{\mathcal{A}}\, |\, \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\, | \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists a constant } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}}\, |\, \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\, |^2 \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (c) \text{ For any } \boldsymbol{\varepsilon} \in S_d, \text{ the mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}) \\ \text{ is Lebesgue measurable on } \Omega. \\ (d) \text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.25}$$

The elasticity operator $\mathcal{F} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{F}} > 0 \text{ such that} \\ |\, \mathcal{F}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x},\boldsymbol{\varepsilon}_2)\, | \leq L_{\mathcal{F}}\, |\, \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\, | \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\varepsilon} \in S_d, \mathbf{x} \to \mathcal{F}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue} \\ \text{measurable on } \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{F}(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.26}$$

The visco-plasticity operator $\mathcal{G} : \Omega \times S_d \times S_d \times \mathbb{R}^m \to S_d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ |\, \mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\mathbf{k}_1) - \mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\mathbf{k}_2)\, | \\ \leq L_{\mathcal{G}}(|\, \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\, | + |\, \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\, | + \mathbf{k}_1 - \mathbf{k}_2\, |) \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d \text{ and } \mathbf{k}_1, \mathbf{k}_2 \in \mathbb{R}^m \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S_d \text{ and } \mathbf{k} \in \mathbb{R}^m, \mathbf{x} \to \mathcal{G}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\mathbf{k}) \\ \text{is Lebesgue measurable on } \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},\mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.27}$$

The function $S : \Omega \times S_d \times S_d \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_S > 0 \text{ such that} \\ \mid S(\mathbf{x}, \boldsymbol{\varepsilon}_1, \boldsymbol{\sigma}_1, \mathbf{k}_1) - S(\mathbf{x}, \boldsymbol{\varepsilon}_2, \boldsymbol{\sigma}_2, \mathbf{k}_2) \mid \\ \leq L_S(\mid \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \mid + \mid \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \mid + \mid \mathbf{k}_1 - \mathbf{k}_2 \mid) \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d \text{ and } \mathbf{k}_1, \mathbf{k}_2 \in \mathbb{R}^m \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S_d \text{ and } \mathbf{k} \in \mathbb{R}^m, \mathbf{x} \to S(\mathbf{x}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{k}) \\ \text{is Lebesgue measurable on } \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to S(\mathbf{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \text{ belongs to } L^2(\Omega)_s^{m \times m}. \end{cases} \tag{3.28}$$

The normal compliance functions $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ $(r = \nu, \tau)$ satisfy

$$\begin{cases} (a) \text{ There exists a constant } L_r > 0 \text{ such that} \\ \mid p_r(\mathbf{x}, u_1) - p_r(\mathbf{x}, u_2) \mid \leq L_r \mid u_1 - u_2 \mid \ \forall u_1, u_2 \in \mathbb{R}, \\ \text{a.e. } \mathbf{x} \in \Gamma_3, \\ (b) \text{ the mapping } \mathbf{x} \to p_r(\mathbf{x}, u) \text{ is measurable on } \Gamma_3 \\ \text{for any } u \in \mathbb{R}, \\ (c) \ p_r(\mathbf{x}, u) = 0 \text{ for all } u \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{cases} \tag{3.29}$$

A simple example of a normal compliance function p_ν which satisfies (3.29) is $p_\nu(r) = c_\nu r_+$ where $c_\nu \in L^\infty(\Gamma_3)$ is a positive function and $p_\tau = \mu p_\nu$ with $\mu \geq 0$. The electric permittivity operator $\mathbf{B} = (b_{ij})$: $\Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \ \mathbf{B}(\mathbf{x})\mathbf{E} = (b_{ij}(\mathbf{x})E_j) \ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \ b_{ij} = b_{ji}, \ b_{ij} \in L^\infty(\Omega). \\ (c) \text{ There exists a constant } m_B > 0 \text{ such that} \\ \mathbf{BE}.\mathbf{E} \geq m_B \mid \mathbf{E} \mid^2 \ \ \forall \, \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega. \end{cases} \tag{3.30}$$

The piezoelectric operator $\mathcal{E} : \Omega \times S_d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \ \mathcal{E}(\mathbf{x})\boldsymbol{\tau} = (e_{ijk}(\mathbf{x})\tau_{jk}) \ \forall \boldsymbol{\tau} = (\tau_{ij}) \in S_d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \ e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{cases} \tag{3.31}$$

The surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_\psi > 0 \text{ such that} \\ \mid \psi(\mathbf{x}, u_1) - \psi(\mathbf{x}, u_2) \mid \leq L_\psi \mid u_1 - u_2 \mid \\ \forall u_1, u_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (b) \text{ There exists a constant } N_\psi > 0 \text{ such that} \\ \mid \psi(\mathbf{x}, u) \mid \leq N_\psi \ \forall u \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ (c) \ \mathbf{x} \to \psi(\mathbf{x}, u) \text{ is measurable on } \Gamma_3 \text{ for all } u \in \mathbb{R}. \\ (d) \ \mathbf{x} \to \psi(\mathbf{x}, u) = 0 \text{ for all } u \leq 0. \end{cases} \tag{3.32}$$

The body forces and surface tractions have the regularity

$$\mathbf{f}_0 \in C(0, T; H), \quad \mathbf{f}_2 \in C(0, T; L^2(\Gamma_2)^d), \tag{3.33}$$

$$q_0 \in C(0, T; L^2(\Omega)), \quad q_2 \in C(0, T; L^2(\Gamma_b)). \tag{3.34}$$

The adhesion coefficients satisfy

$$\gamma_\nu, \gamma_\tau \ \in L^\infty(\Gamma_3),\ \varepsilon_a \in L^2(\Gamma_3),\ \gamma_\nu,\ \gamma_\tau,\ \varepsilon_a \geq 0 \text{ a.e. on } \Omega. \tag{3.35}$$

Finally, we assume that the initial gap function and the initial data satisfy

$$h \in L^2(\Gamma_3), \quad h \geq 0 \text{ a.e. on } \Gamma_3, \tag{3.36}$$

$$\varphi_0 \in L^2(\Gamma_3),\ \mathbf{u}_0 \in V,\ \mathbf{k}_0 \in Y, \tag{3.37}$$

$$\beta_0 \in L^2(\Gamma_3),\ 0 \leq \beta_0 \leq 1 \text{ a.e. on } \Gamma_3. \tag{3.38}$$

We define the three mappings $\mathbf{f} : [0, T] \to V$, $q : [0, T] \to W$ and $\gamma : V \times W \to W$, respectively, for all $\mathbf{u}, \mathbf{v} \in V$, $\varphi, \phi \in W$ and $t \in [0, T]$, by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \,.\, \mathbf{v}\, dx \, + \, \int_{\Gamma_2} \mathbf{f}_2(t) \,.\, \mathbf{v}\, da, \tag{3.39}$$

$$(q(t), \phi)_W = \int_\Omega q_0(t)\, \phi\, dx \, - \, \int_{\Gamma_b} q_2(t)\, \phi\, da, \tag{3.40}$$

$$(\gamma(\mathbf{u}, \varphi), \phi)_W = \int_{\Gamma_3} \psi(u_\nu - h)\phi_L(\varphi - \varphi_0)\phi\, da. \tag{3.41}$$

Also, we define the adhesion functional $j_{ad} : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$, the normal compliance functional $j_{nc} : V \times V \to \mathbb{R}$ and the friction functional $j_{fr} : V \times V \to \mathbb{R}$ by

$$j_{ad}(\beta, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} (-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu \ + \gamma_\tau \beta^2 \mathbf{R}_\tau(u_\tau).\mathbf{v}_\tau) da, \tag{3.42}$$

$$j_{nc}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu - h) v_\nu da, \tag{3.43}$$

$$j_{fr}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\tau(u_\nu - h)\, |\mathbf{v}_\tau|\, da. \tag{3.44}$$

The functional $j_{fr} : V \times V \to \mathbb{R}$ satisfies

$$\text{For all } \mathbf{g} \in V,\ j_{fr}(\mathbf{g}, .) \text{ is proper, convex and lower semi-continuous on } V. \tag{3.45}$$

We note that condition (3.33) and (3.34) imply

$$\mathbf{f} \in C(0, T; V),\ \ q \in C(0, T; W). \tag{3.46}$$

Using standard arguments, we obtain the following variational formulation of the mechanical problem (3.1)-(3.15).

Problem PV. Find a displacement field $\mathbf{u} : [0, T] \to V$*, a stress field* $\boldsymbol{\sigma} : [0, T] \to \mathcal{H}$*, an electric potential field* $\varphi : [0, T] \to W$*, an electric displacement field* $\mathbf{D} : [0, T] \to H$*, a bonding field* $\beta : [0, T] \to L^\infty(\Gamma_3)$ *and an internal state variable field* $\mathbf{k} : [0, T] \to Y$ *such that* for $t \in (0, T)$

$$\boldsymbol{\sigma}\,(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathcal{E}^*\nabla\varphi(t)$$

$$+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}\,(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)) - \mathcal{E}^*\nabla\varphi(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s))\, ds, \tag{3.47}$$

$$\dot{\mathbf{k}}(t) = S(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t) - \mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)), \tag{3.48}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}(t)))_{\mathcal{H}} + j_{ad}(\beta(t), \mathbf{u}(t), \mathbf{v}-\dot{\mathbf{u}}(t)) + j_{nc}(\mathbf{u}(t), \mathbf{v}-\dot{\mathbf{u}}(t))$$
$$+ j_{fr}(\mathbf{u}(t), \mathbf{v}) - j_{fr}(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v}-\dot{\mathbf{u}}(t))_V \ \ \forall \mathbf{v} \in V, \tag{3.49}$$

$$\mathbf{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \mathbf{B}\nabla\varphi(t), \tag{3.50}$$

$$(\mathbf{D}(t), \nabla\phi)_H = -(q(t), \phi)_W \ + \ (\gamma(\mathbf{u}(t), \varphi), \phi)_W \ \ \forall \phi \in W, \tag{3.51}$$

$$\dot{\beta} = -(\beta(\gamma_\nu(R_\nu(u_\nu))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_\tau) \mid^2) - \varepsilon_a)_+ \text{ a.e. } t \in (0, T), \tag{3.52}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \ \beta(0) = \beta_0, \ \mathbf{k}(0) = \mathbf{k}_0 \, . \tag{3.53}$$

The existence of the unique solution of problem PV is stated and proved in the next section.

4 Existence and uniqueness result

Our main result which states the unique solvability of Problem PV is the following.

Theorem 4.1. *Assume that (3.25)-(3.38) and (3.45) hold. Then if* $N_\psi < \frac{m_B}{a_0^2}$, *there exists a unique solution* $\{\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D}, \beta, \mathbf{k}\}$ *to problem PV satisfying*

$$\mathbf{u} \in C^1(0, T; V), \ \boldsymbol{\sigma} \in C(0, T; \mathcal{H}_1), \tag{4.1}$$

$$\varphi \in C(0, T; W), \ \mathbf{D} \in C(0, T; \mathcal{W}), \tag{4.2}$$

$$\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap Z, \tag{4.3}$$

$$\mathbf{k} \in C^1(0, T; Y). \tag{4.4}$$

We conclude that, under the assumptions (3.25)-(3.38) and (3.45) and if $N_\psi < \frac{m_B}{a_0^2}$ is satisfied, the mechanical problem (3.1)-(3.15) has a unique weak solution satisfying the regularities (4.1)-(4.4). The proof of Theorem 4.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 4.1 hold, and we consider that C is a generic positive constant which depends on $\Omega, \Gamma_1, \Gamma_2, \Gamma_3$, $\mathcal{A}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{E}$, S, p_ν, p_τ, L and T and may change from place to place. The proof is based on arguments of time-dependent variational inequalities, differential equations and fixed point arguments.

In the first step we let $\boldsymbol{\eta} = (\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) \in C(0, T; V \times Y)$ and $\mathbf{g} \in C(0, T; V)$ be given and consider the following variational inequality.

Problem $PV_{\eta g}$: *Find a displacement field* $\mathbf{v}_{\eta g} : [0, T] \to V$ *such that* $\forall t \in [0, T]$

$$(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_{\eta g}(t)), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{v}_{\eta g}(t)))_{\mathcal{H}} \ + \ j_{fr}(\mathbf{g}(t), \mathbf{v}) - j_{fr}(\mathbf{g}(t), \mathbf{v}_{\eta g}(t))$$
$$\geq (\mathbf{f}(t) - \boldsymbol{\eta}^1(t), \mathbf{v}-\mathbf{v}_{\eta g}(t))_V \ \ \forall \mathbf{v} \in V. \tag{4.5}$$

In the study of problem $PV_{\eta g}$ we have the following result.

Lemma 4.2. *$PV_{\eta g}$ has a unique weak solution with the regularity*

$$\mathbf{v}_{\eta g} \in C(0,T;V). \tag{4.6}$$

Proof. We define the operator $A : V \to V$ by

$$(A\mathbf{u},\mathbf{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u},\mathbf{v} \in V. \tag{4.7}$$

Moreover using Riesz representation theorem we may define an element $\mathbf{F} \in C(0,T;V)$ by

$$(\mathbf{F}(t),\mathbf{v})_V = (\mathbf{f}(t),\mathbf{v})_V - (\boldsymbol{\eta}^1(t),\mathbf{v})_V.$$

The definition of the operator A given in (4.7), assumption (3.25) on the operator $\mathcal{A}$ combined with assumption (3.45) on j_{fr}, by using classical result on elliptic inequalities (see for example [9]), we conclude that there exists a unique function $\mathbf{v}_{\eta g}(t) \in V$ which satisfies

$$\begin{aligned}(A\mathbf{v}_{\eta g}(t),\mathbf{v}-\mathbf{v}_{\eta g}(t))_V + j_{fr}(\mathbf{g}(t),\mathbf{v}) - j_{fr}(\mathbf{g}(t),\mathbf{v}_{\eta g}(t))\\ \geq (\mathbf{F}(t),\mathbf{v}-\mathbf{v}_{\eta g}(t))_V \quad \forall \mathbf{v} \in V.\end{aligned}$$

The regularity of the functions $\mathbf{f}$, $\mathbf{g}$ and $\boldsymbol{\eta}^1$ show that the regularity (4.6) is satisfied. ■

We consider now the following operator $\Lambda_\eta : C(0,T;V) \to C(0,T;V)$ defined by

$$\Lambda_\eta \mathbf{g}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_{\eta g}\,(s)\,ds \quad \forall \mathbf{g} \in C(0,T;V). \tag{4.8}$$

Lemma 4.3. *The operator Λ_η has a unique fixed point $\mathbf{g}_\eta \in C(0,T;V)$.*

Proof. The proof is based on Banach's fixed point theorem, see for example [26]. ■

Now we consider the following problem.

Problem PV_η. Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ such that $\forall t \in [0,T]$, $\mathbf{u}_\eta(0) = \mathbf{u}_0$ and

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)),\boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + j_{fr}(\mathbf{u}_\eta(t),\mathbf{v}) - j_{fr}(\mathbf{u}_\eta(t),\dot{\mathbf{u}}_\eta(t))\\ +(\boldsymbol{\eta}^1(t),\mathbf{v}-\dot{\mathbf{u}}_\eta(t))_V \geq (\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}_\eta(t))_V \ \ \forall \mathbf{v} \in V.\end{aligned} \tag{4.9}$$

In the study of the problem PV_η we have the following result.

Lemma 4.4. *PV_η has a unique solution satisfying the regularity expressed in (4.1).*

Proof. For $\boldsymbol{\eta} \in C(0,T;V \times Y)$, we denote by $\mathbf{g}_\eta \in C(0,T;V)$ be the fixed point obtained in Lemma 4.3 and let $\mathbf{u}_\eta$ be the function defined by

$$\mathbf{u}_\eta(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_{\eta g_\eta}(s)\,ds \qquad \forall t \in [0,T]. \tag{4.10}$$

We have $\Lambda_\eta \mathbf{g}_\eta = \mathbf{g}_\eta$. From (4.8) and (4.10) it follows that $\mathbf{u}_\eta = \mathbf{g}_\eta$. Therefore, taking $\mathbf{g} = \mathbf{g}_\eta$ in (4.5), we see that $\mathbf{u}_\eta$ is the unique solution of the problem PV_η satisfying the regularity expressed in (4.1). ■

In the second step we let $\boldsymbol{\eta} \in C(0,T;V\times Y)$, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.4 and consider the following variational problem.

Problem QV_η. Find the electric potential field $\varphi_\eta : [0,T] \to W$ such that $\forall t \in [0,T]$

$$(\mathbf{B}\nabla\varphi_\eta(t),\nabla\phi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)),\nabla\phi)_H + (\gamma(\mathbf{u}_\eta(t),\varphi_\eta(t)),\phi)_W$$
$$= (q(t),\phi)_W \quad \forall\phi \in W, \tag{4.11}$$

we have the following result.

Lemma 4.5. *QV_η has a unique solution φ_η which satisfies the regularity expressed in (4.2). Moreover, denote $\mathbf{u}_{\eta_i} = \mathbf{u}_i$ and if $\varphi_{\eta_i} = \varphi_i$ for $i = 1,2$ are the solutions of (4.11) corresponding to $\jmath_i \in C(0,T;V\times Y)$. Then, there exists $C > 0$, such that*

$$|\ \varphi_1(t) - \varphi_2(t)\ |_W \leq C\ |\ \mathbf{u}_1(t) - \mathbf{u}_2(t)\ |_V \ \ \forall\, t \in [0,T]. \tag{4.12}$$

Proof. Let $t \in [0,T]$. We use Riesz representation theorem to define the operator $A_\eta(t) : W \to W$ by

$$(A_\eta(t)\varphi,\phi)_W = (\mathbf{B}\nabla\varphi,\nabla\phi)_W \ - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)),\nabla\phi)_W$$
$$+(\gamma(\mathbf{u}_\eta(t),\varphi),\phi)_W \quad \forall\varphi,\phi \in W. \tag{4.13}$$

Let $\varphi_1, \varphi_2 \in W$. Assumption (3.30), (3.41), (3.32) (a) and the monotonicity of the function ϕ_L give us

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \geq m_B\ |\ \varphi_1 - \varphi_2\ |_W^2, \tag{4.14}$$

then $A_\eta(t)$ is a strongly monotone operator on W. On the other hand, using again (3.30)-(3.32), (3.41) and (3.24) we find

$$|\ A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2\ |_W \leq (C_B + N_\psi a_0^2)\ |\ \varphi_1 - \varphi_2\ |_W, \tag{4.15}$$

where C_B is a positive constant which depends on $\mathbf{B}$, which shows that $A_\eta : W \to W$ is Lipschitz continuous. Since A_η is a strongly monotone and Lipschitz continuous operator on W, we deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$A_\eta(t)\varphi_\eta(t) = q(t). \tag{4.16}$$

We conclude that φ_η is the unique solution of QV_η. We show next that $\varphi_\eta \in C(0,T;W)$. Let $t_1, t_2 \in [0,T]$ and denote $\varphi_\eta(t_i) = \varphi_i$, $u_{\eta\nu}(t_i) = u_i$, $\mathbf{u}_\eta(t_i) = \mathbf{u}_i$, $q(t_i) = q_i$ for $i = 1,2$. Using (4.11), (3.30)-(3.31) and (3.41) we find

$$m_B\ |\ \varphi_1 - \varphi_2\ |_W^2 \leq C_{\mathcal{E}}(|\ \mathbf{u}_1 - \mathbf{u}_2\ |_V + |\ q_1 - q_2\ |_W)\ |\ \varphi_1 - \varphi_2\ |_W$$
$$+ \int_{\Gamma_3} |\ \psi(u_1 - h)\phi_L(\varphi_1 - \varphi_0) - \psi(u_2 - h)\phi_L(\varphi_2 - \varphi_0)\ ||\ \varphi_1 - \varphi_2\ |\ da, \tag{4.17}$$

where $C_{\mathcal{E}}$ is a positive constant which depends on the piezoelectric tensor $\mathcal{E}$. We use now the bounds $|\ \psi(u_i - h)\ | \leq N_\psi$, $|\ \phi_L(\varphi_1 - \varphi_0)\ | \leq L$, the Lipschitz continuity of the functions ψ and ϕ_L, the inequality (3.22) and (3.24). After some algebraic manipulations we obtain

$$m_B \mid \varphi_1 - \varphi_2 \mid_W \leq (C_{\mathcal{E}} + L_\psi L a_0 c_0) \mid \mathbf{u}_1 - \mathbf{u}_2 \mid_V$$
$$+ \mid q_1 - q_2 \mid_W + N_\psi a_0^2 \mid \varphi_1 - \varphi_2 \mid_W . \tag{4.18}$$

It follows from inequality (4.18) and the fact that $N_\psi < \frac{m_B}{a_0^2}$ that

$$\mid \varphi_1 - \varphi_2 \mid_W \leq C(\mid \mathbf{u}_1 - \mathbf{u}_2 \mid_V + \mid q_1 - q_2 \mid_W). \tag{4.19}$$

Since $\mathbf{u} \in C^1(0,T;V)$, $q \in C(0,T;W)$, the inequality (4.19) implies that $\varphi_\eta \in C(0,T;W)$.

Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(0,T;V \times Y)$ and denote $\varphi_{\eta i} = \varphi_i$, $\mathbf{u}_{\eta i} = \mathbf{u}_i$ for $i = 1,2$. We use (4.11) and arguments similar to those used in the proof of (4.18) to obtain

$$m_B \mid \varphi_1(t) - \varphi_2(t) \mid_W$$
$$\leq (C_{\mathcal{E}} + L_\psi L a_0 c_0) \mid \mathbf{u}_1(t) - \mathbf{u}_2(t) \mid_V + N_\psi a_0^2 \mid \varphi_1(t) - \varphi_2(t) \mid_W$$

for all $t \in [0,T]$. This inequality combined with the fact that $N_\psi < \frac{m_B}{a_0^2}$ leads to the estimate (4.12). ∎

In the third step we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.4 and we consider the following initial-value problem.

Problem PV_β. Find the bonding field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that for a.e. $t \in (0,T)$

$$\dot{\beta}_\eta(t) = -(\beta_\eta(t)(\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t)) \mid^2) - \varepsilon_a)_+, \tag{4.20}$$

$$\beta_\eta(0) = \beta_0. \tag{4.21}$$

We have the following result.

Lemma 4.6. *There exists a unique $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z$ solution to problem PV_β.*

Proof. We use similar arguments that those used in [30] . ∎

Define $\mathbf{k}_\eta \in C^1(0,T;Y)$ by

$$\mathbf{k}_\eta(t) = \mathbf{k}_0 + \int_0^t \boldsymbol{\eta}^2(s)\, ds. \tag{4.22}$$

In the fourth step we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.4, the electric potential field φ_η obtained in Lemma 4.5 and $\mathbf{k}_\eta$ defined in (4.22) to construct the following Cauchy problem for the stress field.

Problem $PV\sigma_\eta$. Find a stress field $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \mathbf{k}_\eta(s))ds \;\forall t \in [0,T] . \tag{4.23}$$

In the study of problem $PV\sigma_\eta$ we have the following result.

Lemma 4.7. *There exists a unique solution of problem $PV\sigma_\eta$ and it satisfies $\sigma_\eta \in C^1(0,T;\mathcal{H})$. Moreover, if $\mathbf{u}_i$ and $\boldsymbol{\sigma}_i$ represent the solutions of problem PV_{η_i}, $PV\sigma_{\eta_i}$ respectively, and $\mathbf{k}_i$ is defined in (4.22) for $\boldsymbol{\eta}_i \in C(0,T;V\times Y)$ $i=1,2$, then there exists $C>0$ such that*

$$\mid \boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\mid^2_{\mathcal{H}} \leq C(\mid \mathbf{u}_1(t)-\mathbf{u}_2(t)\mid^2_V$$
$$+\int_0^t \mid \mathbf{u}_1(s)-\mathbf{u}_2(s)\mid^2_V ds + \int_0^t \mid \mathbf{k}_1(s)-\mathbf{k}_2(s)\mid^2_Y ds)\ \forall t\in[0,T]. \tag{4.24}$$

Proof. Let $\Lambda_\eta : C(0,T;\mathcal{H}) \to C(0,T;\mathcal{H})$ be the operator given by

$$\Lambda_\eta \boldsymbol{\sigma}\,(t) = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}\,(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \mathbf{k}_\eta(s))ds, \tag{4.25}$$

for $\boldsymbol{\sigma} \in C(0,T,\mathcal{H})$ and $t\in[0,T]$. For $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in C(0,T,\mathcal{H})$, we use (4.25) and (3.27) to obtain for all $t\in[0,T]$

$$\mid \Lambda_\eta\boldsymbol{\sigma}_1(t) - \Lambda_\eta\boldsymbol{\sigma}_2(t)\mid_{\mathcal{H}} \leq L_{\mathcal{G}} \int_0^t \mid \boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\mid_{\mathcal{H}} ds.$$

It follows from this inequality that for p large enough, the operator Λ_η^p is a contraction on the Banach space $C(0,T;\mathcal{H})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H})$ such that $\Lambda_\eta\boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_\eta$. Moreover, $\boldsymbol{\sigma}_\eta$ is the unique solution of problem $PV\sigma_\eta$ and, using (4.23), the regularity of $\mathbf{u}_\eta$, the regularity of $\mathbf{k}_\eta$ and the properties of the operators $\mathcal{F}$ and $\mathcal{G}$, it follows that $\boldsymbol{\sigma}_\eta \in C^1(0,T,\mathcal{H})$.

Consider now $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(0,T;V\times Y)$ and for $i=1,2$, denote $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$ and $\mathbf{k}_{\eta_i} = \mathbf{k}_i$. We have

$$\boldsymbol{\sigma}_i(t) = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}_i(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\mathbf{u}_i(s)), \mathbf{k}_i(s))\,ds \quad \forall t\in[0,T],$$

and, using the properties (3.26) and (3.27) of $\mathcal{F}$ and $\mathcal{G}$, we find

$$\mid \boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\mid^2_{\mathcal{H}} \leq C(\mid \mathbf{u}_1(t)-\mathbf{u}_2(t)\mid^2_V + \int_0^t \mid \boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\mid^2_{\mathcal{H}} ds$$
$$+\int_0^t \mid \mathbf{u}_1(s)-\mathbf{u}_2(s)\mid^2_V ds + \int_0^t \mid \mathbf{k}_1(s)-\mathbf{k}_2(s)\mid^2_Y ds) \quad \forall t\in[0,T].$$

We use Gronwall argument in the obtained inequality to deduce the estimate (4.24). ∎

Finally as a consequence of these results and using the properties of the operators $\mathcal{F}$, $\mathcal{G}$, $\mathcal{E}$, the function S and the functional j_{ad} and j_{nc} for $t\in[0,T]$, we consider the element

$$\Lambda\boldsymbol{\eta}(t) = (\Lambda^1\boldsymbol{\eta}(t), \Lambda^2\boldsymbol{\eta}(t)) \in V\times Y, \tag{4.26}$$

defined by

$$(\Lambda^1\boldsymbol{\eta}(t), \mathbf{v})_V = (\mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}$$

$$+(\int_0^t \mathcal{G}(\sigma_\eta(s), \varepsilon(\mathbf{u}_\eta(s)), \mathbf{k}_\eta(s))ds, \varepsilon(\mathbf{v}))_{\mathcal{H}}$$
$$+j_{ad}(\beta_\eta(t), \mathbf{u}_\eta(t), \mathbf{v}) + j_{nc}(\mathbf{u}_\eta(t), \mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{4.27}$$
$$\Lambda^2\boldsymbol{\eta}(t) = S(\sigma_\eta(t), \varepsilon(\mathbf{u}_\eta(t)), \mathbf{k}_\eta(t)). \tag{4.28}$$

Here, for every $\boldsymbol{\eta} \in C(0,T;V \times Y)$, $\mathbf{u}_\eta, \varphi_\eta, \beta_\eta, \sigma_\eta$ represent the displacement field, the potential electric field, the bonding field, the stress field obtained in Lemmas 4.4, 4.5, 4.6, 4.7 respectively and $\mathbf{k}_\eta$ is the internal state variable given by (4.22). We have the following result.

Lemma 4.8. *The operator Λ has a unique fixed point $\boldsymbol{\eta}_* \in C(0,T;V \times Y)$ such that $\Lambda\boldsymbol{\eta}_* = \boldsymbol{\eta}_*$.*

Proof. Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(0,T;V \times Y)$. Write $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_{\eta_i} = \mathbf{v}_i, \varphi_{\eta_i} = \varphi_i$, $\beta_{\eta_i} = \beta_i$, $\sigma_{\eta_i} = \sigma_i$, $\mathbf{k}_{\eta_i} = \mathbf{k}_i$ for $i = 1,2$. Using (3.21), (3.26), (3.27), (3.29), (3.31), (4.24), (4.27) and the definition of R_ν, $\mathbf{R}_\tau$ to deduce that

$$| \Lambda^1\boldsymbol{\eta}_1(t) - \Lambda^1\boldsymbol{\eta}_2(t) |_V^2 \leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2$$
$$+\int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds + \int_0^t | \mathbf{k}_1(s) - \mathbf{k}_2(s) |_Y^2 \, ds$$
$$+ | \varphi_1(t) - \varphi_2(t) |_W^2 + | \beta_1(t) - \beta_2(t) |_{L^2(\Gamma_3)}^2). \tag{4.29}$$

On the other hand, from (4.20) and (4.21) we can write

$$\beta_i(t) = \beta_0 - \int_0^t (\beta_i(s)(\gamma_\nu(R_\nu(u_{i\nu}(s)))^2 + \gamma_\tau | \mathbf{R}_\tau(\mathbf{u}_{i\tau}(s)) |^2) - \varepsilon_a)_+ \, ds, \tag{4.30}$$

and then

$$| \beta_1(t) - \beta_2(t) |_{L^2(\Gamma_3)}$$
$$\leq C\int_0^t | \beta_1(s)(R_\nu(u_{1\nu}(s)))^2 - \beta_2(s)(R_\nu(u_{2\nu}(s)))^2 |_{L^2(\Gamma_3)} \, ds$$
$$+C\int_0^t | \beta_1(s) | \mathbf{R}_\tau(\mathbf{u}_{1\tau}(s)) |^2 - \beta_2(s) | \mathbf{R}_\tau(\mathbf{u}_{2\tau}(s)) |^2|_{L^2(\Gamma_3)} \, ds.$$

Using the definition of R_ν and $\mathbf{R}_\tau$ and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we get

$$| \beta_1(t) - \beta_2(t) |_{L^2(\Gamma_3)}$$
$$\leq C(\int_0^t | \beta_1(s) - \beta_2(s) |_{L^2(\Gamma_3)} \, ds + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_{L^2(\Gamma_3)^d} \, ds). \tag{4.31}$$

By Gronwall's inequality and from (3.22), it follows that

$$| \beta_1(t) - \beta_2(t) |_{L^2(\Gamma_3)} \leq C\int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V \, ds. \tag{4.32}$$

We substitute (4.12) and (4.32) in (4.29) to obtain

$$| \Lambda^1\boldsymbol{\eta}_1(t) - \Lambda^1\boldsymbol{\eta}_2(t) |_V^2 \leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2$$

$$+\int_0^t \mid \mathbf{u}_1(s)-\mathbf{u}_2(s)\mid_V^2 ds+\int_0^t \mid \mathbf{k}_1(s)-\mathbf{k}_2(s)\mid_Y^2 ds). \tag{4.33}$$

By similar arguments, from (4.28), (4.24) and (3.28) it follows

$$\mid \Lambda^2\boldsymbol{\eta}_1(t)-\Lambda^2\boldsymbol{\eta}_2(t)\mid_Y^2$$

$$\leq C(\mid \mathbf{u}_1(t)-\mathbf{u}_2(t)\mid_V^2+\int_0^t \mid \mathbf{u}_1(s)-\mathbf{u}_2(s)\mid_V^2 ds$$

$$+\mid \mathbf{k}_1(t)-\mathbf{k}_2(t)\mid_Y^2+\int_0^t \mid \mathbf{k}_1(s)-\mathbf{k}_2(s)\mid_Y^2 ds). \tag{4.34}$$

Therefore,

$$\mid \Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\mid_{V\times Y}^2\leq C(\mid \mathbf{u}_1(t)-\mathbf{u}_2(t)\mid_V^2+\int_0^t \mid \mathbf{u}_1(s)-\mathbf{u}_2(s)\mid_V^2 ds$$

$$+\mid \mathbf{k}_1(t)-\mathbf{k}_2(t)\mid_Y^2+\int_0^t \mid \mathbf{k}_1(s)-\mathbf{k}_2(s)\mid_Y^2 ds). \tag{4.35}$$

Moreover, from (4.5) we obtain that

$$(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_1)-\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_2),\boldsymbol{\varepsilon}(\mathbf{v}_1-\mathbf{v}_2))_{\mathcal{H}}\leq$$

$$j_{fr}(\mathbf{u}_1,\mathbf{v}_2)-j_{fr}(\mathbf{u}_1,\mathbf{v}_1)+j_{fr}(\mathbf{u}_2,\mathbf{v}_1)-j_{fr}(\mathbf{u}_2,\mathbf{v}_2)-(\boldsymbol{\eta}_1^1-\boldsymbol{\eta}_2^1,\mathbf{v}_1-\mathbf{v}_2)_V. \tag{4.36}$$

It follows from (4.36), (3.29) and (3.44) that

$$(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_1)-\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_2),\boldsymbol{\varepsilon}(\mathbf{v}_1-\mathbf{v}_2))_{\mathcal{H}}\leq$$

$$-(\boldsymbol{\eta}_1^1(t)-\boldsymbol{\eta}_2^1(t),\mathbf{v}_1(t)-\mathbf{v}_2(t))_V\ +C\mid \mathbf{u}_1-\mathbf{u}_2\mid_V\mid \mathbf{v}_1-\mathbf{v}_2\mid_V .$$

We integrate this equality with respect to time and use condition (3.25) to find

$$m_{\mathcal{A}}\int_0^t \mid \mathbf{v}_1(s)-\mathbf{v}_2(s)\mid_V^2 ds\leq$$

$$C\int_0^t (\mid \boldsymbol{\eta}_1^1(s)-\boldsymbol{\eta}_2^1(s)\mid+\mid \mathbf{u}_1(s)-\mathbf{u}_2(s)\mid_V)\mid \mathbf{v}_1(s)-\mathbf{v}_2(s)\mid_V ds$$

for all $t\in[0,T]$. Then, using the inequality $2ab\leq\frac{a^2}{\gamma}+\gamma b^2$ we obtain

$$\int_0^t \mid \mathbf{v}_1(s)-\mathbf{v}_2(s)\mid_V^2 ds\leq$$

$$C\int_0^t (\mid \boldsymbol{\eta}_1^1(s)-\boldsymbol{\eta}_2^1(s)\mid_V^2+\mid \mathbf{u}_1(s)-\mathbf{u}_2(s)\mid_V^2)ds\quad \forall t\in[0,T]. \tag{4.37}$$

Since $\mathbf{u}_1(0)=\mathbf{u}_2(0)=\mathbf{u}_0$ we have

$$\mid \mathbf{u}_1(t)-\mathbf{u}_2(t)\mid_V^2\leq C\int_0^t \mid \mathbf{v}_1(s)-\mathbf{v}_2(s)\mid_V^2 ds. \tag{4.38}$$

Using (4.38), (4.37) and applying Gronwall's inequality to find that

$$| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 \leq C \int_0^t | \boldsymbol{\eta}_1^1(s) - \boldsymbol{\eta}_2^1(s) |_V^2 \, ds. \tag{4.39}$$

On the other hand, from (4.22) we have

$$| \mathbf{k}_1(t) - \mathbf{k}_2(t) |_Y^2 \leq C \int_0^t | \boldsymbol{\eta}_1^2(s) - \boldsymbol{\eta}_2^2(s) |_Y^2 \, ds. \tag{4.40}$$

We substitute (4.39) and (4.40) in (4.35) to see that

$$| \Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t) |_{V\times Y}^2 \leq C \int_0^t | \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s) |_{V\times Y}^2 \, ds.$$

Reiterating this inequality n times leads to

$$| \Lambda^n\boldsymbol{\eta}_1 - \Lambda^n\boldsymbol{\eta}_2 |_{C(0,T;V\times Y)}^2 \leq \frac{(CT)^n}{n!} | \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 |_{C(0,T;V\times Y)}^2 .$$

Thus, for n sufficiently large, Λ^n is a contraction on the Banach space $C(0,T;V \times Y)$, and so Λ has a unique fixed point. ■

Now, we have all the ingredients needed to prove Theorem 4.1.

Proof. Let $\boldsymbol{\eta}_* = (\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) \in C(0,T;V\times Y)$ be the fixed point of Λ defined by (4.26)-(4.28) and denote

$$\mathbf{u} = \mathbf{u}_{\eta_*}, \varphi = \varphi_{\eta_*}, \beta = \beta_{\eta_*}, \mathbf{k} = \mathbf{k}_{\eta_*}, \tag{4.41}$$

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{E}^*\nabla\varphi + \boldsymbol{\sigma}_{\eta_*}, \tag{4.42}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{B}\nabla\varphi. \tag{4.43}$$

We prove that $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D}, \beta, \mathbf{k})$ satisfies (3.47)-(3.53) and the regularities (4.1)-(4.4). Indeed, we write (4.23) for $\boldsymbol{\eta} = \boldsymbol{\eta}_*$ and use (4.41)-(4.42) to obtain (3.47). We consider (4.9) and (4.22) for $\boldsymbol{\eta} = \boldsymbol{\eta}_*$ and use the equalities $\Lambda^1(\boldsymbol{\eta}_*) = \boldsymbol{\eta}^1$ and $\Lambda^2(\boldsymbol{\eta}_*) = \boldsymbol{\eta}^2$ combined with (4.27)-(4.28), (3.47) and (4.41) to conclude that (3.48)-(3.49) are satisfied. We write (4.11) for $\boldsymbol{\eta} = \boldsymbol{\eta}_*$ and use (4.41) and (4.43) to obtain (3.50)-(3.51). We write (4.20) for $\boldsymbol{\eta} = \boldsymbol{\eta}_*$ and use (4.41) to obtain (3.52). Next, (3.53) and the regularities (4.1)-(4.4) follow from Lemmas 4.4, 4.5, 4.6 and the relation (4.22). The regularity of $\boldsymbol{\sigma}$ is a consequence of Lemma 4.4, 4.5, 4.7, the relations (4.41)-(4.42) and the assumptions on $\mathcal{A}$ and $\mathcal{E}$. The regularity of $\mathbf{D}$ follows from the regularity of $\mathbf{u}$ and φ given by (4.1)-(4.2), the relation (4.43) and the assumptions on $\mathbf{B}$ and $\mathcal{E}$. The uniqueness of Theorem 4.1 is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.26)-(4.28) and the unique solvability of the problems PV_η, QV_η, PV_β and PV_{σ_η}. ■

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